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APPROXIMATION-SOLVABILITY OF HAMMERSTEIN EQUATIONS

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To the memory of Slobodan Aljančić

1. Introduction.

In this paper, we shall study Hammerstein operator equations of the form

$$x - KFx = f \tag{1.1}$$

where K is linear and F is a nonlinear map. We first study Eq. (1.1) in the operator form using the (pseudo) A-proper mapping approach and the Brouwer degree theory. Then we apply the obtained results to Hammerstein integral equations. There is an extensive literature on Hammerstein equations and we refer to [Kr], [KZ] and [V].

2. Some preliminries on A-proper maps.

Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach spaces X and Y respectively such that $\dim X_n = \dim Y_n$ for each n and $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. Let $P_n : X \to Y_n$ and $Q_n : Y \to Y_n$ be linear projections onto X_n and Y_n respectively such that $P_n x \to x$ for each $x \in X$ and $\delta = \max ||Q_n|| < \infty$. Then $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ is a projection scheme for (X, Y).

Definition 2.1. A map $T : D \subset X \to Y$ is said to be *approximation-proper* (*A-proper* for short) with respect to Γ if (i) $Q_n T : D \cap X_n \to Y_n$ is semicontinuous for each n and (ii) whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $||Q_{n_k} T x_{n_k} - Q_{n_k} f|| \to 0$ for some $f \in Y$, then a subsequence $x_{n_{k(i)}} \to x$ and $Tx = f$. T is said to be *pseudo A-proper* w.r.t. Γ if in (ii) above we do not require that a subsequence of $\{x_{n_k}\}$

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converges to x for which $f \in Tx$. If f is given in advance, we say that T is (pseudo) A-proper at f.

For the developments of the (pseudo) A-proper mapping theory and applications to differential equations, we refer to [Mi-5,8] and [P]. To demonstrate the generality and the unifying nature of the (pseudo) A-proper mapping theory, we state now a number of examples of A-proper and pseudo A-proper maps.

To look at ϕ-condensing maps, we recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0 : D$ has a finite covering by sets of diameter less than $d\}$. The *ball-measure of noncompactness* of D is defined as $\chi(D) = \inf\{r > 0 | D \subset \cup_{i=1}^{n} B(x_i, r),\ x \in X,\ n \in N\}$. Let ϕ denote either the set or the ball-measure of noncompactness. Then a map $N : D \subset X \to X$ is said to be *$k - \phi$ contractive* (*ϕ-condensing*) if $\phi(N(Q)) \leq k\phi(Q)$ (respectively $\phi(N(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$).

Recall that $N : X \to Y$ is K-monotone for some $K : X \to Y^*$ if $(Nx - Ny, K(x - y)) \geq 0$ for all $x,\ y \in X$. It is said to be generalized pseudo-K-monotone (of type (KM)) if whenever $x_n \rightharpoonup x$ and $\limsup(Nx_n, K(x_n - x)) \leq 0$ then $(Nx_n, K(x_n - x)) \to 0$ and $Nx_n \rightharpoonup Nx$ (then $Nx_n \rightharpoonup Nx$). Recall that N is said to be of type (KS_+) if $x_n \rightharpoonup x$ and $\limsup(Nx_n, K(x_n - x)) \leq 0$ imply that $x_n \to x$. If $x_n \rightharpoonup x$ implies that $\limsup(Nx_n\ , K(x_n - x)) \geq 0$, N is said to be of type (KP). If $Y = X^*$ and K is the identity map, then these maps are called monotone, generalized pseudo monotone, of type (M) and (S_+) respectively. If $Y = X$ and $K = J$ the duality map, then J-monotone maps are called accretive. It is known that bounded monotone maps are of type (M). We say that N is demicontinuous if $x_n \to\ x$ in X implies that $Nx_n \rightharpoonup Nx$. It is well known that $I - N$ is A-proper if N is ball-condensing and that K-monotone like maps are pseudo A-proper under some conditions on N and K. Moreover, their perturbations by Fredholm or hyperbolic like maps are A-proper or pseudo A-proper. (see [Mi-5,7].

The following result states that ball-condensing perturbations of stable A-proper maps are also A-proper.

THEOREM 2.1. [Mi-1] *Let $D \subset X$ be closed, $T : X \to Y$ be continuous and A-proper w.r.t. a projectional scheme Γ and a-stable, i.e., for some $c > 0$ and n_0*

$$||Q_nTx - Q_nTy|| \geq c||x - y|| \ \ for\, x, y \in X_n \ and\, n \geq n_0$$

and $F : D \to Y$ be continuous. Then $T + F : D \to Y$ is A-proper w.r.t. Γ if F is k-ball contrctive with $k\delta < c$, or it is ball-condensing if $\delta = c = 1$.

Remark 2.1. The A-properness of T in Theorem 2.2 is equivalent to T being surjective. In particular, as T we can take a c-strongly K- monotone map for a suitable $K : X \to Y^*$, i.e., $(Tx - Ty, K(x - y)) \geq c||x - y||^2$ for all $x, y \in X$. In particular, since c-strongly accretive maps are surjective, we have the following important special case [Mi-1].

