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SLOWLY VARYING SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Asymptotic behavior of slowly varying (in the sense of Karamata) solutions of second order linear differential equations is determined under less restrictive conditions then in the known results.

Introduction

Let $f(x)$ be continuous and positive on a half-axis $[a, \infty)$, $a > 0$. We obtain the precise asymptotic behavior for $x \to \infty$ of slowly varying solutions $y(x)$ of the equation

$$y''(x) - f(x)y(x) = 0, \tag{E}$$

As it is well known, the class of slowly varying solutions was introduced by Karamata [**6**], by the following

Definition. A positive measurable function L defined on $[a, \infty), a > 0$ is said to be slowly varying (s.v.) at infinity if, for each $t > 0$

$$\lim_{x\to\infty} L(xt)/L(x) = 1.$$

The function $g(x) = x^\alpha L(x)$ is said to be regularly varying (r.v.) at infinity of index α.

All functions tending to positive constants as $x \to \infty$, or the function

$$L(x) = \prod_{\nu=1}^{n} (\log_\nu x)^{\xi_\nu}$$

(ξ_ν reals, $\log_\nu$ denotes the ν-th iteration of the logarithm) may serve as the simplest examples of s.v. functions.

From the extensively developed theory of s.v. (and related) functions ([**2**], [**4**], [**8**]) we quote only a few needed here:

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THEOREM A. (The representation theorem, [**6**]) *A function L is s.v. at infinity iff it is of the form*

$$L(x) = c(x)\exp\left(-\int_a^x (\eta(t)/t)\ dt\right),$$

where, for $x \to \infty$, $c(x) \to c > 0$, $\eta(x) \to 0$, $a > 0$.

If especially $c(x) = c > 0$, then L is called normalized and is denoted by L_0. The later class is relevant here.

For differentiable functions (which are of use here) there holds, [**8**], [**1**, vi].

THEOREM B. *A positive differentiable function g is a normalized s.v. one at infinity if*

$$\lim_{x\to\infty} xg'(x)/g(x) = 0; \tag{1.1}$$

conversely if g is s.v. with g' monotone, then (1.1) holds.

In [**7**] we proved the existence of s.v. solutions by the result we reproduce here as the keystone of our analysis in this paper.

THEOREM C. *All positive, decreasing solutions of equation (E) are s.v. functions at infinity iff for $x \to \infty$*

$$x\int_x^\infty f(t)dt \to 0. \tag{1.2}$$

Such solutions are of the form

$$y(x) = L_0(x) = A\exp\left(-\int_a^x (\eta(t)/t)dt\right), \tag{1.3}$$

where $\eta(t)$ is positive and tends to zero as $x \to \infty$ and $A > 0$.

For the sake of completeness we include the proof of Theorem C:

a) Necessity. Let $y(x)$ be s.v. then, since it is also convex by (E), it satisfies (1.1) in virtue of the second part of Theorem B. Hence, (1.3) follows. Also, since $L_0''/L_0 \equiv (L_0'/L_0)' + (L_0'/L_0)^2$, equation (E) becomes

$$(L_0'/L_0)' + (L_0'/L_0)^2 = f(x),$$

or by integrating over (x,∞) and multiplying throughout by x,

$$-xL_0'/L_0 + x\int_x^\infty (tL_0'/L_0)^2 t^{-2}dt = x\int_x^\infty f(t)dt.$$

Now because of (1.1) both integrals converge. Moreover the left-hand side and hence the right-hand one of the above equation tend to zero as $x \to \infty$.

b) Sufficiency. By integrating both sides of (E) over (x,∞) and since $y(x)$ is positive, decreasing and thus such that $y'(x) \to 0$, as $x \to \infty$, one has

$$-xy'/y = x\int_x^\xi f(t)dt = \eta(x).$$

$\eta(x)$ is obviously positive and, due to (1.2), tends to zero as $x \to \infty$. Whence, by the first part of Theorem B, $y(x)$ is s.v.

We remark that a positive s.v. solution of (E) cannot increase. For otherwise, in virtue of its convexity, one would have eventually $y'(x) \geq k$ for some $k > 0$, or by integrating $y(x) \geq kx+l$. Hence $y(x)$ cannot be s.v. since for any $\varepsilon > 0$ $x^{-\varepsilon}L(x) \to 0$ as $x \to \infty$, [**8**, §1.5, 10]. This shows that we are dealing here with all s.v. solutions of (E).

Troughout the text all inequalities hold for $x \geq a$, for some $a > 0$, $n \in N$, and all majorizing (minorizing) constants are denoted by the same letter k.

Results

2.1. We may point out that condition (1.2) is the sole one we use in our analysis. Properties of s.v. functions do the rest.

