PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 58 (72), 1995, 85–92
Slobodan Aljančić memorial volume

GAUSS–PETROVIĆ LINEAR FUNCTIONALS AND SLOW OSCILATION IN NORMED LINEAR SPACES

Č. V. Stanojević, D. L. Graser, and **K. M. Adams**

Dedicated to Slobodan Aljančić

Abstract. In [**6, 8**] slow oscillation is generalized to linear topological spaces. Linear functionals are used to control the distribution of sequences. Consequently, charcterizations of slow oscillation in norm and corresponding Tauberian theorems are obtained.

1. Introduction. In normed linear spaces questions regarding convergence are often resolved by estimates via the triangle inequality. In fact, spaces for which the absolute convergence of series implies the convergence in norm of the series characterize the Banach spaces.

We may then ask, can some information be gained about the absolute convergence of the series from convergence in norm of the series? To answer this question we address the problem of finding inequalities for which the norms of partial sums of the series dominate partial sums of the absolute values of the terms of the series. The following theorem is instructive:

THEOREM A. *Let* $\{z_1, \dots, z_n\}$ *be a set of points in the complex plane* $\mathbb{C}$. *Then there exists a subset* S *of* $\{1, \dots, n\}$ *for which*

$$\left|\sum_{k\in S} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{n} |z_k|.$$

Remark 1. This theorem is well known. The proof requires finding an angle $-\pi \leq \theta < \pi$ which maximizes the sum

$$\sum_{k=1}^{n} |z_k| \cos^+((\arg z_k) - \theta)(\cos^+(u) = \frac{1}{2}(|\cos u| + \cos u)).$$

AMS Subject Classification (1991): Primary 46B99

Key Words and Phrases: Slow oscillation, characterization of slow oscillation in normed linear spaces, Tauberian Theorems

The theorem is unsatisfactory in addressing the question of convergence since the subset S depends on n. It does however provide some insight. It is possible to find a more suitable inequality by controlling the arguments of the points.

From Gauss' Theorem regarding the center of gravity [**1**] follows an important inequality found by Petrović [**2**].

Theorem B. *Let* $\{z_1, \ldots, z_n\}$ *be a set of points in the complex plane* $\mathbb{C}$. *If for* $z_k \neq 0$ $|\arg z_k| \leq \theta_0 < \pi/2, \quad 1 \leq k \leq n$, *then*

$$\left|\sum_{k=1}^{n} z_k\right| \geq \cos\theta_0 \sum_{k=1}^{n} |z_k|.$$

Remark 2. A condensed proof is due to Karamata [**3**]. It can be shown that the theorem is rotationally invariant.

Diaz and Metcalf [**4**] have generalized the Gauss–Petrović inequality for Hilbert and Banach spaces. The Banach space theorem is as follows:

Theorem C. *Let* $(B, \|\cdot\|)$ *be a Banach space and let* F *be a linear functional of unit norm. Suppose the vectors* $\{v_1, \ldots, v_n\} \subset B$. *If* $0 \leq r\|v_k\| \leq \operatorname{Re} F(v_k)$, $1 \leq k \leq n$, *then*

$$r \sum_{k=1}^{n} \|v_k\| \leq \left\|\sum_{k=1}^{n} v_k\right\|.$$

The goal of this paper is to use the Gauss–Petrović inequality, complementary triangle inequalities, and a characterization of slow oscillation of positive sequences to obtain a characterization of slow oscillation in norm.

2. Definitions and observations. R. Schmidt introduced slow oscillation as a Tauberian condition to restore convergence of positive sequences from $(C,1)$ summability. Schmidt's [**5**] definition of slow oscillation of sequences has been extended to normed linear spaces by Č. V. Stanojević [**6**] as follows. Let $(X, \|\cdot\|)$ be a normed linear space and $x = \{x_k\} \subset X$. Denote

$$S_n(x) = \sum_{k=1}^{N} x_k \quad \text{and} \quad S_n(\|x\|) = \sum_{k=1}^{N} \|x_k\|.$$

Definition 2.1. The sequence $\{S_N(x)\} \subset X$ is slowly oscillating in norm if

$$\lim_{\substack{N>M\to\infty \\ N/M\to 1}} \|S_N(x) - S_M(x)\| = 0.$$

In normed linear spaces the interplay between the norm topology and the topology induced by bounded linear functionals is often of interest. This motivates the next definition. Let φ be a linear functional on X.