COROLLARY 2.1. *Let X be a π_1 space, $D \subset X$ be closed, $T : X \to X$ be continuous and c-strongly accretive and $F : D \to X$ be continuous and either k-ball* *contractive with $k < c$, or it is ball-condensing if $c = 1$. Then $T + F : D \to X$ is A-proper w.r.t. Γ.*

To study error estimates of approximate solutions for nondifferentiable maps, we need a notion of a multivalued derivative. Let $U \subset X$ be an open set and $T : \bar{U} \to Y$. A positively homogeneous map $A : X \to 2^Y$, with Ax closed and convex for each $x \in X$, is said to be a *multivalued derivative* of T at $x_0 \in U$ if there is a map $R = R(x_0) : \bar{U} - x_0 \to 2^Y$ such that $||y||/||x - x_0|| \to 0$ as $x \to x_0$ for each $y \in R(x - x_0)$ and

$$Tx - Tx_0 \in A(x - x_0) + R(x - x_0) \text{ for } x \text{ near } x_0.$$

A map $A : X \to 2^Y$ is m-bounded if there is $m > 0$ such that $||y|| \le m||x||$ for each $y \in Ax$, $x \in X$. It is c-coercive if $||y|| \ge c||x||$ for each $y \in Ax$, $x \in X$.

The following result from [Mi-5] will be needed below.

THEOREM 2.2. *Let $T : \overline{U} \subset X \to Y$ be A-proper w.r.t. Γ and x_0 be a solution of $Tx = f$. Suppose that A is an odd multivalued derivative of T at x_0 and there exist constants $c_0 > 0$ and $n_0 \ge 1$ such that*

$$||Q_n u|| \ge c_0||x|| \quad \text{for } x \in X_n,\ u \in Ax,\ n \ge n_0. \tag{2.1}$$

(a) If x_0 is an isolated solution, then the equation $Tx = f$ is strongly approximation solvable in $B_r(x_0)$ for some $r > 0$.
(b) If, in addition, A is c_1-coercive for some $c_1 > 0$, then x_0 is an isolated solution, the conclusion of (a) holds and, for $\epsilon \in (0, c_0)$, approximate solutions x_n satisfy

$$||x_n - x_0|| \le (c_0 - \epsilon)^{-1}||Tx_n - f|| \quad \text{for } \quad n \ge n_1 \ge n_0. \tag{2.2}$$

(c) If x_0 is an isolated solution in $B_r(x_0)$, A is c_2-bounded for some c_2 and

$$Tx - Ty \in A(x - y) + R(x - y) \quad \text{whenever} \quad x - y \in B_r \tag{2.3}$$

and $z/||x - y|| \to 0$ as $x \to x_0$ and $y \to x_0$ for each $z \in R(x - y)$, then the equation $Tx = f$ is uniquely approximation solvable in $B_r(x_0)$ and the unique solutions $x_n \in B_r(x_0) \cap X_n$ of $Q_nTx = Q_nf$ satisfy

$$||x_n - x_0|| \le k||P_n x_0 - x_0|| \le c\,\mathrm{dist}(x_0, X_n), \tag{2.4}$$

where k depends on c_0, c_2, ϵ and δ and $c = 2k\delta_1$, $\delta_1 = \sup ||P_n||$.

3. Hammerstein operator equations

We shall consider (1.1) in a general setting between two Banach spaces. To that end, we shall use two approaches. One is based on applying the Brouwer degree theory directly to the finite dimensional approximations of the map $I - KF$, and the other one is based on splitting first the map K as a product of two suitable maps and then use the Brouwer degree.

A. A direct method. In this section, we shall prove a number of solvability results of (1.1) imposing various types of conditions on K and F.

Theorem 3.1. *Let X and Y be Banach spaces, $K : Y \to X$ be linear and continuous and $N : X \to Y$ be nonlinear and such that $I - KF : X \to X$ is pseudo A-proper w.r.t. $\Gamma = \{X_n, P_n\}$. Suppose that there are some constants a and b such that $\delta a||K|| < 1$, $\delta = \max||P_n||$, and*

$$||Fx|| \leq a||x|| + b \ \text{ for all } ||x|| \geq R.$$

Then Eq. (1.1) *is solvable for each $f \in X$.*

Proof. Consider the homotopy $H(t, x) = x - tKFx - tf$. Then our assumptions imply that for each $f \in X$ there is an $r > R$ and $n_0 \geq 1$ such that

$$P_n H(t, x) \neq tP_n f \ \text{ for all } t \in [0, 1],\ x \in \partial B(0, r) \cap X_n,\ n \geq n_0.$$

By the Brouwer degree properties and the pseudo A-properness of $I - KF$, there is an $x \in X$ such that $x - KFx = f$. □

We say that a map T satisfies condition (+) if whenever $Tx_n \to f$ in Y then $\{x_n\}$ is bounded in X. T satisfies condition (++) if whenever $\{x_n\}$ is bounded and $Tx_n \to f$, then $Tx = f$ for some $x \in X$.

Let $\sigma(K)$ denote the spectrum of K. Our next result involves a suitable Leray-Schauder type of condition.

