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SLOW OSCILLATION IN NORM AND STRUCTURE OF LINEAR FUNCTIONALS

Časlav V. Stanojević

САЊИ

Abstract. Slow oscillation extended to linear topological spaces yields information about bounded linear functionals. For instance bounded linear functional on slow oscillating sequences are Fourier coefficients of functions in various function spaces.

1. Introduction. A divergent limiting process that (i) converges in a generalized manner regular with respect to the initial limiting process, and (ii)controlled in growth and/or regularity may yield convergence, provided (i) and (ii) are compatible. That is, (i) is a regular summability method and (ii) is a corresponding Tauberian condition through which convergence is restored out of summability. A classical example of such a situation is the inverse Abelian theorem(s): Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < 1$, and let $\lim\limits_{z\to 1-0} f(z)$ exists. If $\lim\limits_{\substack{N>M\\ N/M\to 1}} \sum\limits_{k=M+1}^{N} a_k = 0$, then the series $\sum_{n=0}^{\infty} a_n$ converges to $\lim\limits_{z\to 1-0} f(z)$.

The condition

$$\lim_{\substack{N>M\\ N/M\to 1}} \sum_{k=M+1}^{N} a_k = 0 \tag{1.1}$$

is a special form of Cauchy's condition. It was introduced by Schmidt [**1**] and it is called slow oscillation of the sequence $S_n(a) = \sum_{k=0}^{n} a_k$ of partial sums of the series $\sum_{n=0}^{\infty} a_n$. There are two important special cases of (1.1) the first one

$$V_n(|a|,p) = \frac{1}{n}\sum_{k=1}^{n} k^p |a_k|^p = O(1), \quad n\to\infty, \quad p>1 \tag{1.2}$$

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is due to Hardy–Littlewood [**2**] as a conjecture, and proved by Szasz [**3**]. The other one comes from the fact that

$$V_n(|a|,1) = \frac{1}{n}\sum_{k=1}^{n} k|a_n| = O(1), \quad n \to \infty, \tag{1.3}$$

is not a Tauberian condition. This was observed by Renyi [**4**] who showed that the existence of the limit $\lim_n V_n(|a|,1)$ is a Tauberian condition that is neither implied by (1.2) nor implies (1.2).

Karamata [**5, 6**] and Avakumović [**7**] introduced another way of controlling the divergence of positive sequence $\{R(n)\}$. In general a positive sequence $\{R(n)\}$ is O-regularly varying if $\overline{\lim}_n R([\lambda n])/R(n)$ is finite for $\lambda > 1$; if

$$\lim_{\lambda\to 1+0} \overline{\lim_n} \frac{R([\lambda n])}{R(n)} \le 1,$$

the sequence $\{R(n)\}$ is $*$-regularly varying [**8**]; if, in particular,

$$\lim_n \frac{R([\lambda n])}{R(n)} = \lambda^\theta, \quad \lambda > 1, \quad \theta \in (-1,\infty)$$

the sequence $\{R(n)\}$ is regularly varying of index θ; a special case, $\theta = 0$ is of considerable interest, $\lim_n R([\lambda n])/R(n) = 1$, and defines slowly varying sequences.

Since $f \in L^1$ is equivalent to $(C,1)$-summability in norm of

$$S_n(\hat f) = S_n(\hat f, t) = \sum_{|k|\le n} \hat f(k) e^{ikt}, \quad t \in T = R/2\pi Z \tag{1.4}$$

the problem of convergence in L^1-norm of (1.4) is a Tauberian problem. The main difficulty in solving this Tauberian problem is that the proper setting is not only space L^1 but spaces L^p, $p > 1$, and there interpolation properties, [**9**]. The condition needed for the recovery of convergence in L^1-norm is obtained in [**8, 10**]. It has the form

$$\lim_{\lambda\to 1+0} \overline{\lim_n} \sum_{|k|=n+1}^{[\lambda u]} |k|^{p-1} |\Delta \hat f(k)|^p < \infty, \quad p \in (1,2], \tag{1.5}$$

which is equivalent to $V_n(|\Delta \hat f|, p) = O(1), \quad n \to \infty$.

However (1.5) is equivalent to

$$\left\{ \exp\left(\sum_{|k|\le n} |k|^{p-1} |\Delta \hat f(k)|^p \right) \right\}$$

being O-regularly varying sequence. Hence, via Riesz' theorem [**11**] and Carleson's theorem, [**12**] from the condition (1.5) the structure of the sequence $\{\hat f(n)\}$ is obtained and consequently a.e convergence of (1.4). The in-depth study of (1.5) and its various equivalent and more general forms can be found in [**10**]. The most general Tauberian condition of this kind is obtained in [**13, 14**], i.e. $\| S_n(\Delta \hat{f}) \|_p = O(1)$, $n \to \infty$, $p > 1$ and it is based on the compacticity principle.