Definition 2.2. The sequence $\{S_N(x)\} \subset X$ is φ-slowly oscillating if for some φ,

$$\lim_{\substack{N>M\to\infty \\ N/M\to 1}} |\varphi(S_N(x)) - \varphi(S_M(x))| = 0.$$

If $\{S_N(x)\}$ is φ-slowly oscillating for each bounded linear functional on X the $\{S_N(x)\}$ is weakly slowly oscillating.

The next definition is needed for the characterization result. Denote

$$V_N(x) = \frac{1}{N}\sum_{k=1}^{N} kx_k \quad \text{and} \quad V_N(\|x\|) = \frac{1}{N}\sum_{k=1}^{N} k\|x_k\|.$$

Definition 2.3. The sequence $\{V_N(x)\} \subset X$ is strongly slowly oscillating in norm if

$$\lim_{\substack{N>M\to\infty \\ N/M\to 1}} \left\| \frac{N}{M} V_N(x) - V_M(x) \right\| = 0.$$

Remark 3. A Renyi [**7**] has shown that for a sequence of positive numbers $a = \{a_k\}$ that $V_N(a) = N^{-1}\sum_{k=1}^{N} ka_k = O(1)$, $N \to \infty$ is not a Tauberian condition.

The following observations were made in [**8**].

1. Let $a = \{a_k\}$ be a sequence of positive numbers. Then $\{S_N(a)\}$ is slowly oscillating if and only if $\{V_N(a)\}$ is strongly slowly oscillating. This is evident from the following inequality.

$$S_N(a) - S_M(a) \le \frac{N}{M} V_N(a) - V_M(a) \le \frac{N}{M}[S_N(a) - S_M(a)]$$

for positive integers $N > M$.

2. Let $b = \{b_k\}$ be a sequence of positive numbers and let $V_N(b) = O(1)$, $N \to \infty$. Then $\{S_N(b)\}$ slowly oscillating is equivalent to $\{V_N(b)\}$ slowly oscillating.

Theorem B motivates the next definition.

Definition 2.4. Let $-\pi \le \theta_0 < \pi$ and $0 \le \gamma < \pi/2$. The set $M_{\theta_0} = \{z \in \mathbb{C} \mid |\arg z - \theta| < \gamma\}$, is the Gauss-Petrović cone at θ_0, with index γ. If $\{x_k\} \subset B$ is a sequence in the normed linear space $(B, \|\cdot\|)$ and φ is a linear functional on B such that:

(2.4.1) (i) $|\varphi(x_k)| \ge r\|x_k\|$, $r > 0$, for all k;

(2.4.2) (ii) $\varphi(x_k) \in M_{\theta_0}$, for all k where M_{θ_0} is a Gauss-Petrović cone

then φ is said to be a Gauss-Petrović linear functional on the sequence $\{x_k\} \subset B$.

Č. V. Stanojević has shown [**10**] that linear functionals of this type can be used to obtain structural information.

3. Results. Let $(B, \|\cdot\|)$ be a normed linear space and $x = \{x_k\} \subset B$. The following theorem uses a Gauss–Petrović linear functional to control the distribution of the sequence.