Theorem 3.2. *Let $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$, $F : X \to X$ be nonlinear, $T_p = pI - (I - \lambda K)^{-1}K(F - \lambda I) : X \to X$ for $p \geq 1$, T_1 satisfy condition (+) and either F is odd or, for some $R > 0$,*

$$K(F - \lambda I)x \neq t(I - \lambda K)x \ \text{ for } ||x|| \geq R, t > 1. \tag{3.1}$$

a) If T_1 is A-proper w.r.t. Γ, then Eq. (1.1) *is approximation solvable for each $f \in X$.*
b) If T_p is A-proper w.r.t. Γ for each $p > 1$ and T_1 satisfies condition (++), then Eq. (1.1) *is solvable for each $f \in X$.*

Proof. Eq. (1.1) is equivalent to

$$Ax - Nx = f \tag{3.2}$$

where $A = I - \lambda K$ and $N = K(F - \lambda I)$. It is easy to see that (3.1) implies that

$$Nx \neq tAx \ \text{ for } ||x|| \geq R, t > 1.$$

Hence, the conclusion follows from Theorem 3.1 in [Mi-2]. □

Corollary 3.1. *Let $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$, $F : X \to X$ be nonlinear, $T_p = pI - (I - \lambda K)^{-1}K(F - \lambda I) : X \to X$ for $p \geq 1$, and*

$$\limsup_{||x|| \to \infty} ||Fx - \lambda x||/||x|| < ||(I - \lambda K)^{-1}K||^{-1}. \tag{3.3}$$

a) If T_1 is A-proper w.r.t. Γ, then Eq. (1.1) *is approximation solvable for each $f \in X$.*

b) If T_p is A-proper w.r.t. Γ for each $p > 1$ and T_1 satisfies condition (++), then Eq. (1.1) *is solvable for each $f \in X$.*

COROLLARY 3.2. *Let X be a uniformly convex space with a scheme $\Gamma = \{X_n, P_n\}$, $\max ||P_n|| = 1$, $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$ and $F : X \to X$ be nonlinear such that $(I - \lambda K)^{-1}K(F - \lambda I) : X \to X$ is nonexpensive and* (3.3) *hold. Then Eq.* (1.1) *is solvable for each $f \in X$.*

Let us now look at some special cases.

THEOREM 3.3. *Let $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$, $d = ||(I - \lambda K)^{-1}K||^{-1}$ and $F : X \to X$ be nonlinear and continuous.*
a) Let, for some $k \in (0, d)$

$$||Fx - \lambda x - (Fy - \lambda y)|| \leq k||x - y|| \text{ for all } x, y \in H. \tag{3.4}$$

Then Eq. (1.1) *is uniquely solvable for each $f \in X$ and the solution is the limit of the iteration process*

$$x_n - \lambda K x_n = KFx_{n-1} - \lambda K x_{n-1} + f. \tag{3.5}$$

b) If, in addition, either K is compact or $\delta = \max ||P_n|| = 1$ and $k||(I - \lambda K)^{-1}|| \times ||K|| < 1$, then Eq. (1.1) *is approximation solvable w.r.t. Γ for each $f \in X$ and the approximate solutions $\{x_n \in X_n\}$ of $x - P_nKFx = P_nf$ satisfy*

$$||x_n - x|| \leq c||x_n - KFx_n - f|| \text{ for some } c \text{ and all large } n. \tag{3.6}$$

and

$$||x_n - x|| \leq c||P_n x - x|| \leq c_1 \operatorname{dist}(x, X_n). \tag{3.7}$$

c) If condition (3.4) *holds with $k = d$, X is a uniformly convex space with $\delta = 1$ and*

$$||Fx - \lambda x|| \leq a||x|| + b \text{ for some } a < k, b > 0, x \in X. \tag{3.8}$$

then Eq. (1.1) *is solvable for each $f \in X$.*

Proof. Eq. (1.1) is equivalent to (3.2) with $A = I - \lambda K$ and $N = K(F - \lambda I)$. Hence, it is easy to show that $A^{-1}N$ is $k_1 = k||A^{-1}K||$-contractive with $k_1 < 1$. Thus, part a) follows from the contractive fixed point principle and c) follows from Corollary 3.2. Regarding part b), we need only to show that condition (2.1) of Theorem 2.2 holds. Assume first that K is compact. Then $I - KF$ is A-proper w.r.t. Γ. Set $B_1 x = \{K(y - \lambda x) \mid ||y - \lambda x|| \leq k||x||\}$ and $Bx = Ax - B_1 x$ for $x \in X$. Then B is homogeneous with Bx convex for each $x \in X$ and $A(x-y) - (Nx - Ny) \in B(x - y)$ for each $x, y \in X$. Moreover, if $0 \in Bx$, then $Ax = K(y - \lambda x)$ for some y and

$$||x|| \leq ||A^{-1}K|| \, ||y - \lambda x|| < ||x||.$$

Hence, $x = 0$. Since B_1 is upper semicontinuous and compact, $B = A - B_1$ is A-proper w.r.t. Γ and satisfies (2.1) by Lemma 2.2 in [Mi-2]. Since also $Nx - Ny \in B_1(x - y)$, the conclusions follow from Theorem 2.2.

Next, let $\delta = 1$ and $k||(I-\lambda K)^{-1}||\ ||K|| < 1$. Then $I - KF$ is A-proper w.r.t. $\Gamma = \{X_n, P_n\}$. Indeed, let $\{x_n \in X_n\}$ be bounded and $x_n - P_nKFx_n \to f$. Set $y_n = (I-\lambda K)x_n$. Then $y_n - P_nK(F-\lambda I)(I-\lambda K)^{-1}y_n \to f$ and the map $F_1 = (F-\lambda I)(I-\lambda K)^{-1}$ is an l-contraction with $l < 1$. Hence, $I - F_1$ is A-proper w.r.t. Γ and therefore, a subsequence $y_{n_k} \to y$ and $y - F_1y = f$. Hence, $x - KFx = f$ with $x = (I-\lambda K)^{-1}y$, proving that $I - KF$ is A-proper.

