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ON THE BEHAVIOR NEAR THE ORIGIN OF SINE SERIES WITH CONVEX COEFFICIENTS

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In memory of Professor Slobodan Aljančić

Abstract. Let a numerical sequence $\{a_k\}$ tend to zero and be convex. We obtain estimates of

$$g(x) := \sum_{k=1}^{\infty} a_k \sin kx$$

for $x \to 0$ expressed in terms of the coefficients a_k. These estimates are of order- or asymptotic character. For example, the following order equality is true:

$$g(x) \sim m a_m + \frac{1}{m} \sum_{k=1}^{m-1} k a_k,$$

where

$$x \in \left(\frac{\pi}{m+1}, \frac{\pi}{m} \right].$$

We consider the sine series

$$\sum_{k=1}^{\infty} a_k \sin kx, \tag{1}$$

with coefficients tending to zero and such that the sequence $\{a_k\}$ is monotone or convex, i.e. $\Delta a_k := a_k - a_{k+1} \geq 0$ or, respectively, $\Delta^2 a_k := \Delta a_k - \Delta a_{k+1} = a_k - 2a_{k+1} - a_{k+2} \geq 0$ for all k. It is well known that under such conditions the series (1) converges for all x. Denote $g(x)$ its' sum.

We obtain estimates of $g(x)$ for $x \to 0$ expressed in terms of the coefficients a_k. These estimates are of order- or asymptotic character. Apparently, it was Young [**1**] who was the first to consider this problem, and he was concerned solely

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about estimates of $|g(x)|$ from above. Papers by other authors are quoted in the sequel, when we compare our estimates with the known results. Only those results are quoted here which are directly connected with ours.

For brevity sake, let

$$I_m := \left(\frac{\pi}{m+1}, \frac{\pi}{m} \right], \quad m = 1, 2, \ldots \tag{2}$$

Everywhere in the sequel, the constants in O-expressions are absolute. Numerical factors in order relations are also absolute constants.

Two theorems are proved. The first of them deals with the series with monotone coefficients, and the second – with convex ones.

THEOREM. *Assume that $a_k \downarrow 0$. Then for $x \in I_m$ the following estimate is valid*

$$g(x) = \sum_{k=1}^{m} k a_k x + O\left(\frac{1}{m^3} \sum_{k=1}^{m} k^3 a_k \right). \tag{3}$$

Proof. Applying Abel's transform we obtain

$$g(x) = \sum_{k=1}^{\infty} \Delta a_k \widetilde{D}_k(x), \tag{4}$$

where $\widetilde{D}_k(x)$ are the conjugate Dirichlet kernels. Since $\widetilde{D}_k(x) = O(1/x)$ and $x \in I_m$, we see that

$$\sum_{k=m+1}^{\infty} \Delta a_k \widetilde{D}_k(x) = O\left(\sum_{k=m+1}^{\infty} \Delta a_k \frac{1}{x} \right) = O(m a_m).$$

Furthermore,

$$\sum_{k=1}^{m} \Delta a_k \widetilde{D}_k(x) = \sum_{k=1}^{m} a_k \sin kx - a_{m+1} \widetilde{D}_m(x) = \sum_{k=1}^{m} a_k \sin kx + O(m a_m).$$

Therefore,

$$g(x) = \sum_{k=1}^{m} a_k \sin kx + O(m a_m). \tag{5}$$

The estimate (3) follows from (5), since

$$\sum_{k=1}^{m} a_k \sin kx = \sum_{k=1}^{m} k a_k x + O\left(\sum_{k=1}^{m} k^3 a_k x^3 \right)$$

and by virtue of the monotonicity of a_k,

$$m a_m \le \frac{4}{m^3} \sum_{k=1}^{m} k^3 a_m \le \frac{4}{m^3} \sum_{k=1}^{m} k^3 a_k.$$

This completes the proof.

Let's compare the estimate (3) with the theorem of Hartman and Wintner [**2**]. They proved that if $a_k \downarrow 0$, then

$$\lim_{x\to 0} \frac{g(x)}{x} = \sum_{k=1}^{\infty} k a_k, \tag{6}$$

no matter whether the series

$$\sum_{k=1}^{\infty} k a_k \tag{7}$$

converges or diverges.

