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# MEASURING ASYMPTOTIC CONVEXITY

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**Abstract**. We study a class of functions which are almost convex in a certain sense for large values of the argument. For this class of functions an Abel–Tauber theorem is proved.

## 0. Introduction

The theory of regular variation, including second order regular variation (the class $\Pi$) is well established by now. Basic properties were developed by Karamata in the thirties in order to define a suitable class of functions in connection with Tauberian theorems. In the first order theory basically functions $f$ are studied which are slowly varying. These are measurable, eventually positive and satisfy

$$f(tx)/f(t) \to 1 \quad (t \to \infty) \quad \text{for} \quad x > 0. \tag{0.1}$$

The next step is a second order theory: One considers the class of functions $f$ for which there exists a positive function $a$ such that $\lim_{t\to\infty}\{f(tx)-f(t)\}/a(t)$ exists. The most interesting case is the class $\Pi$, for which

$$\{f(tx) - f(t)\}/a(t) \to \log x \quad (t \to \infty) \quad \text{for} \quad x > 0. \tag{0.2}$$

A third order class, connected with the class $\Pi$, is defined by the relation

$$\{f(tx) - f(t) - a(t)\log x\}/a_1(t) \to \frac{1}{2}(\log x)^2 \quad (t \to \infty) \quad \text{for} \quad x > 0.$$

or equivalently,

$$\{f(txy) - f(tx) - f(ty) + f(t)\}/a_1(t) \to (\log x)(\log y) \quad (t \to \infty) \quad \text{for} \quad x, y > 0. \tag{0.3}$$

The relations (0.1) and (0.2) are discussed in [**3**] and [**4**]. The third order relation (0.3) is discussed in [**2**] and [**6**]. Note the relation with convexity: If $f$ satisfies (0.3), there exists a function $f_1$ such that $f_1(e^x)$ is convex and $f_1(t) - f(t) = o(a_1(t)) \quad (t \to \infty)$. See the appendix in [**2**].

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If in the defining relation (0.1) for slowly varying functions the existence of the limit is replaced by a boundedness condition one obtains the concept of $O$-regular variation. This concept was introduced in the paper [**1**] by Aljančić and Aranđelović in 1977. For more recent references the reader is referred to [**3**] and [**4**]. A measurable, eventually positive function $f$ is $O$-regularly varying ($f \in \mathrm{RO}$) if

$$\overline{\lim_{t\to\infty}} f(tx)/f(t) < \infty \quad \text{for} \quad x > 0. \tag{0.1a}$$

Similarly if in the defining relation (0.2) of the class $\Pi$ the existence of the limit is replaced by boundedness conditions, one obtains the class AB of asymptotically balanced functions. In the paper by de Haan and Resnick [**5**] this class is used in the study of extreme values in probability theory. For a more restricted definition the reader is referred to [**3**, Ch. 3.11].

*Definition* 0.1. A measurable function $f$ is asymptotically balanced ($f \in \mathrm{AB}$ or $f \in \mathrm{AB}(\sigma)$) if there exists a positive function $\sigma$ such that

$$\overline{\lim_{t\to\infty}} \{f(tx) - f(t)\}/\sigma(t) < \infty \quad \text{for} \quad x > 1 \tag{0.2a}$$

$$\underline{\lim_{t\to\infty}} \{f(tx) - f(t)\}/\sigma(t) > -\infty \quad \text{for} \quad x > 0 \tag{0.2b}$$

and if there exists $x_0 > 1$ such that

$$\underline{\lim_{t\to\infty}} \{f(tx) - f(t)\}/\sigma(t) > 0 \quad \text{for all} \quad x > x_0. \tag{0.2c}$$

The class AB is related to the class RO in the sense that if $f \in \mathrm{AB}(\sigma)$, then $\sigma \in \mathrm{RO}$. See [**4**, lemma 3.10]. In the defining relation (0.3) we shall now replace the existence of a limit by appropriate boundedness conditions. The following class of functions results.