THEOREM 3.1. *Let φ be a Guass-Petrović linear functional on $x = \{x_k\} \subset B$. Then*

(i) *φ-slow oscillation of $\{S_N(x)\}$ implies norm slow oscillation of $\{S_N(x)\}$,*
(ii) *φ bounded implies norm slow oscillation of $\{S_N(x)\}$ is equivalent to strong slow oscillation of $\{V_N(\|x\|)\}$.*

Proof of 3.1 (i). Since φ is a Gauss-Petrović linear functional on $\{x_k\}$ then there exists $0 \le \alpha_0 < \pi/2$ and by theorem B we have for positive integers $N > M$,

$$\left|\sum_{k=M+1}^{N} \varphi(x_k)\right| \ge \cos\alpha_0 \sum_{k=M+1}^{N} |\varphi(x_k)|. \tag{3.1.1}$$

From (2.4.1) together with (3.1.1) we obtain,

$$\left|\sum_{k=M+1}^{N} \varphi(x_k)\right| \ge r\cos\alpha_0 \sum_{k=M+1}^{N} \|x_k\|. \tag{3.1.2}$$

Since φ is linear we have,

$$|\varphi(S_N(x)) - \varphi(S_M(x))| \ge r\cos\alpha_0 \sum_{k=M+1}^{N} \|x_k\|. \tag{3.1.3}$$

Then (i) follows from (3.1.3).

Proof of 3.1 (ii). The continuity of φ together with (3.1.3) and Observation 1 yields (ii).

In the following theorem the linear functional has range which allows use of Theorem C.

THEOREM 3.2 *Let ψ be a linear functional. If*

$$0 \le r\|x_k\| \le \operatorname{Re}\psi(x_k), \quad 1 \le k < \infty, \tag{3.2.1}$$

then,

(i) *ψ-slow oscillation of $\{S_N(x)\}$ implies the norm slow oscillation of $\{S_N(x)\}$,*
(ii) *ψ bounded implies norm slow oscillation of $\{S_N(x)\}$ is equivalent to strong slow oscillation of $\{V_N(\|x\|)\}$.*

The proof is similar to the proof of Theorem 3.1.

Remark 4. The functionals in Theorems 3.1 and 3.2 control the distribution of the sequences in the complex plane.

Remark 5. There is an analogous theorem to Theorem 3.2 for the imaginary case of condition (3.2.1).

In the next theorem we shall show that the condition for equivalence of slow oscillation in norm and strong slow oscillation is also a Tauberian condition for the recovery of convergence in norm out of the $(C, 1)$ summability in norm.

Theorem 3.3. *Let $\{\varphi_k\} \subset B^*$ each of unit norm. If there exists a sequence of positive numbers $\{r_k\}$ bounded and bounded away from zero such that for every pair of non-negative integers $N > M$,*

$$\big|\varphi_k(S_{j+1}(x))\big| - \big|\varphi_k(S_j(x))\big| \geq r_k\|x_k\|, \quad M \leq j, k \leq N, \tag{3.3.1}$$

then

(i) *norm slow oscillation of $\{S_N(x)\}$ is equivalent to strong slow oscillation of $\{V_N(\|x\|)\}$,*

(ii) *$x = \{x_k\}$ $(C,1)$ summable in norm implies $x = \{x_k\}$ converges in norm to the $(C,1)$-limit of $x = \{x_k\}$.*

Proof. Part (i), from (3.3.1) and summing on j from M to $N-1$ we have,

$$|\varphi(S_N(x))| - |\varphi_k(S_M(x))| \geq (N-M)r_k\|x_k\|, \quad M \leq k \leq N. \tag{3.3.2}$$

Majorizing the difference in (3.3.2) and the fact that each φ_k is continuous we obtain,

$$\Big\| \sum_{l=M+1}^{N} x_l \Big\| \geq (N-M)r_k\|x_k\|, \quad M \leq k \leq N. \tag{3.3.3}$$

We now divide by r_k and $(N-M)$ and sum on k from M to N. Hence,

$$\Big\| \sum_{l=M+1}^{N} x_l \Big\| \frac{1}{(M-N)} \sum_{k=M+1}^{N} \frac{1}{r_k} \geq \sum_{k=M+1}^{N} \|x_k\|. \tag{3.3.4}$$

Part (i) follows from (3.3.4), $\dfrac{1}{N-M} \displaystyle\sum_{k=M+1}^{N} \dfrac{1}{r_k}$ is bounded and bounded away from zero, and Observation 1.