Now, let $y \in P_n(Ax - B_1x)$ for some $x \in X_n$. Then $y = P_n(Ax - Kv) = Ax - P_nKv$ for some v with $||v|| \le k||x||$ and $x = A^{-1}P_n(y + Kv)$. Hence,

$$||x|| \le \delta ||A^{-1}||(||y|| + k||K||\ ||x||)$$

and

$$(1 - k||A^{-1}||\ ||K||))||x|| \le ||A^{-1}||\ ||y||$$

which implies that A is c-coercive. Thus, Theorem 2.2 applies. $\square$

Let us now specialize this to a Hilbert spase H setting. Let $\Sigma(K)$ be the set of characteristic values of K, i.e., $\Sigma(K) = \{\mu \mid 1/\mu \in \sigma(K)\}$.

THEOREM 3.4. *Let $K : H \to H$ be a selfadjoint map, $\lambda \notin \Sigma(K)$, $F : H \to H$ be nonlinear and continuous and $T_p = pI - (I-\lambda K)^{-1}K(F-\lambda I) : H \to H$ for $p \ge 1$. Suppose that for some k with $k\delta < d = \operatorname{dist}(\lambda, \Sigma(K))$*

$$\limsup_{||x|| \to \infty} ||Fx - \lambda x||/||x|| < k.$$

a) If T_1 is A-proper w.r.t. Γ, then Eq. (1.1) *is approximation solvable for each $f \in H$.*
b) If T_p is A-proper w.r.t. Γ for each $p > 1$ and T_1 satisfies condition (++), then Eq. (1.1) *is solvable for each $f \in H$.*

Proof. Eq. (1.1) is equivalent to $x = (I-\lambda K)^{-1}K(F-\lambda)x + (I-\lambda K)^{-1}f$. Since $(I-\lambda K)^{-1}K = -1/\lambda + 1/\lambda(I-\lambda K)^{-1}$, we have that ([K]) $||(I-\lambda K)^{-1}K|| = \sup_{\mu\in\sigma(K)} |-1/\lambda + 1/\lambda(1-\lambda\mu)^{-1}| = \sup_{\mu\in\sigma(K)} |(\mu-\lambda)^{-1}| = d^{-1}$. Then the concluions follow from Corollary 3.1 $\square$

Let $\mu^* = \inf\{\mu \mid \mu \in \Sigma(K) \cap (0,\infty)\}$. For $c \in \Sigma(K) \cap (-\infty, \mu^*]$, define $d_c^- = \operatorname{dist}(c, \Sigma(K) \cap (-\infty, c))$.

THEOREM 3.5. *Let $K : H \to H$ be a selfadjoint map, $F : H \to H$ be nonlinear and continuous and*
(i) $(Fx - Fy, x - y) \ge \alpha||x-y||^2$ for all $x, y \in H$,
(ii) $||Fx - Fy|| \le \beta||x - y||$ for all $x, y \in H$.
(a) If (i)–(ii) hold and $\beta^2 < \alpha d_c^- + c(d_c^- - c - 2\alpha)$ for some $c \le \mu^*$, then Eq. (1.1) *is uniquely approximation solvable for each $f \in H$ and* (3.6)–(3.7) *hold.*
(b) If $\beta^2 \le \alpha d_c^- + c(d_c^- - c - 2\alpha)$ and, for some $a < \lambda = c - d_c^-/2$ and $b > 0$,

$$||Fx - \lambda x|| \le a||x|| + b \ \ \textit{for all}\ x \in H$$

then Eq. (1.1) *is solvable for each $f \in H$.*

Proof. Let $\lambda = c - d_c^-/2$. Then $\lambda \notin \Sigma(K)$ and $d = \mathrm{dist}(\lambda, \Sigma(K)) > 0$ with $d^{-1} = ||(I - \lambda K)^{-1} K||$. Using conditions (i)–(ii), we get

$$||Fx + \lambda x - (Fy + \lambda y)|| \leq (\beta^2 + \lambda^2 + 2\alpha\lambda)^{1/2} ||x - y||.$$

By our choice of λ and the condition on β, we get

$$\beta^2 + \lambda^2 + 2\alpha\lambda = \beta^2 + \alpha d_c^- + c(d_c^- - c - 2\alpha) + (d_c^-/2)^2 < (d_c^-/2)^2 = d^2.$$

Hence, the conclusions follow from Theorem 3.3. $\square$

THEOREM 3.6. *Let $K : H \to H$ be selfadjoint, $F : H \to H$ be a gradient map and $B^{\pm} : H \to H$ be selfadjoint maps such that*
(i) $(B^-(x - y), x - y) \leq (Fx - Fy, x - y) \leq (B^+(x - y), x - y)$ for all $x, y \in H$.
(ii) $\delta ||B^{\pm} - \lambda I|| \leq d = \min\{|\mu| \mid \mu \in \sigma(I - \lambda K)^{-1} K\}$.
(a) If the inequality is strict in (ii), then Eq. (1.1) *is uniquely approximation solvable w.r.t. Γ for H for each $f \in H$ and the approximate solutions satisfy* (3.6)–(3.7).
(b) If, in addition, there are $0 < a < d$ and $b \geq 0$ such that

$$||Fx - \lambda x|| \leq a||x|| + b \ \textit{ for all } x \in H$$

then Eq. (1.1) *is solvable for each $f \in H$.*

Proof. Since C is a gradient of the functional $x \to (Cx, x)/2$, $N - C$ is a gradient map and

$$-||B^- - \lambda I|| \, ||x - y||^2 \leq ((B^- - \lambda I)(x - y), x - y),$$
$$((B^+ - \lambda I)(x - y), x - y) \leq ||B^+ - \lambda I|| \, ||x - y||^2.$$