Put

$$\int_x^\infty f(t)dt = F(x); \tag{2.1}$$

then there holds

Theorem 1. *Any s.v. solution of equation (E) is of the form*

$$y(x) = A\exp\left(-\int_a^x (\eta(t)/t)dt\right). \tag{2.2}$$

Here A is a constant, $\eta(t) > 0$ for $t \geq a$, $\eta(t) \to 0$, as $t \to \infty$ and $\eta(x) = \lim_{n\to\infty} \eta_n(x)$ uniformly in $[a, \infty)$, where $\eta_n(x)$ is defined recursively by

$$\eta_0(x) = xF(x), \quad \eta_n(x) = x\{F(x) - \int_x^\infty (\eta_{n-1}(t)/t)^2 dt\}, \tag{2.3}$$

or by

$$\eta_n(x) = x\Big\{F(x) - \int_x^\infty [F(x_0) - \int_{x_0}^\infty F(x_1) - \ldots - \int_{x_{n-2}}^\infty F^2(x_{n-1})dx_{n-1}]^2 \ldots]^2 dx_1]dx_0\Big\}. \tag{2.4}$$

2.2. The asymptotic formula for the considered solution is obtained using Theorem 1, in terms of functions $\eta_n(x)$ in the following

Theorem 2. *Put*

$$u_1(x) = \int_x^\infty F^2(t)dt, \quad u_n(x) = 2\int_x^\infty F(t)u_{n-1}(t)dt, \; n \geq 2. \tag{2.5}$$

Then each $\eta_n(x)/x$ for $n \geq 1$, contains the term $u_n(x)$ such that for $x \to \infty$

$$\eta_n(x)/x - \eta_{n-1}(x)/x = (-1)^n u_n(x) + o(u_n(x)) \tag{2.6}$$

and

$$u_{n+1}(x) = o(u_n(x)). \tag{2.7}$$

If there exists a positive integer n *such that*

$$\int^{\infty} u_n(t)dt < \infty, \tag{2.8}$$

then for any s.v. solution of the equation (E) the following asymptotic formula holds for $x \to \infty$, *and for some* A

$$\text{a)}\quad y(x) \sim A \exp\left(-\int_a^x (\eta_{n-1}(t)/t)dt\right), \qquad \text{b)}\quad xy'(x)/y(x) \to 0. \tag{2.9}$$

Notice that the behaviour of the second linearly independent solution $\tilde{y}(x)$ is obtained by applying to the integral in $\tilde{y}(x) = y(x)\int_a^x y^{-2}(t)dt$ Karamata's theorem, [**8**, Th. 1.4], to obtain for $x \to \infty$ $\tilde{y}(x) \sim x/y(x)$. Hence $\tilde{y}(x)$ is a regularly varying function of index 1. In addition a direct calculation gives $x\tilde{y}'(x)/\tilde{y}(x) \to 1$.

2.3. Condition (2.8) might be cumbersome to verify. It can, however, be replaced by a simpler, in general a cruder one, as it is done in the Corollary following proof of Theorem 2.

Proofs

3.1. *Proof of Theorem* 1. By Theorem C there is a solution of the form

$$y(x) = \exp\left(-\int_a^x (\eta(t)/t)dt\right), \tag{3.1}$$

where $\eta(t) \to 0$ as $t \to \infty$. Also, $\eta(t) > 0$ due to the concluding remark in the Introduction. Obviously all other s.v. solutions are of the form $Ay(x)$. By substituting $y(x)$ from (3.1) into (E) one obtains the Riccati equation

$$\{\eta(t)/t\}' - \{\eta(t)/t\}^2 + f(t) = 0. \tag{3.2}$$

We shall reduce it to a corresponding integral one where condition (1.2) occurs explicitely. By integrating (3.2) over (x, ∞) one obtains

$$\eta(x) = x\int_x^{\infty} f(t)dt - x\int_x^{\infty} (\eta(t)/t)^2 dt. \tag{3.3}$$

Now observe that the sequence (2.3) is the one of successive approximations for (3.3) which we shall show to converge uniformly on $[a, \infty)$ to a solution $\eta(x)$ of (3.3) with desired properties. To that end put into (3.3) and (2.3) respectively

$$m(x) = \eta(x)/xF(x), \;\; m_n(x) = \eta_n(x)/xF(x), \; n = 0, 1, \ldots \tag{3.4}$$

to obtain

$$m(x) = 1 - F^{-1}(x)\int_x^{\infty} m^2(t)F^2(t)dt \tag{3.5}$$

and

$$m_0(x) = 1, \;\; m_n(x) = 1 - F^{-1}(x) \int_x^\infty m_{n-1}^2(t) F^2(t) dt. \tag{3.6}$$

Also notice that condition (1.2) implies that for any $\varepsilon > 0$ there exists $a > 0$ such that for $x \geq a$

$$\int_x^\infty F^2(t) dt \leq \varepsilon F(x). \tag{3.7}$$

For, by a partial integration,

$$\int_x^\infty F^2(t) dt = -x F^2(x) + 2 \int_x^\infty t F(t) f(t) dt$$

and (3.7) follows by (1.2).