From the above results it follows that (1.2) is equivalent to

$$\Big\{\exp\Big(\sum_{k=1}^{n} k^{p-1} |a_k|^p\Big)\Big\}$$

being O-regularly varying, and consequently $f(z) = \sum_{n=0}^{\infty} a_n |z|^n$, $|z| < 1$ belongs to H^q, $1/p + 1/q = 1$ which determines the structure of Taylor coefficients $\{a_n\}$.

The following questions seem to be relevant and of considerable interest:

(1) Is there a reasonable analogue to slow oscillation in linear topological space, more specifically in normed linear spaces?
(2) Would such an analogue imply structural and convergence information in those spaces?
(3) What would be ramifications of positive answers in (1) and (2) for the classical function spaces?

In this work questions (1) and (2) are addressed, and the question (3) will be considered in a subsequent study.

2. Slow oscillation in linear topological spaces. A natural extension of slow oscillation to linear topological spaces goes via linear functionals. Let B be a complex linear topological space and B^* the space of bounded linear functionals.

Definition 2.1. Let $\{x_n\} \subset B$ and $S_n(x) = \sum_{k=1}^{n} x_k$. Then $\{S_n(x)\}$ is φ-slowly oscillating if for some $\varphi \in B^*$

$$\varphi(S_N(x) - S_M(x)) = o(1), \quad N > M \to \infty, \quad N/M \to 1. \tag{2.1}$$

If it is φ-slow oscillating for every $\varphi \in B^*$, then $\{S_n(x)\} \subset B$ is weakly slow oscillating.

LEMMA 2.1. *Let $\{S_n(x)\}$ be weakly slow oscillating. Then the series $\sum_{n=1}^{\infty} \varphi(x_n)/n$ converges for every $\varphi \in B^*$.*

Proof. For every $\varphi \in B^*$ and every $N > M \to \infty$, $\dfrac{N}{M} \to 1$, $\sum\limits_{k=M+1}^{N} \varphi(x_k) = o(1)$. Choose $M = n$ and $N = [\lambda n]$, $\lambda > 1$; then

$$\lim_{\lambda \to 1+0} \overline{\lim}_n [\lambda n]/n = 1, \quad \text{i.e.} \quad \lim_{\substack{N > M \to \infty \\ N/M \to 1}} \frac{N}{M} = 1.$$

Consider the sequence $\Big\{\exp\Big|\sum\limits_{k=1}^{n} \varphi(x_k)\Big|\Big\}$ of positive numbers. Then

$$\exp\Big|\sum_{k=1}^{[\lambda n]} \varphi(x_k)\Big| \Big/ \exp\Big|\sum_{k=1}^{n} \varphi(x_k)\Big| \le \exp\Big|\sum_{k=n}^{[\lambda n]} \varphi(x_k)\Big|,$$

and

$$\lim_{\lambda\to 1+0}\overline{\lim_{n}}\exp\Big|\sum_{k=n+1}^{[\lambda n]}\varphi(x_k)\Big|\le \exp\Big(\lim_{\lambda\to 1+0}\overline{\lim_{n}}\Big|\sum_{k=n+1}^{[\lambda n]}\varphi(x_k)\Big|\Big)=1.$$

Hence the sequence $\Big\{\exp\Big|\sum\limits_{k=1}^{n}\varphi(x_k)\Big|\Big\}$ is $*$-regularly varying. For some $*$-regularly varying $\{R(n)\}$ and $p\in(1,2]$

$$\Big|\sum_{k=1}^{n}\varphi(x_k)\Big|^p=\lg^p R(n),\quad n\ge 1.$$

Since $\{\lg^p R(n)\}$ is slowly varying, the series

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{n}\varphi(x_k)\Big|^p n^{-p}$$

converges for every $p>1$.

This implies that there is $h\in L^q$, $1/p+1/q=1$ such that

$$\sum_{k=1}^{n}\varphi(x_k)=n\hat h(n)\quad\text{and}\quad\frac{\varphi(x_n)}{n}=\hat h(n)-\hat h(n-1)+\frac{\hat h(n-1)}{n}.$$

Thus the series $\sum_{n=1}^{\infty}\varphi(x_n/n)$ converges for every $\varphi\in B^*$.