Hence, by Lemma 1 in [Mi-3],

$$||Fx - \lambda x - (Ny - \lambda y)|| \leq k||x - y|| \ \text{ for all } x, y \in H$$

where $k = \max(||B^- - \lambda I||, ||B^+ - \lambda I||)$. Since $d = ||(I - \lambda K)^{-1} K||^{-1}$ ([K]), the conclusions follow from Theorem 3.3. $\square$

For $c \in \Sigma(K) \cap (\mu^*, \infty)$, define $d_c^+ = \mathrm{dist}(c, \Sigma(K) \cap (c, \infty))$. We have the following sharper version of Theorem 3.5.

THEOREM 3.7. *Let $K : H \to H$ be selfadjoint, $F : H \to H$ be a gradient map and α, $\beta \in R$ be such that*

$$\alpha||x - y||^2 \leq (Fx - Fy, x - y) \leq \beta||x - y||^2 \quad \textit{for} \quad x, y \in H.$$

(a) If either $c \in \Sigma(K) \cap (-\infty, \mu^*]$ and $-c < \alpha \leq \beta < -c + d_c^-$, or $c \in \Sigma(K) \cap (\mu^*, \infty)$ and $-c - d_c^+ < \alpha \leq \beta < -c$, then Eq. (1.1) *is uniquely approximation solvable for each $f \in H$ and* (3.6)–(3.7) *hold.*
(b) If the conditions in (a) hold with each "$<$" sign replaced by "$\leq$" and, for some $a < \lambda$ with $\lambda = c - d_c^-/2$ if $c \leq \mu^*$ and $\lambda = c + d_c^+/2$ if $c > \mu^*$, and $b > 0$,

$$||Fx - \lambda x|| \leq a||x|| + b \ \textit{ for all } x \in H$$

then Eq. (1.1) *is solvable for each* $f \in H$.

Proof. As above, we have that

$$||Fx + \lambda x - Fy - \lambda y|| \leq \max(|\alpha + \lambda|, |\beta + \lambda|)||x - y||.$$

By our choice of λ as given in b), we conclude that $|\alpha + \lambda| \leq d = \operatorname{dist}(\lambda, \Sigma(K)) = d_c^{\pm}/2$ and $|\beta + \lambda| \leq d$ with the inequalities being strict in part a). Hence, Theorem 3.3 is applicable. $\square$

B. A splitting method. In this section, we shall study Eq. (1.1) by using a suitable splitting of K. We shall look at Hammerstein equtions with asymptotically linear and $\{B_1, B_2\}$-quasilinear nonlinerities F.

B1. Hammerstein equations with asymptotically linear nonlinearities. Recall that a Banach space X is embeddable if there is a Hilbert space H such that $X \subset H \subset X^*$ with each inclusion being dense and $\langle y, x\rangle = (y, x)_H$ for each $y \in H$ and $x \in X$, where $\langle\cdot\rangle$ is the duality pairing of X and X^*.

For asymptotically linear nonlinearity F, we have the following basic result.

THEOREM 3.8. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive definite bounded selfadjoint map and $C = K_H^{1/2}$, where K_H is the resrtiction of K to H, and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Suppose that $F : X \to X^*$ and $F_\infty : X \to X^*$ is a linear map such that*
(i) the homotopy $H_t = I - (1-t)TF_\infty C - tTFC : H \to H$ is A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ for each $t \in [0,1]$
(ii) there are positive constants a, b and R such that

$$||Fx - F_\infty x|| \leq a||x|| + b \quad \textit{for } ||x|| \geq R$$

(iii) $1 \notin \sigma(KF_\infty)$ and $a||K|| < ||(I - TF_\infty C)^{-1}||^{-1}$.

Then Eq. (1.1) *is solvable in X for each $f \in C(H) \subset X$.*

Proof. We know that the positive square root $K_H^{1/2}$ can be extended to a bounded linear map $T : X^* \to H$ such that $K = T^*T$, where the adjoint of T is $T^* = K_H^{1/2} = C : H \to X$ and $C^* = T$ (cf. [V]). Hence, we can write $K = CT$. Define the homotopy $H(t,x) = x - (1-t)TF_\infty Cx - tTFCx$ on $[0,1] \times H$. Let $f \in C(H) \subset X$, $f = Ch$, be fixed. Then there is an $r > R$ such that

$$H(t,x) \neq th \text{ for } x \in \partial B(0,r),\ t \in [0,1].$$

If not, then there would exist $x_n \in H$, $t_n \in [0,1]$ such that $||x_n|| \to \infty$ and

$$x_n - TF_\infty Cx_n = t_n(TFCx_n - TF_\infty Cx_n + h).$$

But $I - TF_\infty C$ is invertible if and only if $I - KF_\infty$ is invertible and so $(I - TF_\infty C)^{-1}$ exists by (ii). Then

$$\begin{aligned} ||(I - TF_\infty C)^{-1}||^{-1}||x_n|| &\leq ||(I - TF_\infty C)x_n|| \\ &\leq ||T(F - F_\infty)Cx_n|| + ||h|| \leq ||T||\,(a||C||\,||x_n|| + b) + ||h||. \end{aligned}$$

