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AN ESTIMATE FOR COEFFICIENTS OF POLYNOMIALS IN L^2 NORM. II

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Dedicated to the memory of Professor S. Aljančić

Abstract. Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{k=0}^n a_k x^k$ of degree at most n and $\|P\|_{d\sigma} = (\int_{\mathbb{R}} |P(x)|^2 d\sigma(x))^{1/2}$, where $d\sigma(x)$ is a nonnegative measure on $\mathbb{R}$. We determine the best constant in the inequality $|a_k| \le C_{n,k}(d\sigma)\|P\|_{d\sigma}$, for $k = 0, 1, \dots, n$, when $P \in \mathcal{P}_n$ and such that $P(\xi_k) = 0$, $k = 1, \dots, m$. The cases $C_{n,n}(d\sigma)$ and $C_{n,n-1}(d\sigma)$ were studed by Milovanović and Guessab [**6**]. In particular, we consider the case when the measure $d\sigma(x)$ corresponds to generalized Laguerre orthogonal polynomials on the real line.

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum\limits_{k=0}^{n} a_k x^k$ of degree at most n. The first inequality of the form $|a_k| \le C_{n,k}\|P\|$ was given by Markov [**3**]. Namely, if $\|P\| = \|P\|_\infty = \max_{x\in[-1,1]} |P(x)|$ and $T_n(x) = \sum\limits_{\nu=0}^{n} t_{n,\nu} x^\nu$ denotes the n-th Chebyshev polynomial of the first kind, then Markov proved that

$$|a_k| \le \begin{cases} |t_{n,k}| \cdot \|P\|_\infty & \text{if} \quad n-k \quad \text{is even}, \\ |t_{n-1,k}| \cdot \|P\|_\infty & \text{if} \quad n-k \quad \text{is odd}. \end{cases} \tag{1.1}$$

For $k = n$ (1.1) reduces to the well-known Chebyshev inequality

$$|a_n| \le 2^{n-1}\|P\|_\infty. \tag{1.2}$$

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Using a restriction on the polynomial class like $P(1) = 0$ or $P(-1) = 0$, Schur [**8**] found the following improvement of (1.2)

$$|a_n| \le 2^{n-1}\left(\cos\frac{\pi}{4n}\right)^{2n}\|P\|_\infty.$$

This result was extended by Rahman and Schmeisser [**7**] for polynomials with real coefficients, which have at most $n-1$ distinct zeros in $(-1, 1)$.

Similarly in L^2 norm,

$$\|P\| = \|P\|_2 = \left(\int_{-1}^{1}|P(x)|^2\,dx\right)^{1/2},$$

Tariq [**10**] improved the following result of Labelle [**2**]

$$|a_k| \le \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!}\left(k+\frac{1}{2}\right)^{1/2}\binom{[(n-k)/2]+k+1/2}{[(n-k)/2]}\|P\|_2. \tag{1.3}$$

for $P \in \mathcal{P}_n$ and $0 \le k \le n$, where the symbol $[x]$ denotes as usual the integral part of x. Equality in this case is attained only for the constant multiplies of the polynomial

$$\sum_{\nu=0}^{[(n-k)/2]}(-1)^\nu(4\nu+2k+1)\binom{k+\nu-1/2}{\nu}P_{k+2\nu}(x),$$

where $P_m(x)$ denotes the Legendre polynomial of degree m.

Under restriction $P(1) = 0$, Tariq [**10**] proved that

$$|a_n| \le \frac{n}{n+1}\cdot\frac{(2n)!}{2^n(n!)^2}\left(\frac{2n+1}{2}\right)^{1/2}\|P\|_2, \tag{1.4}$$

with equality case $P(x) = P_n(x) - \dfrac{1}{n^2}\displaystyle\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(x)$. Also, he obtained that

$$|a_{n-1}| \le \frac{(n^2+2)^{1/2}}{n+1}\cdot\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}\left(\frac{2n-1}{2}\right)^{1/2}\|P\|_2, \tag{1.5}$$

with equality case

$$P(x) = \frac{2n+1}{n^2+2}P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2}\sum_{\nu=0}^{n-2}(2\nu+1)P_\nu(x).$$

In the absence of the hypothesis $P(1) = 0$ the factor $(n^2+2)^{1/2}/(n+1)$ appearing on the right-hand side of (1.5) is to be dropped.