If the series (7) is convergent, then

$$\frac{1}{m^2} \sum_{k=1}^{m} k^3 a_k \to 0 \quad \text{as } m \to \infty.$$

This follows by standard arguments. For an arbitrary N one has:

$$\frac{1}{m^2} \sum_{k=1}^{m} k^3 a_k \le \frac{1}{m^2} \sum_{k=1}^{N} k^3 a_k + \sum_{k=N+1}^{\infty} k a_k.$$

We first choose, for a given $\varepsilon > 0$, the number $N = N(\varepsilon)$ so that

$$\sum_{k=N+1}^{\infty} k a_k < \frac{\varepsilon}{2}.$$

Then for all sufficiently large m

$$\frac{1}{m^2} \sum_{k=1}^{m} k^3 a_k < \varepsilon.$$

Thus, if the series (7) converges, then the estimate (3) provides an asymptotic equality sharpening the result (6) of Hartman and Wintner.

However, (3) can be also used to obtain asymptotic expression for $g(x)$ as $x \to 0$ in certain cases, when the series (7) is divergent. For example, if $a_k = k^{-2}$, then according to (3) for $x \in I_m$ we have

$$g(x) = x \log m + O(1/m) = x \log(1/x) + O(x).$$

Note that Salem [**3**; 4, theorem 1] proved that if the sequence $k a_k$ is monotone decreasing, then the following order equality is true:[1]

$$g(x) \sim \sum_{k=1}^{m} k a_k x, \quad x \in I_m, \quad x \to 0. \tag{8}$$

[1] We write $u(x) \sim v(x), x \to 0$ if there are absolute positive constants c and C such that $c \le u(x)/v(x) \le C$ is in a neighborhood of the point $x = 0$, and we write $u(x) \approx v(x), x \to 0$ if $lim_{x\to 0} u(x)/v(x) = 1$.

This result is not completely covered by our estimate (3).

THEOREM *Let $a_k \to 0$ and let the sequence $\{a_k\}$ be convex. If $x \in I_m$, where $m \geq 11$, then the following estimate holds true:*

$$\frac{a_m}{2} \cot \frac{x}{2} + \frac{1}{2m} \sum_{k=1}^{m-1} k^2 \Delta a_k \leq g(x) \leq \frac{a_m}{2} \cot \frac{x}{2} + \frac{6}{m} \sum_{k=1}^{m-1} k^2 \Delta a_k. \tag{9}$$

Proof. For $x \in (0, \pi]$ and $k = 0, 1, 2, \ldots$, introduce the functions

$$\varphi_k(x) := -\frac{\cos(k + 1/2)x}{2 \sin x/2} \tag{10}$$

and

$$\psi_k(x) := \sum_{i=0}^{k} \varphi_i(x) = -\frac{\sin(k+1)x}{4 \sin^2 x/2}. \tag{11}$$

According to (4)

$$\begin{aligned} g(x) &= \sum_{k=1}^{m-1} \Delta a_k \widetilde{D}_k(x) + \sum_{k=m}^{\infty} \Delta a_k \Big(\frac{1}{2} \cot \frac{x}{2} + \varphi_k(x) \Big) \\ &= \frac{a_m}{2} \cot \frac{x}{2} + \sum_{k=1}^{m-1} \Delta a_k \widetilde{D}_k(x) + \sum_{k=m}^{\infty} \Delta a_k \varphi_k(x). \end{aligned}$$

After applying Abel's transform to the extreme sum on the right-hand side we obtain

$$g(x) = \frac{a_m}{2} \cot \frac{x}{2} + A_m(x) + B_m(x), \tag{12}$$

where for the sake of brevity we introduced the notations

$$A_m(x) := \sum_{k=1}^{m-1} \Delta a_k \widetilde{D}_k(x) \tag{13}$$

$$B_m(x) := \sum_{k=m}^{\infty} \Delta^2 a_k (\psi_k(x) - \psi_{m-1}(x)). \tag{14}$$

We will make use of the representation (12) for $x \in I_m$.

From now and till the end of the proof of theorem 2 we assume that $x \in I_m$, and will not remind of it.