Since $||T|| = ||C|| = ||K||^{1/2}$, we get that

$$||(I - TF_\infty C)^{-1}||^{-1} \leq a||K|| + (b + ||h||)/||x_n||.$$

Passing to the limit, we get that $||(I - TF_\infty C)^{-1}||^{-1} \leq a||K||$, a cotradiction. Hence, $H(t,x) \neq tf$ on $[0,1] \times \partial B(0,r)$ for some $r > R$. Since $H(t,x)$ is A-proper, there is an $n_0 \geq 0$ such that

$$P_n H(t,x) \neq tP_n h \ \text{ for } x \in \partial B(0,r) \cap H_n,\ t \in [0,1],\ n \geq n_0.$$

Hence

$$\deg(I - P_n TFC, B(0,r) \cap H_n, P_n h) = \deg(I - P_n TF_\infty C, B(0,r) \cap H_n, 0) \neq 0$$

for all $n \geq n_0$. This and the A-properness of $I - TFC$ imply that $y - TFCy = h$ for some $y \in H$. Applying C and using the fact that $K = CT$, we get that $x - KFx = f$ with $x = Cy \in X$. $\square$

B2. Hammerstein equtions with $\{B_1, B_2\}$-quasilinear nonlinearities. In this section we shall study Eq. (1.1) with $\{B_1, B_2\}$-quasilinear nonlinearities N, where $B_1, B_2 : H \to H$ are selfadjoint maps with $B_1 \leq B_2$, i.e. $(B_1 x, x) \leq (B_2 x, x)$ for $x \in H$. A fixed point theory for such maps has been developed by Perov [Pe] and Krasnoselskii-Zabreiko (cf. [KZ]) assuming that $\{B_1, B_2\}$ is a regular pair. These maps have been studied extensively in the context of semilinear equations by the author [Mi-1,5,6,7].

Definition 3.1. a) A map $K : H \to H$ is $\{B_1, B_2\}$-*quasilinear* on a set $S \subset H$ if for each $x \in S$ there exists a selfadjoint map $B : H \to H$ such that $B_1 \leq B \leq B_2$ and $Bx = Kx$; b) A map $N : H \to H$ is said to be *asymptotically* $\{B_1, B_2\}$-*quasilinear* if there is a $\{B_1, B_2\}$-quasilinear outside some ball map K such that

$$|N - K| = \limsup_{||x|| \to \infty} \frac{||Nx - Kx||}{||x||} < \infty.$$

This class of maps is rather large. For example, let $N : H \to H$ have a selfadjoint weak Gateaux derivative $N'(x)$ on H. Assume that $B_1 \leq N'(x) \leq B_2$ for each x and some selfadjoint maps B_1 and B_2. Then N is asymptotically $\{B_1, B_2\}$-quasilinear with $|N - K| = 0$ (cf. [Mi-4,5]). In the nondifferentiable case, if $Nx = B(x)x + Mx$ for some nonlinear map M with the quasinorm $|M| < \infty$ and selfadjoint maps $B(x) : H \to H$ with $B_1 \leq B(x) \leq B_2$ for each $x \in H$, then N is asymptotically $\{B_1, B_2\}$-quasilinear.

The pair $\{B_1, B_2\}$ is said to be regular if 1 is not in the spectrum $\sigma(B_1) \cup \sigma(B_2)$, $\sigma(B_1) \cap (1, \infty) = \{\lambda_1, ..., \lambda_k\}$, $\sigma(B_2) \cap (1, \infty) = \{\mu_1, ..., \mu_m\}$, where the λ_i's and the μ_j's are eigenvalues of B_1 and B_2 respectively of finite multiplicities and the sum of the multiplicities of the λ_i's is equal to the sum of multiplicities of the μ_j's. It has been shown in [KZ] that if $\{B_1, B_2\}$ is a regular pair, then there is a constant $c > 0$ such that for each selfadjoint map C with $B_1 \leq C \leq B_2$ we have that

$$||x - Cx|| \geq c||x|| \ \text{ for all } x \in H.$$

Using this fact, we have proved in [Mi-4] the following extension of the fixed point theorem of Perov [P] and its extension in [KZ] for compact maps.

THEOREM 3.9. *Let $\{B_1, B_2\}$ be a regular pair, $M, N : H \to H$ be bounded and N be asymptotically $\{B_1, B_2\}$- quasilinear with $|M + N - K| < c$. Suppose that for some selfadjoint map $C_0 : H \to H$ with $B_1 \leq C_0 \leq B_2$, the map $H_t = I - (1-t)C_0 - t(M+N)$ is A-proper w.r.t. $\Gamma = \{X_n, P_n\}$ for each $t \in [0,1)$ and H_1 is either pseudo A-proper w.r.t. Γ or satisfies condition (++). Then $(I-M-N)(H) = H$.*

If C_0 and N are compact maps, $M = 0$ and $|N - K| = 0$, we obtain the result of Perov [P] and [KZ]. If N is k-ball contractive, M is c_1-strongly monotone and C_0 is a k_1-ball contractive with $k + k_1 < c_1$, then H_t is A-proper for each $t \in [0,1]$ and Theorem 3.9 is applicable. Or, we can take N and C_0 to be compact and M such that $(Mx - My, x - y) \geq -||x - y||^2$.