This result was extended by Milovanović and Guessab [**4**] for polynomials with real coefficients, which have m zeros on real line.

In this paper we consider more general problem including L^2 norm of polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$. The generalized Laguerre measure is is also included.

2. Main results

Let $d\sigma(x)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k \, d\sigma(x)$, $k = 0, 1, \dots$, exist and are finite, and $\mu_0 > 0$. In that case, there exist a unique set of orthonormal polynomials $\pi_n(\cdot) = \pi_n(\cdot; d\sigma)$, $n = 0, 1, \dots$, defined by

$$\begin{aligned} &\pi_n(x) = b_n^{(n)}(d\sigma)x^n + b_{n-1}^{(n)}(d\sigma)x^{n-1} + \cdots + b_0^{(n)}(d\sigma), \qquad b_n^{(n)}(d\sigma) > 0, \\ &(\pi_n, \pi_m) = \delta_{nm}, \qquad n, m \geq 0, \end{aligned}$$

where

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} \, d\sigma(x) \qquad (f, g \in L^2(\mathbb{R})). \tag{2.1}$$

For $P \in \mathcal{P}_n$, we define

$$\|P\|_{d\sigma} = \sqrt{(P, P)} = \left(\int_{\mathbb{R}} |P(x)|^2 \, d\sigma(x) \right)^{1/2}. \tag{2.2}$$

Also, for $\xi_k \in \mathbb{C}$, $k = 1, \dots, m$, we define a restricted polynomial class

$$\mathcal{P}_n(\xi_1, \dots, \xi_m) = \{P \in \mathcal{P}_n \mid P(\xi_k) = 0, \ k = 1, \dots, m\} \qquad (0 \leq m \leq n).$$

In the case $m = 0$ this class of polynomials reduces to $\mathcal{P}_n$. The case $m = n$ is trivial. If $\xi_1 = \cdots = \xi_k = \xi$ $(1 \leq k \leq m)$ then the restriction on polynomials at the point $x = \xi$ becomes $P(\xi) = P'(\xi) = \cdots = P^{(k-1)}(\xi) = 0$.

Let

$$\prod_{i=1}^{m} (x - \xi_i) = x^m - s_1 x^{m-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m$$

where s_k denotes elementary symmetric functions of $\xi_1, \dots, \xi_m$, i.e.,

$$s_k = \sum \xi_1 \cdots \xi_k \qquad \text{for} \quad k = 1, \dots, m. \tag{2.3}$$

For $k = 0$ we have $s_0 = 1$, and $s_k = 0$ for $k > m$ or $k < 0$.

THEOREM 2.1. *Let $P \in \mathcal{P}_n(\xi_1, \dots, \xi_m)$ and $s_1, \dots, s_m$ be given by (2.3). If the measure $d\hat{\sigma}(x)$ is given by*

$$d\hat{\sigma}(x) = \prod_{k=1}^{m} |x - \xi_k|^2 d\sigma(x) \tag{2.4}$$

and $\|P\|_{d\sigma}$ is defined by (2.2), then

$$|a_{n-k}| \le \left(\sum_{j=0}^{k} \left(\sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}_{n-m-i}^{(n-m-j)} \right)^2 \right)^{1/2} \|P\|_{d\sigma}, \tag{2.5}$$

for $k = 0, 1, \dots, n$, where $\hat{b}_\nu^\mu = b_\nu^\mu(d\hat{\sigma})$, $\nu = 0, 1, \dots, \mu$, are the coefficients in the orthonormal polynomial $\hat{\pi}_\mu(\cdot) = \pi_\mu(\cdot; d\hat{\sigma})$.