*Definition* 0.2. Suppose the function $f : \mathbf{R}^+ \to \mathbf{R}$ is measurable. The function $f$ is asymptotically balanced of second order and we write $f \in \mathrm{AB}_2$ or $f \in \mathrm{AB}_2(\sigma)$ if there exists a positive function $\sigma$ and a constant $y_0 > 1$ such that the function $\rho_{x,y}(t)$ defined by

$$\rho_{x,y}(t) := \frac{f(txy) - f(tx) - f(ty) + f(t)}{\sigma(t)}$$

satisfies

$$\overline{\lim_{t\to\infty}} \rho_{x,y}(t) < \infty \qquad \text{for} \quad x > 1,\ y \geq y_0 \tag{0.4}$$

$$\underline{\lim_{t\to\infty}} \rho_{x,y}(t) > -\infty \quad \text{for} \quad x > 0,\ y \geq y_0 \tag{0.5}$$

$$\underline{\lim_{t\to\infty}} \rho_{x,y}(t) > 0 \qquad \text{for} \quad x > y_0,\ y \geq y_0 \tag{0.6}$$

Notation: $f \in \mathrm{AB}_2(\sigma)$ or $f \in \mathrm{AB}_2$.

In this paper we study the relation between the classes AB and $\mathrm{AB}_2$. In section 1 we consider the case where $\psi(t) := f(e^t)$ is convex. Note that the relation

$$f(txy) - f(tx) - f(ty) + f(t) > 0 \quad \text{for} \quad x, y > 1,\ t > 0$$

is equivalent to $\psi$ being convex. This is not implied by definition 0.2, but relation (0.6) in the definition can be seen as a form of asymptotic convexity. The size of the function $\sigma$ (which is an element of RO) in the denominator can be seen as a measure of the asymptotic convexity of $\psi$. We show that if $\psi$ is convex, the condition $f \in \mathrm{AB}_2$ is equivalent to: $tf'(t) \in \mathrm{AB}$. This is similar to the connection between the class $\Pi$ and slow variation: If $f$ is concave, then $f \in \Pi$ if and only if $tf'(t)$ is slowly varying. In section 2 the convexity condition on the function $\psi$ is replaced by the weaker condition of asymptotic convexity (see (2.2) below). In that case the connection between the classes AB and $\mathrm{AB}_2$ runs via fractional integrals rather than derivatives. More specifically, using the function $\gamma_r$ defined by

$$\gamma_r(t) := f(t) - rt^{-r} \int_{t_0}^{t} s^{r-1} f(s)\, ds$$

it follows that $f \in \mathrm{AB}_2(\sigma)$ if and only if $\gamma_r \in \mathrm{AB}(\sigma)$ for $r$ sufficiently large. We close the section with an Abel–Tauber theorem for the class $\mathrm{AB}_2$.

## 1. Asymptotic balance with convexity

In this section we assume that $\psi(t) := f(e^t)$ is convex. Then $\psi$ has a non-decreasing Radon–Nikodym derivative $\varphi = \psi'$. We shall prove

THEOREM 1.1. *Suppose $\psi$ is convex with derivative $\varphi$. The function $f(s) := \psi(\log s)$ is asymptotically balanced of second order if and only if $g(s) := \varphi(\log s)$ is asymptotically balanced (of first order).*

For the proof of the theorem we need two propositions in which it is shown that the convexity assumption allows us to describe the concepts AB and $\mathrm{AB}_2$ in terms of the asymptotic behaviour of certain sequences.

PROPOSITION 1.1. *Suppose $\varphi$ is a non-decreasing function. Equivalent are:*

1. *The function $g$ defined by $g(s) := \varphi(\log s)$ is asymptotically balanced.*
2. *There exists a constant $c > 0$ such that $\log(a_n/a_{n+1})$ is bounded $(n \to \infty)$ where $a_n := \varphi((n+1)c) - \varphi(nc)$.*

*Proof.* For the proof of $1 \Rightarrow 2$ note that in definition 0.1 we may replace $\sigma(t)$ by either $\sigma(ty)$ for any $y > 0$ (since $\sigma$ is RO, see [**4**, Lemma 3.10]) or by $f(tz) - f(t)$ for any $z > x_0$ (obvious from the definition). It follows that $\{\varphi(t+z) - \varphi(t)\}/\{\varphi(t+z_0) - \varphi(t)\}$ is bounded away from zero and infinity $(t \to \infty)$ for $z > 0$ and $z_0$ sufficiently large.