Next, we shall apply Theorem 3.9 to Hammerstein equations with TFC asymptotically $\{B_1, B_2\}$-quasilinear.

THEOREM 3.10. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $\{B_1, B_2\}$ be a regular pair of selfadjoint maps in H, $K : X^* \to X$ be a positive definite bounded selfadjoint map and $C = K_H^{1/2}$, where K_H is the resrtiction of K to H, and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Suppose that $F : X \to X^*$ is such that TFC is asymptotically $\{B_1, B_2\}$ -quasilinear and, for some selfadjoint map C_0 with $B_1 \leq C_0 \leq B_2$, the homotopy $H_t = I - (1-t)TC_0C - tTFC : H \to H$ is A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ for each $t \in [0,1)$ and H_1 is pseudo A-proper w.r.t. Γ. Then Eq.* (1.1) *is solvable in X for each $f \in C(H) \subset X$.*

Proof. As before, we can write $K = CT$. Let $f \in C(H) \subset X$, $f = Ch$, be fixed. Then, by Theorem 3.9, there is an $y \in H$ such that $y - TFCy = h$. Applying C and using the fact that $K = CT$, we get that $x - KFx = f$ with $x = Cy \in X$. $\square$

Next, we shall look at the case when K is not positive definite. Let X be an embeddable reflexive Banach space ($X \subset H \subset X^*$) and $K : X^* \to X$ be a bounded linear map whose restriction K_H to H is a selfadjoint map in H. Define

$$K_H^+ = 1/2(|K_H| + K_H), \;\; K_H^- = 1/2(|K_H| - K_H)$$

$$A = (K_H^+)^{1/2} + (K_H^-)^{1/2}, \;\; C = (K_H^+)^{1/2} - (K_H^-)^{1/2}$$

where $|K_H|$ is the absolute value of K_H and $(.)^{1/2}$ is the positive square root of the corresponding positive selfadjoint map in H. C is known as the principal square root of K_H.

Recall that K is said to be regular if K_H^+ and K_H^- have bounded extensions K^+ and K^- from X^* to X. Note that if K_H is quasinegative, i.e., the subspace $H_1 \subset H$, determined by the positive part of the spectrum of K_H, has positive finite dimension, then K is a regular map.

It is known that [V] if X is an embeddable reflexive Banach space and $K : X^* \to X$ is a regular bounded linear map, then K can be represented in the form $K = V^*W = W^*V$, where V and W are bounded extensions of A and C, respectively, from X^* to H, $V^* = A$ and $W^* = C$.

THEOREM 3.11. *Let X be an embeddable reflexive Banach space ($X \subset H \subset X^*$) and $K : X^* \to X$ be a regular bounded selfadjoint map and $F : X \to X^*$ be such that VFA is a bounded asymptotically $\{B_1, B_2\}$-quasilinear map. Let P be the projection from H onto the subspace $H_1 \subset H$ determined by the positive spectrum of K_H, $Q = I - P$ and the pair $\{B_1 - 2P, B_2 - 2P\}$ be regular. Suppose that C_0 is a selfadjoint map with $B_1 - 2P \leq C_0 \leq B_2 - 2P$, $H_t = I - (1-t)C_0 - t(VFA - 2P)$ is A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ for H for each $t \in [0, 1)$ and H_1 is pseudo A-proper w.r.t. Γ. Then Eq.* (1.1) *is solvable in X for each $f = Ah$ with $h \in H$.*

Proof. The map $VFA - 2P$ is asymptotically $\{B_1 - 2P, B_2 - 2P\}$- quasilinear. Then, for each $f = Ah$ with $h \in H$, the equation $u - 2Pu + VFAu = Qh$ is solvable in H by Theorem 3.9. Since $P - Q = 2P - I$ and $(P - Q)A = C$ [V], we have that $(P - Q)V = W$. Applying $2P - I$ to the above equation, we get $u - WFAu = h$. Applying $V^* = A$ to this equation and setting $x = Au$, we get that $x - KFx = f$. $\square$

4. Hammerstein integral equations

Let $Q \subset R^n$ be a bounded domain, $k(t, s) : Q \times Q \to R$ be measurable and $f(s, u) : Q \times R \to R$ is a Caratheodory function. We consider the problem of a solution $u \in L_2(Q)$ of the Hammerstein integral equation

$$u(t) = \int_Q k(t, s) f(s, u(s)) ds + g(t) \tag{4.1}$$

where g is a measurable function. There is a vast literature on the solvability of (4.1) and we just mention the books by Krasnoselskii [K] and Vainberg [V]. Define the linear map

$$Ku(t) = \int_Q k(t, s) u(s) ds$$

in $H = L_2(Q)$. Define $Fu = f(s, u(s))$ and note that Eq. (4.1) can be written in the form $u - KFu = g$.