Let B be a complex linear normed space with norm $\|\cdot\|$.

Definition 2.2. Let $\{S_n(x)\}\subset B$. If

$$\lim_{\substack{N>M\\ N/M\to 1}}\Big\|\sum_{k=M+1}^{N}x_k\Big\|=0,$$

the sequence $\{S_n(x)\}\subset B$ is slowly oscillating in norm, (or $\|\cdot\|$-slowly oscillating).

The next lemma is a corollary to Lemma 2.1.

LEMMA 2.2. *Let $\{S_n(x)\}$ be $\|\cdot\|$-slowly oscillating. Then the series* $\sum\limits_{n=1}^{\infty}\dfrac{\varphi(x_n)}{n}$ *converges for every $\varphi\in B^*$.*

Proof. Slow oscillation in $\|\cdot\|$-norm implies weak slow oscillation.

THEOREM 2.1. *Let B be a linear topological space and let $\{x_n\}\subset B$. If $\{S_n(x)\}$ is weakly slow oscillating in B, then the series*

$$\sum_{n=1}^{\infty}\frac{\varphi(x_n)}{n}e^{int}$$

is the Fourier series of a function in L^r, $r\ge 2$.

Proof. Using Lemma 2.1 one can prove a more precise form of Theorem 2.1. Namely for every $\varphi \in B^*$ there exists $\alpha_\varphi \in AC$, (where AC denotes the space of function with absolutely convergent Fourier series) such that

$$\text{(i)} \quad \alpha' \in L^r, \quad r \geq 2 \qquad \text{(ii)} \quad g(t) = \sum_{n=1}^{\infty} \varphi\Big(\frac{x_n}{n}\Big) e^{int} = \alpha_\varphi(t) + i(1 - e^{-it})\alpha'_\varphi(t)$$

The conclusion of Theorem 2.1 follows.

The next theorem is a consequence of Theorem 2.1.

THEOREM 2.2. *Let B be a normed linear space and $\{x_n\} \subset B$. If $\{S_n(x)\}$ is $\|\cdot\|$-slow oscillating; then the series*

$$\sum_{n=1}^{\infty} \frac{\varphi(x_n)}{n} e^{int}$$

is the Fourier series of a function in L^r, $r \geq 2$.

Proof. Lemma 2.2 implies that for every $\varphi \in B^*$ there exists $\alpha_\varphi \in AC$ with properties (i) and (ii) as in the proof of Theorem 2.1.

3. Generalizations of slow oscillation in norm. Throughout this section B denotes a normed complex linear space and B^* the space of bounded linear functionals.

Definition 3.1. Let $\{S_n(x)\} \subset B$. The sequence $\{S_n(x)\}$ is moderately divergent in $\|\cdot\|$-norm if for $s > 1$

$$\|S_n(x)\| = o(n^{s-1}), \quad n \to \infty \qquad \sum_{n=1}^{\infty} \frac{\|S_n(x)\|}{n^s} < \infty.$$

It is clear that every $\|\cdot\|$-slowly oscillating sequence is moderately divergent since there is $\{R(n)\}$, *-regularly varying, such that $\|S_n(x)\| = \lg R(n)$, as in the proof of Lemma 2.2. The converse is not true as shown by counterexamples in [**15**].

THEOREM 3.1. *Let $\{S_n(x)\} \subset B$ be moderately divergent. Then the series*

$$\sum_{n=1}^{\infty} \frac{\varphi(x_n)}{n} e^{int}$$

is the Fourier series of a function in L^r, $r \geq 2$.

Proof. Since $\{S_n(x)\}$ is moderately divergent in $\|\cdot\|$-norm, for $\varepsilon \in (0, 1/2]$

$$\|S_n(x)\| = o(n^\varepsilon), n \to \infty \qquad \sum_{n=1}^{\infty} \frac{\|S_n(x)\|}{n^{1+\varepsilon}} < \infty$$

Hence

$$\sum_{n=1}^{\infty} \Big(\frac{\|S_n(x)\|}{n^{1-\varepsilon/(s-1)}}\Big)^{s-1} \frac{\|S_n(x)\|}{n^{1+\varepsilon}} < \infty \quad \text{for} \quad s \in (1/(1-\varepsilon), 2]$$

Therefore for every $\varphi \in B^*$

$$\sum_{n=1}^{\infty} \Big| \sum_{k=1}^{n} \varphi(x_k) \Big|^s n^{-s} < \infty.$$

The rest of the proof goes as in Theorem 2.1.