Next we prove the implication $2 \Rightarrow 1$. Note that for $n = [t/c]$ and $k \geq 1$ integer we have

$$\frac{\sum_{i=1}^{k-1} a_{n+i}}{\sum_{i=0}^{2} a_{n+i}} \leq \frac{\varphi(t+kc) - \varphi(t)}{\varphi(t+2c) - \varphi(t)} \leq \frac{\sum_{i=0}^{k} a_{n+i}}{a_{n+1}}.$$

Now (0.2c) follows by taking the $\liminf$ as $t \to \infty$. Moreover (0.2a) follows with $x = \exp(kc)$ and by monotonicity (0.2a) is also true for any $x > 1$. Similarly (0.2b) is a consequence of the inequality

$$\frac{\varphi(t-kc)-\varphi(t)}{\varphi(t+2c)-\varphi(t)} \geq -\frac{\sum_{i=0}^{k} a_{n-i}}{a_{n+1}},$$

valid for $n = [t/c]$ and $k \geq 1$.

Proposition 1.2. *Suppose $\psi$ is convex. Equivalent are*

1. *The function $f$ defined by $f(s) := \psi(\log s)$ is asymptotically balanced of second order.*
2. *There exists a constant $c > 0$ such that*

$$\varlimsup_{t\to\infty} \frac{\Delta(nc,2c)}{\Delta(nc,c)} < \infty \tag{1.1}$$

*where $\Delta(t,x) := \psi(t+x) - 2\psi(t) + \psi(t-x)$.*

*Proof.* We prove the implication $1 \Rightarrow 2$. Set $s(t) = \sigma(e^t)$. The conditions (0.4) and (0.6) imply the conditions

$$\varlimsup_{t\to\infty} \Delta(t,x)/s(t) < \infty \quad \text{for} \quad x \geq x_0$$
$$\varliminf_{t\to\infty} \Delta(t,x)/s(t) > 0 \quad \text{for} \quad x \geq x_0.$$

It follows that we may choose $s(t) = \Delta(t,x_1)$ for any fixed $x_1 > x_0$. This gives (1.1) with $c = x_0$. For the converse implication one can use similar arguments as in the proof of Proposition 1.1.

*Proof of Theorem* 1.1. Suppose $f \in \mathrm{AB}_2$. Set $d_n = \Delta(nc,c)$. Then

$$\Delta(nc,2c) = d_{n+1} + 2d_n + d_{n-1}. \tag{1.2}$$

Divide by $d_n$. Proposition 1.2 ensures that $d_{n+1}/d_n$ and $d_{n-1}/d_n$ are bounded. Since $\varphi = \psi'$ it follows that

$$\begin{aligned} d_{n+1} &= \Delta((n+1)c,c) = \psi((n+2)c) - 2\psi((n+1)c) + \psi(nc) \qquad (1.3)\\ &= \int_{(n+1)c}^{(n+2)c} \varphi(s)\,ds - \int_{nc}^{(n+1)c} \varphi(s)\,ds = \int_0^c \{\varphi(c(n+1)+s) - \varphi(nc+s)\}\,ds. \end{aligned}$$

Using monotonicity of $\varphi$ gives

$$c\{\varphi((n+2)c) - \varphi(nc)\} \geq d_{n+1}. \tag{1.4}$$

Similarly we find

$$c\{\varphi((n+2)c) - \varphi(nc)\} \leq \int_{(n+2)c}^{(n+3)c} \varphi(s)\,ds - \int_{(n-1)c}^{nc} \varphi(s)\,ds = d_n + d_{n+1} + d_{n+2}. \tag{1.5}$$

If we replace $n$ by $2n$ and $c$ by $c/2$ in (1.4) and (1.5) we see that the conditions of Proposition 1.1 are satisfied, hence $g$ is asymptotically balanced. The proof of the converse statement is an immediate consequence of the inequalities (1.4) and (1.5) and Propositions 1.1 and 1.2.

## 2. Asymptotic balance with asymptotic convexity

In this section we do not assume that $\psi(s) = f(e^s)$ is convex. However in order to obtain non-trivial results we have to impose condition (2.2) below which can be seen as an asymptotic convexity condition. First we consider the possible order of growth of the function $\sigma$ in definition 0.2.

LEMMA 2.1. *If $f \in \mathrm{AB}_2(\sigma)$, then $\overline{\lim}_{t\to\infty} \sigma(at)/\sigma(t) < \infty$ for all $a > 0$. Moreover we may take $\sigma$ measurable, hence $\sigma \in \mathrm{RO}$.*

*Proof.* Take $a > 0$ arbitrary. Observe that

$$\sigma(at)/\sigma(t) = \{\rho_{ay,x}(t) - \rho_{a,x}(t)\}/\rho_{x,y}(at). \tag{2.1}$$

Note that $\overline{\lim}_{t\to\infty} \sigma(at)/\sigma(t) < \infty$ if we choose $x > y_0$, $y > \max(a^{-1}, y_0)$ and use definition 0.2. We may choose $\sigma(t) = f(ty_0^2) - 2f(ty_0) + f(t)$ which is measurable.