THEOREM 4.1. *Let $K : H \to H$ be compact and selfadjoint, $\Sigma(K) = \{\lambda \mid \lambda^{-1} \in \sigma(K)\}$ and assume that either one of the following conditions holds*
(i) Let $\lambda \notin \Sigma(K)$ and $a < \operatorname{dist}(\lambda, \Sigma(K))$ be such that for some $h \in L_2(Q)$

$$|f(s, u) - \lambda u| \leq a|u| + h(s) \ \ \textit{for all } s \in Q, \ u \in R,$$

(ii) There are $\lambda, \mu \in \Sigma(K)$ such that $(\lambda, \mu) \cap \Sigma(K) = \emptyset$ and $\lambda < \alpha < \beta < \mu$ and $\epsilon > 0$ such that for $s \in Q$

$$\alpha + \epsilon \leq f_-(s) = \liminf_{|u| \to \infty} (f(s, u)/u) \leq f_+(s) = \limsup_{|u| \to \infty} (f(s, u)/u) \leq \beta - \epsilon.$$

Then Eq. (4.1) *is solvable in L_2 for each $g \in L_2$.*

Proof. We shall show first that (ii) implies (i). From (ii), we get that there is $R > 0$ such that

$$\alpha < f_-(s) - \epsilon \leq f(s,u)/u \leq f_+(s) + \epsilon < \beta, \text{ for all } s \in Q \text{ and } |u| \geq R.$$

Hence, for each $s \in Q$,

$$\Big|\frac{f(s,u)}{u} - \frac{\lambda+\mu}{2}\Big| \leq \min\Big(f_+(s) + \epsilon - \frac{\lambda+\mu}{2}, \frac{\lambda+\mu}{2} - f_-(s) + \epsilon\Big)$$
$$\leq \min\Big(\beta - \frac{\lambda+\mu}{2}, \frac{\lambda+\mu}{2} + \alpha\Big) = a < \frac{\mu-\lambda}{2} = \operatorname{dist}\Big(\frac{\lambda+\mu}{2}, \Sigma(K)\Big).$$

Thus, (i) holds and the conclusion holds by Theorem 3.4. □

THEOREM 4.2. *Let $K : H \to H$ be continuous, $\lambda \notin \Sigma(K)$ and $d^{-1} = \operatorname{dist}(\lambda, \Sigma(K))$. Let $\delta = \max ||P_n||$ and for some $k \in (0, d/k)$,*

$$|f(s,u) - \lambda u - (f(s,v) - \lambda v)| \leq k|u-v| \text{ for } s \in Q,\ u,v \in R.$$

Then Eq. (4.1) *is uniquely solvable for each $g \in L_2$ and* (3.6)-(3.7) *hold.*

Proof. It follows from Theorem 3.3. □

THEOREM 4.3. *Let $K : H \to H$ be selfadjoint and for some $\alpha, \beta \in R$,*

$$\alpha|u-v|^2 \leq (f(s,u) - f(s,v))(u-v) \leq \beta|u-v|^2 \text{ for } s \in Q,\ u,v \in R$$

(i) If $-c < \alpha \leq \beta < -c + d_c^+$ for some $c \in \Sigma(K) \cap (-\infty, \mu^*)$ or $-c - d_c^+ < \alpha \leq \beta < -c$ for some $c \in \Sigma(K) \cap (\mu^*, \infty)$, then Eq. (5.1) *is uniquely solvable for each $g \in L_2$.*
(ii) If $<$ is replaced by $\leq$ in (i) and if, for some $a < \lambda$ with $\lambda = c - d_c^-/2$ if $c \leq \mu^*$ and $\lambda = c + d_c^+/2$ if $c > \mu^*$, and some $b \in L_2$, we assume

$$|f(s,u) - \lambda u| \leq a|u| + b(s) \quad \text{for } s \in Q,\ u \in R$$

then Eq. (4.1) *is solvable for each $g \in L_2$ and* (3.6)–(3.7) *hold.*

Proof. It follows from Theorem 3.7. □

Part (i) of this theorem extends a result of Dolph [Do]. For asymptotically linear nonlinearities F, we have

THEOREM 4.4. *Let $K : L_2(Q) \to L_2(Q)$ be compact, selfadjoint and positive definite and also acts from $L_{p'}(Q)$ into $L_p(Q)$ with $2 < p \leq \infty$ and $p' = p/(p-1)$. Assume that $f(s,u)$ is a Caratheodory function and*
(i) There are $a(s) \in L_{p'}(Q)$ and $b \geq 0$ such that

$$|f(s,u)| \leq a(s) + b|u|^{p-1} \text{ for } s \in Q,\ u \in R.$$

(ii) There are functions $f_\infty(s) \in L_{p-2}(Q)$, $b(s) \in L_{p'}(Q)$ and $a \geq 0$ such that

$$|f(s,u) - f_\infty(s)u| \leq b(s) + a|u| \text{ for } s \in Q,\ u \in R.$$

(iii) For the linear map $F_\infty u(s) = f_\infty(s)u(s)$ *in* $L_2(Q)$ *and the decomposition* $K = CT$*, assume that*

$$1 \notin \sigma(KF_\infty) \ \ and\, a||K|| < ||(I - TF_\infty C)^{-1}||^{-1}.$$

Then Eq. (4.1) *is solvable in* $L_p(Q)$ *for each* $g \in C(L_2(Q)) \subset L_p(Q)$.

Proof. By our assumptions on K, it can be written in the form (see [K]) $K = CT$, where $C = K_H^{1/2}$ is the selfadjoint positive definite square root of K, $C = T^* : L_2(Q) \to L_p(Q)$ and $T = C^*$ acting from $L_{p'}(Q)$ to $L_2(Q)$. Since C is compact ([K]), it follows that $I - (1-t)TF_\infty C - tTFC : L_2(Q) \to L_2(Q)$ is A-proper w.r.t. to any scheme $\Gamma = \{H_n, P_n\}$ for $L_2(Q)$. Hence, the theorem follows from Theorem 3.8. $\square$

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