To prove the first inequality (9) we will make use of the following estimate, which is true in view of the monotonous decay of Δa_k and the positivity of $\widetilde{D}_k(x)$ for $k \leq m$:

$$A_m(x) \geq \Delta a_m \sum_{k=1}^{m-1} \widetilde{D}_k(x) = \Delta a_m \sum_{k=0}^{m-1} \Big(\frac{1}{2} \cot \frac{x}{2} + \varphi_k(x) \Big)$$

$$= \Delta a_m\Big(\frac{m}{2}\cot x/2 + \psi_{m-1}(x)\Big) = \frac{\Delta a_m}{4\sin^2 x/2}(m\sin x - \sin mx). \qquad (15)$$

The following estimate from above holds for $B_m(x)$:

$$|B_m(x)| \le \sum_{k=m}^{\infty} \Delta^2 a_k |\psi_k(x) - \psi_{m-1}(x)| \le \frac{\Delta a_m}{4\sin^2 x/2}(1+\sin mx). \qquad (16)$$

From (15) and (16) we obtain

$$\begin{aligned}\frac{1}{2}A_m(x) + B_m(x) &\ge \frac{\Delta a_m}{8\sin^2 x/2}(m\sin x - \sin mx - 2(1+\sin mx))\\ &= \frac{\Delta a_m}{8\sin^2 x/2}(m\sin x - 2 - 3\sin mx).\end{aligned}$$

But

$$m\sin x - 2 - 3\sin mx \ge m\sin\frac{\pi}{m+1} - 2 - 3\sin\frac{m\pi}{m+1} = (m-3)\sin\frac{\pi}{m+1} - 2.$$

The latter expression is positive for $m \ge 11$. Indeed, it is easy to verify that it is increasing along with m. Thus, it is enough to prove its' positivity for $m = 11$:

$$8\sin\frac{\pi}{12} - 2 = 8\cdot\frac{1}{2\sqrt{2+\sqrt{3}}} - 2 > 0.$$

Thus, for $m \ge 11$

$$\frac{1}{2}A_m(x) + B_m(x) \ge 0. \qquad (17)$$

Furthermore, if $k < m$, then

$$\widetilde{D}_k(x) = \sum_{i=1}^{k}\sin ix \ge \sum_{i=1}^{k}\frac{2}{\pi}ix = \frac{k(k+1)}{\pi}x \ge \frac{k(k+1)}{m+1} > \frac{k^2}{m}.$$

Therefore,

$$\frac{1}{2}A_m(x) \ge \frac{1}{2m}\sum_{k=1}^{m-1}k^2\Delta a_k. \qquad (18)$$

The first inequality (9) follows from (12), (17) and (18).

For the proof of the second inequality (9), we will use the following estimate:

$$\widetilde{D}_k(x) \le \sum_{i=1}^{k} ix \le k^2 x \le \pi\frac{k^2}{m}.$$

Thus,

$$A_m(x) \le \frac{\pi}{m}\sum_{k=1}^{m-1}k^2\Delta a_k. \qquad (19)$$

Next we estimate the factor by Δa_m on the right-hand side of (16):

$$\frac{1+\sin mx}{4\sin^2 x/2} \le \frac{1}{2\sin^2 x/2} \le \frac{\pi^2}{2x^2} \le \frac{(m+1)^2}{2}.$$

Note that for $m \ge 11$

$$\frac{(m+1)^2}{2} < \frac{3(m+1)^2}{2(m-1)^3}\sum_{k=1}^{m-1} k^2 < \frac{2,4}{m}\sum_{k=1}^{m-1} k^2.$$

Therefore, because of the monotonicity of Δa_m it follows from (16), that

$$|B_m(x)| \le \frac{2,4}{m}\sum_{k=1}^{m-1} k^2 \Delta a_k. \tag{20}$$

The second inequality (9) follows from (12), (19) and (20).

This completes the proof of the theorem.