The basic result in this section relates second order asymptotic balance of a function $f$ to first order asymptotic balance of the transform $\gamma_r$ of $f$.

THEOREM 2.1. *Suppose $f : \mathbf{R}^+ \to \mathbf{R}$ is measurable and suppose there exists a positive function $\sigma$ such that*

$$\underline{\lim}_{t\to\infty} \rho_{x,y}(t) = \underline{\lim}_{t\to\infty} \frac{f(txy) - f(tx) - f(ty) + f(t)}{\sigma(t)} \geq 0 \quad \textit{for all} \quad x, y > 1. \tag{2.2}$$

*Define the functions $\gamma_r(t)$ and $s_t(x)$ by*

$$\gamma_r(t) := f(t) - rt^{-r} \int_{t_0}^{t} s^{r-1} f(s)\, ds \qquad (t > t_0) \tag{2.3}$$

$$s_t(x) := \frac{f(tx) - f(t) - r \log x \gamma_r(t)}{\sigma(t)} \tag{2.4}$$

*Consider the following statements:*

(i) $f \in \mathrm{AB}_2(\sigma)$

(ii) *there exist $t_0$, $r$ such that $\gamma_r(t)$ is well defined for $t > t_0$ and $\gamma_r(t) \in \mathrm{AB}(\sigma)$*

(iii) *there exist $t_0$, $r$ such that the function $\gamma_r(t)$ is well defined for $t > t_0$ and $s_t(x)$ satisfies the conditions*

$$\overline{\lim}_{t\to\infty} |s_t(x)| < \infty \quad \textit{for all} \quad x > 0 \tag{2.5}$$

$$\underline{\lim}_{t\to\infty} \{s_t(y) - s_t(x)\} \geq 0 \quad \textit{for all} \quad y > x \geq 1 \tag{2.6}$$

*and there exists a constant* $x_0 > 0$ *such that*

$$\varliminf_{t\to\infty} s_t(x) > 0 \quad \textit{for all} \quad x > x_0. \tag{2.7}$$

*Moreover* $\sigma \in \mathrm{RO}$.

*Statement* (i) *implies* (ii) *for all sufficiently large* $r$. *For fixed* $r > 0$ *the statements* (ii) *and* (iii) *are equivalent and imply statement* (i).

In order to be able to formulate the proof of this theorem we need the following class of functions: a measurable, eventually positive function $f$ is of bounded and positive increase ($f \in \mathrm{BI} \cap \mathrm{PI}$) if $f \in \mathrm{RO}$ with lower Matuszewska index positive. See [**3**, Chapter 2.1] or [**4**, Chapter 3]. In order to prove the theorem we need an auxiliary result on ordinary AB functions which is an analogue of Theorem 3.13 in [**4**].

Lemma 2.2. *Suppose* $f : \mathbf{R}^+ \to \mathbf{R}$ *is measurable. Consider the following statements:*

(i) *There is a (positive) function* $\sigma$ *such that* $f \in \mathrm{AB}(\sigma)$ *and*

$$\varliminf_{t\to\infty} \frac{f(tx) - f(t)}{\sigma(t)} \geq 0 \quad \textit{for all} \quad x > 1, \tag{2.8}$$

(ii) *For some* $t_0 > 0$

$$g_r(t) := t^r f(t) - r \int_{t_0}^{t} s^{r-1} f(s)\, ds$$

*is well defined for* $t > t_0$ *and in* $BI \cap PI$. *Moreover*

$$\varliminf_{t\to\infty} \frac{f(tx) - f(t)}{t^{-r} g_r(t)} \geq 0 \quad \textit{for all} \quad x > 1. \tag{2.9}$$

*Statement* (i) *implies* (ii) *for all sufficiently large* $r$. *If statement* (ii) *is true for some* $r > 0$, *then* (i) *holds with* $\sigma(t) := t^{-r} g_r(t)$.