Now, by a standard induction argument we show that for any $\varepsilon > 0$ and $x \in [a, \infty)$ one has

$$|m_n(x) - m_{n-1}(x)| \leq 2^{n-1} \varepsilon^n. \tag{3.8}$$

Obviously, due to (3.7), one obtains from (3.6)

$$|m_1(x) - m_0(x)| \leq \varepsilon.$$

Also,

$$m_{n+1}(x) - m_n(x) = F^{-1}(x) \int_x^\infty (m_n^2(t) - m_{n-1}^2(t)) F^2(t) dt. \tag{3.9}$$

Assuming (3.8) holds and since by (3.6), $m_n(x) < 1$, there follows from (3.9)

$$|m_{n+1}(x) - m_n(x)| \leq 2^n \varepsilon^{n+1}.$$

The uniform convergence over $[a, \infty)$ of sequence (3.6) to a function $m(x)$ follows from

$$m_n(x) = m_0 + \sum_{\nu=1}^{n} (m_\nu(x) - m_{\nu-1}(x))$$

and (3.8), by choosing $\varepsilon < 1/2$. In addition, due to $0 < m_n(x) < 1$ and the convergence of $\int^\infty F^2 dt$, there follows by the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \int_x^\infty m_{n-1}^2(t) F^2(t) dt = \int_x^\infty m^2(t) F^2(t) dt,$$

so that $m(x)$ is a solution of equation (3.5). The same then applies to the sequence $\eta_n(x) = x F(x) m_n(x)$.

3.2. *Proof of Theorem* 2. Formula (2.6) holds for $n = 1$ trivially due to (2.3) and (2.5). Assume its validity for some n, then, again because of (2.3) and (2.5), one has

$$\eta_{n+1}(x)/x - \eta_n(x)/x = (-1)^{n+1} u_{n+1}(x) + \sum_{\nu=1}^{5} R_\nu(x),$$

where

$$R_1(x) = 2(-1)^n \int_x^\infty u_n(t) \int_t^\infty (\eta_{n-2}(s)/s)^2 ds dt,$$
$$R_2(x) = \int_x^\infty (\eta_{n-1}(t)/t) o(u_n(t)) dt, \quad R_3(x) = -\int_x^\infty u_n^2(t) dt,$$
$$R_4(x) = \int_x^\infty u_n(t) o(u_n(t)) dt, \qquad R_5(x) = \int_x^\infty [o(u_n(t))]^2 dt.$$

Since, by (2.3), $\eta_n(x)/x \le F$, one has, by (3.7), $|R_1(x)| \le 2\varepsilon \int_x^\infty F(t)u_n(t)dt$ and so by (2.5) for $x \to \infty$, $R_1(x) = o(u_{n+1}(x))$. The estimate for $R_2(x)$ follows analogously. To treat $R_\nu(x)$, $\nu = 3,4,5$, observe that (2.5) implies for any $\varepsilon > 0$, and for sufficiently large x, $u_n(x) \le \varepsilon F(x)$, which leads to the desired estimate.

Formula (2.7) follows by an easy induction argument: For $n = 1$ this is true due to inequality (3.7). Assume for any $\varepsilon > 0$ and $x \in [a, \infty)$

$$u_{n+1}(x) \le \varepsilon u_n(x). \tag{3.10}$$

Then, because of (2.5) and (3.10),

$$u_{n+2}(x) \le 2\varepsilon \int_x^\infty F(t) u_n(t) dt = \varepsilon u_{n+1}(x),$$

so that (2.7) holds. In addition, by a repeated use of (3.10) one obtains for all $n, p \in N$

$$u_{n+p}(x) \le \varepsilon^p u_n(x). \tag{3.11}$$

On the other hand, due to (2.6), one has for $x \in [a, \infty)$, for some $a > 0$,

$$|\eta_n(x)/x - \eta_{n-1}(x)/x| \le 2u_n(x). \tag{3.12}$$

To complete the proof, bearing in mind Theorem 1, put

$$\eta(x)/x = \eta_{n-1}(x)/x + r_n(x), \tag{3.13}$$
$$r_n(x) = \sum_{\nu=n}^\infty (\eta_\nu(x)/x - \eta_{\nu-1}(x)/x).$$

To estimate the remainder one makes use of inequalities (3.12) and (3.11) to obtain

$$|r_n(x)| \le 2 \sum_{\nu=n}^\infty u_\nu(x) \le 2(1-\varepsilon)^{-1} u_n(x). \tag{3.14}$$

In virtue of (3.1), (3.13), (3.14) and (2.8), the proof of (2.9) a) is completed. Also b) is a direct consequence of the second part of Theorem B.