It follows from (9) that for $x \in I_m$ in a sufficiently small neighbourhood of the origin we have

$$g(x) = \frac{a_m}{2}\cot\frac{x}{2} + O\left(\frac{1}{m}\sum_{k=1}^{m-1} k^2\Delta a_k\right) \tag{21}$$

and if for $m \to \infty$

$$\frac{1}{m}\sum_{k=1}^{m-1} k^2\Delta a_k = o(ma_m),$$

then (21) provides an asymptotical representation of $g(x)$ for $x \to 0$. For example, this is true if $a_k = \log^{-\gamma}(k+1), \gamma > 0$. Shogunbekov [**5**] obtained a somewhat less sharp result, than estimate (21).

Aljančić, Bojanić and Tomić [**6**, theorem 2] give asymptotic expession for $g(x)$ as $x \to 0$, when the coefficients a_k are convex and can be represent as the values $A(k)$ of a slowly varying (in Karamata's sense) function $A(u)$, i.e. for each $t > 0$

$$\lim_{u\to\infty}\frac{A(tu)}{A(u)} = 1. \tag{22}$$

Their result is equivalent to the following statement which can be deduced from (21).

Corollary. *Suppose that the coefficients of the series (1) satisfy the conditions of Theorem 2 and that $a_k = A(k)$, for a slowly varying function $A(u)$. Then the following asymptotic equality holds true:*

$$g(x) \approx a_m\frac{1}{x}, \quad x \in I_m, \quad x \to 0. \tag{23}$$

We apply the same reasoning by which the estimate of Σ_2 was obtained in [**6**, pp. 118 – 119]. Let $n := \left[\frac{k+1}{2}\right]$, then $2n-1 \le k \le 2n$. Since the first differences decrease monotonically we have

$$a_n - a_{2n} = \sum_{i=n}^{2n-1} \Delta a_i \geq n\Delta a_{2n-1} \geq \frac{k}{2}\Delta a_k. \tag{24}$$

On the other hand,

$$a_n - a_{2n} = a_{2n}\left(\frac{a_n}{a_{2n}} - 1\right) = a_{2n}\varepsilon_k, \tag{25}$$

where according to (22) we have $\varepsilon_k \to 0$ as $k \to \infty$. Applying (24) and (25) we obtain

$$k\Delta a_k \leq 2a_k\varepsilon_k. \tag{26}$$

Thus

$$\sum_{k=1}^{m-1} k^2\Delta a_k \leq 2\sum_{k=1}^{m-1} ka_k\varepsilon_k.$$

Since $A(u)$ is a slowly varying function, the sequence ka_k is almost increasing. i.e. $ka_k \leq Cma_m$ for all k, m, with $k < m$, C being a constant. Therefore we have

$$\sum_{k=1}^{m-1} k^2\Delta a_k \leq 2Cma_m \sum_{k=1}^{m-1} \varepsilon_k = o(m^2 a_m), \quad m \to \infty,$$

and now (23) follows from (21).

Salem [**3**; 4, theorem 1] proved, that if the sequence $\{a_k\}$ is convex and the sequence $\{ka_k\}$ is growing monotonically , then for $x \to 0$ the following order equality holds true:

$$g(x) \sim ma_m, \quad x \in I_m. \tag{27}$$

This result is presented in Bary's monograph [**7**] and in the first edition of Zygmund's [**8**].

It follows from Theorem 2 that if we assume only the convexity of a_k, then

$$g(x) \sim ma_m + \frac{1}{m}\sum_{k=1}^{m-1} k^2\Delta a_k, \quad x \in I_m, \quad x \to 0. \tag{28}$$

Relation (28) can be rewritten in a simpler form. To this end, estimate the latter sum in (28) as follows:

$$\frac{1}{m}\sum_{k=1}^{m-1}(2k-1)a_k - ma_m \leq \frac{1}{m}\sum_{k=1}^{m-1} k^2\Delta a_k \leq \frac{1}{m}\sum_{k=1}^{m-1}(2k-1)a_k.$$

The following statement is a corollary from this estimate and (28).

COROLLARY. *Let the sequence* $\{a_k\}$ *tend to zero and be convex. Then the following order equality is true:*

$$g(x) \sim m a_m + \frac{1}{m} \sum_{k=1}^{m-1} k a_k, \quad x \in I_m, \quad x \to 0. \tag{29}$$

Note, that the above proof of estimates (9) and (29) is easier, than Salem's proof of (27).

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