*Proof.* (i) $\to$ (ii) Since $f \in \mathrm{AB}(\sigma)$ we may choose $\sigma \in \mathrm{RO}$ (see [**4**, Lemma 3.10]). Then $t^r\sigma(t) \in \mathrm{BI} \cap \mathrm{PI}$ for any $r > r_0 := -\beta(\sigma)$, the lower Matuszewska index of $\sigma$ (see [**3**, Chapter 2.2] or [**4**, Chapter 3]). We prove that $t^r\sigma(t) \asymp g_r(t)$ for $r > r_0$ as $t \to \infty$. Note that this proves $g_r \in \mathrm{BI} \cap \mathrm{PI}$ ($r > r_0$) and the implication (2.8) $\to$ (2.9). Since $f \in \mathrm{AB}(\sigma)$ there exist $c$, $\alpha$, $t_0 > 0$ such that $|f(t)| < ct^\alpha$ for $t > t_0$ (see [**4**, Lemma 3.12]). We have

$$\frac{g_r(t)}{t^r\sigma(t)} = r \int_{t_0/t}^{1} \frac{f(t) - f(ts)}{\sigma(ts)} \frac{\sigma(ts)}{\sigma(t)} s^{r-1} ds + \frac{t_0^r f(t)}{t^r\sigma(t)}.$$

Hence $g_r(t)$ is finite for $t > t_0$ and if we choose $r$ sufficiently large, then $f(t)/t^r\sigma(t) \to 0$ as $t \to \infty$. Since $\sigma \in \mathrm{RO}$ we can use Lemma 3.12 in [**4**] together with Fatou's lemma to find that

$$\varliminf_{t\to\infty} \frac{g_r(t)}{t^r\sigma(t)} \geq r \int_0^1 \varliminf_{t\to\infty} \frac{f(t/s) - f(t)}{\sigma(t)} \varliminf_{t\to\infty} \frac{\sigma(ts)}{\sigma(t)} s^{r-1} ds.$$

Now by (2.8) and the definition of $\mathrm{AB}(\sigma)$

$$\varliminf_{t\to\infty} \frac{f(t/s)-f(t)}{\sigma(t)} \begin{cases} \geq 0 & \text{for all } 0<s<1 \\ >0 & \text{for } s<x_0^{-1} \end{cases}$$

It follows that $\varliminf_{t\to\infty} g_r(t)/\{t^r\sigma(t)\} > 0$.

Similarly using the inequality

$$\left|\frac{f(t)-f(ts)}{\sigma(ts)}\right|\frac{\sigma(ts)}{\sigma(t)} \leq c_1 s^{-\alpha_1} c_2 s^{\alpha_2} \qquad \text{(see [}\mathbf{4}\text{])}$$

for $t_0/t<s<1$ where $c_i$, $\alpha_i$ are positive constants, we have $\varlimsup\limits_{t\to\infty} g_r(t)/\{t^r\sigma(t)\} < \infty$ if we choose $r>\alpha_1-\alpha_2$. This proves $g_r \in \mathrm{BI}\cap\mathrm{PI}$ for $r$ sufficiently large.

(ii) $\to$ (i) From the definition of $g_r$, for $x>1$ we have

$$\frac{g_r(tx)-g_r(t)}{g_r(t)} = \frac{f(tx)-f(t)}{t^{-r}g_r(t)} + r\int_1^x \frac{f(tx)-f(ts)}{(ts)^{-r}g_r(ts)}\frac{(ts)^{-r}g_r(ts)}{t^{-r}g_r(t)}s^{r-1}ds.$$

Application of Fatou's lemma (using again Lemma 3.12 in [**4**] and (2.9)) shows that

$$\varliminf_{t\to\infty} g_r(tx)/g_r(t) \geq 1 \quad \text{for} \quad r>r_0,\ x>1. \tag{2.10}$$

From the definition of $g_r(t)$ it follows that

$$f(t) = t^{-r}g_r(t) + r\int_{t_0}^t g_r(s)s^{-r-1}ds$$

for $t>t_0$, hence

$$\frac{f(tx)-f(t)}{t^{-r}g_r(t)} = r\int_1^x \frac{g_r(tu)}{g_r(t)}u^{-r-1}du + \frac{(xt)^{-r}g_r(tx)}{t^{-r}g_r(t)} - 1. \tag{2.11}$$