3.3. Estimate (3.7) implies the existence of a positive decreasing function $c(x)$ tending to zero as $x \to \infty$ and such that

$$u_1(x) = \int_x^\infty F^2(t) dt \le c(x) F(x)/2. \tag{3.15}$$

Thus inequality (3.10), with $c(x)$ replacing ε, leads to

$$u_n(x) \le c^n(x)F(x).$$

Hence, Theorem 2 implies the following

COROLLARY. *If for some n*

$$\int^\infty c^n(t)F(t)dt < \infty \tag{3.16}$$

then asymptotic formula (2.9) holds.

Remarks and examples

4.1. Hartman and Wintner proved, [**5**, Ch. XI, Ex. 9.9.b], the following

THEOREM D. *Let $f(t)$ be a continuous complex function defined for $t \ge a$. If for some $p \in [1,2]$*

$$\int^\infty t^{2p-1}|f^p(t)|dt < \infty \tag{4.1}$$

then equation (E) has a pair of solutions such that for $x \to \infty$

$$y_1(x) \sim \exp\left(-\int_a^x tf(t)dt\right) \qquad \textit{and} \qquad xy_1'(x)/y_1(x) \to 0 \tag{4.2}$$

$$y_2(x) \sim t\exp\left(\int_a^x tf(t)dt\right) \qquad \textit{and} \qquad xy_2'(x)/y_2(x) \to 1. \tag{4.3}$$

For positive f Theorem 2 gives, for $n = 1$, the behaviour (4.2) (hence also (4.3)) by integrating partially in (2.9) a) and then using (1.2). Instead of (4.1) we have the condition $\int^\infty tF^2dt < \infty$ (obtained from (2.8) again by a partial integration). These two conditions are not comparable in general. However, for the rather general example $f(x) = \varepsilon(x)/x^2$ where $\varepsilon(x)$ is almost decreasing (meaning that $x_2 > x_1$ implies $\varepsilon(x_2) \le k\varepsilon(x_1)$ for some $k > 1$), condition (4.1) is reduced to $\int^\infty \varepsilon^p t^{-1}dt < \infty$ and (2.8) to $\int^\infty \varepsilon^2 t^{-1}dt < \infty$. They coincide for $p = 2$ whereas for the remaining values of p it might happen that the later is fulfilled but the former is not.

For another result in this direction see [**3**].

4.2. As an example take $f(x) = x^{-2}\ln^{-\alpha}x$ with $\alpha > 0$. Notice that -2 is the only exponent relevant here. For any smaller one s.v. solutions tend to constants (e.g. by (4.2) in Theorem D), whereas for any larger one condition (1.2) is not satisfied so $y(x)$ cannot be s.v. Here as $x \to \infty$

$$F(x) = \int_x^\infty t^{-2}\ln^{-\alpha} t dt \sim x^{-1}\ln^{-\alpha} x \quad \text{and} \quad \int_x^\infty F^2(t)dt \sim \ln^{-\alpha} xF(x)$$

so that one can take in (3.15) $c(x) = k\ln^{-\alpha} x$ for some $k > 0$. Therefore, condition (3.16) becomes

$$\int^{\infty} t^{-1}(\ln t)^{-(n+1)\alpha} dt < \infty \tag{4.4}$$

and the Corollary may be applied.

To illustrate how it works take e.g. $1/4 < \alpha \leq 1/3$, then in (4.4) one has to take $n = 3$ and the asymptotic formula (2.9) a) gives for $x \to \infty$

$$y(x) \sim A \exp\left\{-\int_a^x t^{-1}\ln^{-\alpha} t dt + \int_a^x t^{-1}\ln^{-2\alpha} t dt - 2\int_a^x t^{-1}\ln^{-3\alpha} t dt\right\}. \tag{4.5}$$

Notice that for $1/3 < \alpha \leq 1/2$ one takes $n = 2$ and in (4.5) only the first two integrals remain. Finally, if $1/2 < \alpha \leq 1$ one takes $n = 1$ and only the first integral remains. In this case Theorem D is also applicable.

4.3 Each of the exponentials in (4.5) is a normalized s.v. function L_i, $i = 1, 2, 3$ and such that $L_{i+1} = o(L_i)$. Thus by (4.5) for $x \to \infty$, $y(x) \sim AL_1(x)L_2(x)L_3(x)$. If, however, the number n in Theorem 2 does not exist then we can represent s.v. solutions as infinite products of the form

$$y(x) = \prod_{i=1}^{\infty} L_i(x)(1 + o(1))$$

but we have no approximation formula. These considerations apply also to the general case (2.2) (and thus to (2.9)).

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