There is also a theorem analogous to the comment in the proof of Theorem 2.1.

THEOREM 3.2. *Let $\{S_n(x)\} \subset B$ be moderately divergent. Then for every $\varphi \in B^*$ there exists $\alpha_\varphi \in AC$ such that*

$$\alpha'_\varphi \in L^r, r \geq 2$$

$$g(t) = \sum_{n=1}^{\infty} \varphi\Big(\frac{x_n}{n}\Big) e^{int} = \alpha_\varphi(t) - i(1 - e^{int})\alpha'_\varphi(t).$$

Since a bounded sequence in $\|\cdot\|$ is not necessarily $\|\cdot\|$-slowly oscillating it remains to consider the case $\|S_n(x)\| = O(1)$. The next theorem provides some information in this case.

THEOREM 3.3. *Let $\{x_k\} \subset B$ and let $\varphi \in B^*$. If $\Big\| \sum\limits_{k=1}^{n} e^{i \arg(\varphi(x_k))} x_k \Big\| = O(1)$, $n \to \infty$ then there exists $\alpha_\varphi \in AC$ such that*

$$\sum_{n=1}^{\infty} \varphi(x_k) e^{int}$$

is the Fourier series of α_φ.

Proof. Notice that

$$\|\varphi\|^* \Big\| \sum_{k=1}^{n} e^{i \arg(\varphi(x_k))} x_k \Big\| \geq \Big| \varphi\Big(\sum_{k=1}^{n} e^{i \arg(\varphi(x_k))} x_k \Big) \Big| = \Big| \sum_{k=1}^{n} e^{i \arg(\varphi(x_k))} \varphi(x_k) \Big|$$
$$= \sum_{k=1}^{n} |\varphi(x_k)|$$

where $\|\cdot\|^*$ is the dual norm. Since $\| \sum\limits_{k=1}^{n} e^{i \arg(\varphi(x_k))} \| = O(1)$, $n \to \infty$ the series $\sum\limits_{n=1}^{\infty} |\, \varphi(x_n) \,|$ converges. Hence there exists $\alpha_\varphi \in AC$ such that $\varphi(x_n) = \hat{\alpha}_\varphi(n)$ for all n.

Let $\varphi \in B^*$, $\|\, \varphi \,\|^* = 1$ and $\mathrm{Re}\{\varphi(x_k)\} \geq r\|\, x_k \,\|$, $r < 1$ for all k. In this case (see [**16, 17**]) we have necessary and sufficient conditions that the series $\sum\limits_{n=1}^{\infty} \varphi(x_n) e^{int}$ is the Fourier series of some function in AC.

THEOREM 3.4. *Let* $\{x_k\} \subset B$ *and let* $\varphi \in B^*$, $\| \varphi \|^* = 1$ *such that for each* k $\mathrm{Re}\{\varphi(x_k)\} \geq r\| x_k \|$, $0 < r < 1$. *Then*

(1) (i) *the series* $\sum_{n=1}^{\infty} \varphi(x_n)e^{ikt}$ *is the Fourier series of some function in* AC *if and only if* $\left\| \sum_{k=1}^{n} e^{i\arg(\varphi(x_k))}x_k \right\| = O(1)$, $n \to \infty$;

(2) (ii) $\left\{ \sum_{k=1}^{n} |\varphi(x_n)| \right\}$ *is slowly oscillating if and only if* $\left\{ \sum_{k=1}^{n} e^{i\arg(\varphi(x_k))}x_k \right\}$ *is slowly oscillating.*

Proof. From the inequality in the proof of Theorem 3.3 i.e.

$$\left\| \sum_{k=1}^{n} e^{i\arg(\varphi(x_k))}x_k \right\| \geq \sum_{k=1}^{n} |\varphi(x_k)|$$

we have

$$\begin{aligned}\left\| \sum_{k=1}^{N} e^{i\arg(\varphi(x_k))}x_k \right\| &\geq \sum_{k=1}^{N} |\varphi(x_k)| \geq \sum_{k=1}^{N} \mathrm{Re}\,\varphi(x_k) \geq r\sum_{k=1}^{N} \| x_k \| \\ &= r\sum_{k=1}^{N} \| e^{i\arg(\varphi(x_k))}x_k \| \geq r\left\| \sum_{k=1}^{N} e^{i\arg(\varphi(x_k))}x_k \right\|.\end{aligned}$$

Hence (i) and (ii) follow.

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Department of Mathematics
Universityof Missouri-Rolla
Rolla, MO 65401
U.S.A.

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