Using the inequalities $c^{-1}x^\beta \leq g_r(tx)/g_r(t) \leq cx^\alpha$ for $x\geq 1$, $t>t_0$ where $\alpha,\beta>0$, $c>1$ (see [**4**, Theorem 3.5]) we see that (2.8) holds,

$$\varlimsup_{t\to\infty} \frac{f(tx)-f(t)}{t^{-r}g_r(t)} < \infty \quad \text{for } x>1 \quad \text{and} \quad \varliminf_{t\to\infty} \frac{f(tx)-f(t)}{t^{-r}g_r(t)} > -\infty \ \text{ for } x>0.$$

It remains to prove that $\varliminf_{t\to\infty}\{f(tx)-f(t)\}/\{t^{-r}g_r(t)\} > 0$ for $x>x_0$. By (2.10)

$$\varliminf_{t\to\infty} r\int_1^{x_0} \frac{g_r(tu)}{g_r(t)}u^{-r-1}du \geq r\int_1^{x_0} u^{-r-1}du. \tag{2.12}$$

Moreover, since

$$\varliminf_{t\to\infty} g_r(tx)/g_r(t) > 1 \quad \text{for} \quad x>x_0, \quad r>r_0,$$

we get for $x > x_0$

$$\underline{\lim}_{t\to\infty} r\int_{x_0}^{x} \frac{g_r(tu)}{g_r(t)}u^{-r-1}du + \frac{(xt)^{-r}g_r(tx)}{t^{-r}g_r(t)} - 1 > r\int_{x_0}^{x} u^{-r-1}du + x^{-r} - 1 \qquad (2.13)$$

Combination of (2.11), (2.12) and (2.13) gives the claimed result.

*Remark.* Note that the lemma fails if the assumptions (2.8) and (2.9) are omitted. Take e.g. $f(t) = \log t + \sin t$.

*Proof of Theorem* 2.1. Without loss of generality we may assume that $f(t) = 0$ on a neighborhood of zero.

(i) $\Leftrightarrow$ (ii) Define $\tilde{f}_y(t) := f(ty) - f(t)$. From the definitions of AB and $\mathrm{AB}_2$ it follows that (i) is equivalent to: $\tilde{f}_y \in \mathrm{AB}(\sigma)$ for all $y > y_0$. Application of Lemma 2.3 shows that (i) holds if and only if there exists $y_0$ such that for $y \geq y_0$, $r \geq r_0(y)$

$$\begin{gathered} t^r\{\gamma_r(ty) - \gamma_r(t)\} = t^r\tilde{f}y(t) - r\int_{t_0}^{t} \tilde{f}_y(s)s^{r-1}ds \quad \text{is in BI} \cap \text{PI} \\ \gamma_r(ty) - \gamma_r(t) \asymp \sigma(t) \quad (t\to\infty). \end{gathered} \qquad (2.14)$$

Since $\sigma$ is positive, the convexity condition (2.2) implies that the functions

$$\psi_y(x) := \underline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)} \quad \text{and} \quad \Psi_y(x) := \overline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)}$$

are non-decreasing in $x$ and $y$ for all $x, y > 0$. Indeed this follows since $\Psi_y(x) = \Psi_x(y)$ and $\Psi_y(x)$ is non-decreasing in $x$ since for $x \in (0, u)$ we have

$$\Psi_y(x) \leq \overline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(tu)}{\sigma(tx)}\frac{\sigma(tx)}{\sigma(t)} + \overline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tu) - \tilde{f}_y(t)}{\sigma(t)}. \qquad (2.15)$$

Note that the convexity condition (2.2) is equivalent to

$$\underline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)} \geq 0 \quad \text{for} \quad x > 1.$$

Hence (2.15) is at most

$$-\underline{\lim}_{t\to\infty} \frac{\tilde{f}_y(tu) - \tilde{f}_y(tx)}{\sigma(tx)}\, \overline{\lim}_{t\to\infty} \frac{\sigma(tx)}{\sigma(t)} + \Psi_y(u) \leq \Psi_y(u) \leq \infty$$

and a similar argument for $\psi_y(x)$. Hence for all $x, y > 1$ we have $0 \leq \psi_y(x) \leq \Psi_y(x) \leq \Psi_{\max(y_0,y)}(x) < \infty$. Applying Lemma 3.12 in [**4**] we get

$$\Big|\frac{\tilde{f}_y(tx) - \tilde{f}_y(t)}{\sigma(t)}\Big| \leq c_1(y)x^{\alpha_1(y)} \quad \text{for} \quad x \geq 1,\ t \geq t_0.$$