Part (ii). Fix $M = 0$ in (3.3.2) and we have,

$$\big|\varphi_k(S_N(x))\big| \geq Nr_k\|x_k\|, \quad . \tag{3.3.5}$$

Letting $k = N$ then

$$\big|\varphi_k(S_N(x))\big| \geq Nr_N\|x_N\|. \tag{3.3.6}$$

So,

$$\frac{1}{N}\Big\| \sum_{k=1}^{N} x_k \Big\| \geq r_N\|x_N\|.$$

Then (ii) follows from (3.3.6).

In Theorems 3.1, 3.2, and 3.3 we have analogous results where we require that $V_N(\|x\|) = O(1)$, $N \to \infty$ and then $\{V_N(\|x\|)\}$ strongly slowly oscillating is replaced by slow oscillating in the conclusion.

In Theorem 3.3 the condition (3.3.1) may be replaced by,

$$\big|\varphi_k(S_N(x)) - \varphi_k(S_M(x))\big| \geq r_k\|x_k\|, \quad M \leq k \leq N. \tag{3.3.1$'$}$$

Also, corollaries follow Theorem 3.3 where φ_k is replaced by the real or imaginary parts of φ_k in (3.3.1).

The following theorems are the Hilbert space analogues to the preceding theorems. Let H be a Hilbert space with inner product denoted $(\cdot,\cdot)$ and let $y = \{y_k\} \subset H$.

THEOREM 4.1. *For $a \in H$ let $K_{\theta_0}(\alpha)$ be a Gauss-Petrović cone such that $(y_k, a) \in K_{\theta_0}(\alpha)$ for each $k \in \mathbb{N}$. If*

$$0 \leq r\|y_j\| \leq (y_j, a), \quad 1 \leq j < \infty, \tag{4.1.1}$$

then norm slow oscillation of $\{S_N(y)\}$ is equivalent to strong slow oscillation of $\{V_N(\|y\|)\}$.

THEOREM 4.2. *Let $b \in H$. If*

$$0 \leq r\|y_k\| \leq Re(y_k, b), \quad 1 \leq k < \infty, \tag{4.2.1}$$

then norm slow oscillation of $\{S_N(y)\}$ is equivalent to strong slow oscillation of $\{V_N(\|y\|)\}$.

THEOREM 4.3. *Let $\{a_k\} \subset H$, $\|a_k\| = 1$, $1 \leq k < \infty$. If there exists a sequence of positive numbers $\{r_k\}$ bounded and bounded away from zero such that,*

$$|(S_{j+1}(y), a_k)| - |(S_j(y), a_k)| \geq r_k\|y_k\|, \quad M \leq j, k \leq N, \tag{4.3.1}$$

then

(i) *norm slow oscillation of $\{S_N(y)\}$ is equivalent to strong slow oscillation of $\{V_N(\|y\|)\}$,*
(ii) *$y = \{y_k\}$ $(C, 1)$ summable in norm implies $y = \{y_k\}$ converges in norm to the $(C, 1)$-limit of $y = \{y_k\}$.*

Remark 6. The proofs of 4.1–4.3 follow by making the appropriate choice of linear functionals and applying Theorems 3.1–3.3.

We now focus on a more general space. H. Rubin and M. H. Stone have set down postulates for a linear space that carries a positive definite, Hermitian symmetric, bilinear form [**9**]. The following postulates establish such a space we shall call a Rubin–Stone space.

Postulate 1. X is a linear space over the complex field.

Postulate 2. On X there is defined a non-negative real function q such that

$$q(x + y) + q(x - y) = 2q(x) + 2q(y).$$

Postulate 3. As a function of the real number α the quantity $q(\alpha x)$ is bounded in some neighborhood of $\alpha = 0$ for each x.

Postulate 4. The relation $q(x) = q(ix)$ holds for all x.

Postulate 5. If $q(x) = 0$, then $x = 0$.