It follows that for arbitrary $y > 1$ there exist $c$, $\alpha$ such that

$$\varlimsup_{t\to\infty} \frac{\gamma_r(ty)-\gamma_r(t)}{\sigma(t)} = \varlimsup_{t\to\infty} r \int_{t_0/t}^{1} \frac{\tilde{f}_y(t)-\tilde{f}_y(ts)}{\sigma(t)} s^{r-1} ds$$

$$\leq r \int_0^1 \Psi_y(1/s) \varlimsup_{t\to\infty} \frac{\sigma(ts)}{\sigma(t)} s^{r-1} ds \leq c \int_0^1 s^{\alpha+r-1} ds \leq \infty \tag{2.16}$$

if $r > -\alpha$. The proof of $\varliminf_{t\to\infty}\{\gamma_r(ty)-\gamma_r(t)\}/\sigma(t) > -\infty$ for $y > 0$, $r > r_0$ is similar. Hence (i) implies (ii) for all sufficiently large $r$. The implication (ii) $\to$ (2.14) is trivial.

(ii) $\to$ (iii) From (2.3) it follows that

$$f(t) = \gamma_r(t) + r \int_{t_0}^{t} \gamma_r(s) \frac{ds}{s}, \quad t > t_0$$

hence

$$s_t(x) = \frac{f(tx)-f(t)-r\gamma_r(t)\log x}{\sigma(t)} = \frac{\gamma_r(tx)-\gamma_r(t)}{\sigma(t)} + r\int_1^x \frac{\gamma_r(ts)-\gamma_r(t)}{\sigma(t)} \frac{ds}{s}. \tag{2.17}$$

The last expression together with application of Lemma 3.12 in [**4**] and

$$\varliminf_{t\to\infty} \{\gamma_r(tx)-\gamma_r(t)\}/\sigma(t) \geq 0 \quad \text{for} \quad x > 1$$

(which follows as in (2.16)), shows that (ii) implies (iii).

(iii) $\to$ (ii) Define $q_t(x) := \{\gamma_r(tx)-\gamma_r(t)\}/\sigma(t)$. From (2.4) it follows that for $y > x > 0$

$$q_t(x) = \frac{s_t(y)-s_t(x)-s_{tx}(y/x)\sigma(tx)/\sigma(t)}{r\log y/x}. \tag{2.18}$$

Hence by the assumptions on the functions $s_t(x)$ and $\sigma$ it follows that $\varlimsup_{t\to\infty}|q_t(x)|<\infty$ for $x > 0$. Application of Lemma 3.12 in [**4**] then shows that

$$|q_t(x)| \leq cx^{\varepsilon} \quad \text{for} \quad x > 1,\ t > t_0, \tag{2.19}$$

where $\varepsilon, c > 0$. Hence using (2.17), i.e.

$$s_t(x) = q_t(x) + r\int_1^x q_t(s)\frac{ds}{s}, \tag{2.20}$$

it follows that $s_t(x)$ satisfies the inequality

$$|s_t(x)| < c_0 x^{\varepsilon_0} \quad \text{for} \quad x \geq 1,\ t \geq t_0, \tag{2.21}$$

where $c_0$ and $\varepsilon_0$ are constants. From (2.20) it follows that the function $q_t(x)$ satisfies the relation

$$q_t(x) = rx^{-r} \int_1^x (s_t(x) - s_t(u))u^{r-1}du + x^{-r}s_t(x). \tag{2.22}$$

The proof of $\underline{\lim}_{t\to\infty} q_t(x) > 0$ for $x > x_0$ follows by application of Fatou's lemma to the integral in (2.14) (use (2.6) and (2.7)). Note that by (2.6) for $x > 1$ we have $\underline{\lim}_{t\to\infty} s_t(x) = \underline{\lim}_{t\to\infty}\{s_t(x) - s_t(1)\} > 0$.

In order to formulate our next result we need the following notion. The functions $f, f_0 : \mathbf{R}^+ \to \mathbf{R}$ are $O$-inversely asymptotic if there exist constants $a > 1$ and $t_0$ such that $f(t) \le f_0(at)$ and $f_0(t) < f(at)$ for $t \ge t_0$. Notation: $f \overset{O}{\sim} f_0$ or $f(t) \overset{O}{\sim} f_0(t)$ $(t \to \infty)$. Observe that if $f$, $f_0$ are increasing and unbounded, then $f \overset{O}{\sim} f_0$ if and only if the inverse functions satisfy $f^{\leftarrow} \asymp f_0^{\leftarrow}$, which explains the terminology.

Theorem 2.2. *Suppose $f : \mathbf{R}^+ \to \mathbf{R}$ is measurable and suppose*

$$\hat{f}(s) := s \int_0^\infty e^{-st} f(t)\,dt < \infty \quad \textit{for} \quad s > 0.$$

*Then*

$$f \in \mathrm{AB}_2(\sigma) \quad \textit{with} \quad \beta(\sigma) > -1 \tag{2.23}$$

*implies*

$$\hat{f}(1/t) \in \mathrm{AB}_2(\sigma) \quad \textit{with} \quad \beta(\sigma) > -1. \tag{2.24}$$

*If there exists $t_0$ such that*

$$f(e^t) \ \textit{is convex for}\ t > t_0 \tag{2.25}$$

*then the converse holds:* (2.24) *implies* (2.23). *Moreover if the function $f$ in* (2.23) *satisfies* (2.2), *then there exist $r_0$, $x_0$ such that the transforms $\gamma_r$ and $\gamma_r^*$ satisfy*

$$r\gamma_r(t)\log x \overset{O}{\sim} f(tx) - f(t) \overset{O}{\sim} \hat{f}(1/tx) - \hat{f}(1/t) \overset{O}{\sim} r\gamma_r^*(t)\log x \tag{2.26}$$

*as $t \to \infty$ for $r > r_0$, $x > x_0$, where $\gamma_r(t)$ is as defined in theorem* 2.2 *and*

$$\gamma_r^*(t) = \hat{f}(t^{-1}) - rt^{-r}\int_{t_0}^t s^{r-1}\hat{f}(s^{-1})\,ds.$$

*In particular we have for $r > r_0$*

$$\gamma_r(t) - \gamma_r^*(t) = O(\sigma(t)) \quad (t \to \infty). \tag{2.27}$$

*Proof.* By the definitions of AB and $\mathrm{AB}_2$ it follows that $f \in \mathrm{AB}_2(\sigma)$ is equivalent to $\tilde{f}_y(t) = f(ty) - f(t) \in \mathrm{AB}(\sigma)$ for all $y \ge y_0$. Application of theorem 4.2 in

[**4**] shows that this implies $\hat{\tilde{f}}_y(t) = \hat{f}(1/ty) - \hat{f}(1/t) \in \mathrm{AB}(\sigma)$ for $y \geq y_0$ which is equivalent to $\hat{f}(1/t) \in \mathrm{AB}_2(\sigma)$. A converse statement is true if $\tilde{f}_y(t)$ is eventually non-decreasing in $t$ which is condition (2.25). In order to prove (2.26) note that for $x > x_0$, $r > r_0$, there exists $t_0 = t_0(x, r)$ such that $f(tx) - f(t) > r \log x \gamma_r(t)$ for $t > t_0$ by (2.7).

For a converse inequality, fix $x > x_0$, $r > r_0$. Since $\gamma_r \in \mathrm{AB}(\sigma)$ we have by (2.17) for $y > x$ sufficiently large

$$\begin{aligned}\overline{\lim_{t\to\infty}} \frac{f(tx) - f(t) - r\gamma_r(ty)\log x}{\sigma(t)} &\leq c_1 + r\int_1^x \overline{\lim_{t\to\infty}} \frac{\gamma_r(ts) - \gamma_r(ty)}{\sigma(t)} \frac{ds}{s} \\ &\leq c_1 - rc_2 \int_1^x \frac{ds}{s} \end{aligned} \tag{2.28}$$

where $c_1, c_2 > 0$ are constants (depending on $r$, see (2.4)). The right-hand side in (2.28) is negative if we choose $x > x_0$ sufficiently large, then $y > x$ sufficiently large in order to ensure the validity of (2.28). Hence $r\gamma_r(t)\log x \overset{O}{\sim} f(tx) - f(t)$. The statements $f(tx) - f(t) \overset{O}{\sim} \hat{f}(1/tx) - \hat{f}(1/t)$ and (2.27) follow from [**4**, theorem 4.2]. The proof of $\hat{f}(1/tx) - \hat{f}(1/t) \overset{O}{\sim} r\gamma_r^*(t)\log x \quad (t \to \infty)$ follows as above.

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