Rubin and Stone have formed an inner product on X by defining the quantity $\langle x,y\rangle \in \mathbb{C}$ by the relation

$$\langle x,y\rangle = \frac{q(x+y)-q(x-y)}{4} + i\frac{q(x+iy)-q(x-iy)}{4}.$$

It was shown [**9**] that the inner product $\langle x,y\rangle$ is Hermitian symmetric, linear in the first variable and conjugate linear in the second variable, and positive definite for all $x \in X$. It also obeys a Schwarz inequality of the form $|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle$. Postulate 5 requires that $\|x\| = \langle x,x\rangle$ is a norm on X. If a space has Postulates 1–4 and not 5 then the space carries a semi-norm.

Let S be a Rubin–Stone space and $z = \{z_k\} \subset X$.

THEOREM 5.1. *For $c \in S$ let $K_{\theta_0}(\alpha)$ be a Gauss–Petrović cone such that $\langle z_k, c\rangle \in K_{\theta_0}(\alpha)$ for each $k \in \mathbb{N}$. If*

$$0 \le r\|z_k\| \le \langle z_k, c\rangle, \quad 1 \le k < \infty, \tag{5.1.1}$$

then norm slow oscillation of $\{S_N(z)\}$ is equivalent to strong slow oscillation of $\{V_N(\|z\|)\}$.

THEOREM 5.2. *Let $d \in S$. If*

$$0 \le r\|z_k\| \le Re\langle z_k, d\rangle, \quad 1 \le k < \infty, \tag{5.2.1}$$

then norm slow oscillation of $\{S_N(z)\}$ is equivalent to strong slow oscillation of $\{V_N(\|z\|)\}$.

THEOREM 5.3. *Let $\{\alpha_k\} \subset S$, $\|\alpha_k\| = 1$. If there exists a sequence of positive numbers $\{r_k\}$ bounded and bounded away form zero, such that for positive integers $N > M$,*

$$\left|\langle S_{j+1}(z), \alpha_k\rangle\right| - \left|\langle S_j(z), \alpha_k\rangle\right| \ge r_k\|z_k\|, \quad M \le j,k \le N, \tag{5.3.1}$$

then

(i) *norm slow oscillation of $\{S_N(z)\}$ is equivalent to strong slow oscillation of $\{V_N(\|z\|)\}$,*
(ii) *$z = \{z_k\}$ $(C,1)$ summable in norm implies $z = \{z_k\}$ converges in norm to the $(C,1)$-limit of $z = \{z_k\}$.*

Remark 7. The proofs of 5.1–5.3 are identical to the proofs, of 4.1–4.3 with $(\cdot,\cdot)$ replaced by $\langle\cdot,\cdot\rangle$.

REFERENCES

1 F. Gauss, *Werke, III,* Göttingen Ges. Wiss., 1866

2 M. Petrović, *Theoreme sur les integrals curvilignes,* Publ. Math. Univ. Beograd, II, CTO (1933), 45–59

3 J. Karamata, *Theorey and Applications of Stieltjes Integrals,* Serbian Academy of Sciences, Belgrade, 1949

4 J.B. Diaz, and F.T. Metcalf, *A complementary triangle inequality in Hilbert and Banach spaces,* Proc. Amer. Math. Soc. **17** (1966), 88–97

5 R. Schmidt, *Uber divergente Folgen und lineare Mittelbildungen,* Math. Z. **22** (1924), 89–152

6 Č.V. Stanojević, *Research seminar at UMR, Tauberian theorems and their applications I,* (1992),

7 A. Reni, *On a Tauberian theorem of O. Szász,* Acta Sci. Math. **11** (1946), 119–123

8 Č.V. Stanojević, *Research seminar at UMR, Tauberian theorems and their applications II,* (1993),

9 H. Rubin, and M.M. Stone, *Postulates for generalizations of Hilbert space,* Proc. Amer. Math. Soc. **4** (1953), 611–616

10 Č.V. Stanojević, *Slow oscillation in norm and the structure of linear functionals,* Preprint

Department of Mathematics
University of Missouri-Rolla
Rolla, MO 65401
U.S.A.

(Received 24 04 1995)