Inequality (2.5) is sharp and becomes an equality if and only if $P(x)$ is a constant multiple of the polynomial

$$\left(\sum_{j=0}^{k} \hat{\pi}_{n-m-j}(x) \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}_{n-m-i}^{(n-m-j)} \right) \prod_{k=1}^{m} (x - \xi_k).$$

Proof. At first we consider the inner product (2.1). Then the polynomial $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu \in \mathcal{P}_n$ can be represented in the form $P(x) = \sum_{\nu=0}^{n} \alpha_\nu \pi_\nu(x; d\sigma)$, where $\alpha_\nu = (P, \pi_\nu)$, $\nu = 0, 1, \dots, n$. Then we have

$$a_{n-k} = \sum_{i=0}^{k} \alpha_{n-i} b_{n-k}^{(n-i)}(d\sigma) = \left(P, \sum_{i=0}^{k} b_{n-k}^{(n-i)}(d\sigma) \pi_{n-i} \right), \quad k = 0, 1, \dots, n, \tag{2.6}$$

where $\pi_\nu(\cdot) = \pi_\nu(\cdot; d\sigma)$.

Suppose now that $P \in \mathcal{P}_n(\xi_1, \dots, \xi_m)$. Then we can write

$$P(x) = Q(x) \prod_{k=1}^{m} (x - \xi_k), \tag{2.7}$$

where $Q(x) = a'_{n-m} x^{n-m} + a'_{n-m-1} x^{n-m-1} + \dots + a'_0 \in \mathcal{P}_{n-m}$. Also, we have

$$\prod_{i=1}^{m} (x - \xi_i) = x^m - s_1 x^{m-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m$$

where s_k, $k = 0, 1, \dots, m$, denotes elementary simmetric functions (2.3). Now, putting this in (2.7), we obtain

$$P(x) = \sum_{i=0}^{n-m} \sum_{\nu=0}^{m} a'_i (-1)^\nu s_\nu x^{m+i-\nu} = \sum_{k=0}^{n} a_{n-k} x^{n-k},$$

where

$$a_{n-k} = \sum_{i=0}^{k} a'_{n-m-i}(-1)^{k-i} s_{k-i}, \qquad k = 0, 1, \ldots, n, \tag{2.8}$$

and $a'_k = 0$ for $k < 0$ and $k > n - m$.

Now, the corresponding equalities (2.6) for polynomial Q in the measure $d\hat{\sigma}(x)$, given by (2.4), become

$$a'_{n-m-i} = \Big(Q, \sum_{j=0}^{i} \hat{b}_{n-m-i}^{(n-m-j)} \hat{\pi}_{n-m-j}\Big), \quad i = 0, 1, \ldots, n-m, \tag{2.9}$$

where $\hat{\pi}_\nu(\cdot) = \pi_\nu(\cdot; d\hat{\sigma})$.

According to (2.7), we have

$$a_{n-k} = \sum_{i=0}^{k} (-1)^{k-i} s_{k-i} \left(Q, \sum_{j=0}^{i} \hat{b}_{n-m-i}^{(n-m-j)} \hat{\pi}_{n-m-j}\right) = (Q, W_{n-m}) \tag{2.10}$$

where

$$\begin{aligned} W_{n-m}(x) &= \sum_{i=0}^{k} (-1)^{k-i} s_{k-i} \sum_{j=0}^{i} \hat{b}_{n-m-i}^{(n-m-j)} \hat{\pi}_{n-m-j}(x) \\ &= \sum_{j=0}^{k} \hat{\pi}_{n-m-j}(x) \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}_{n-m-i}^{(n-m-j)} \end{aligned}$$

and $\hat{b}_\nu^{(\mu)} = 0$ for $\nu < 0$. Now, using Cauchy inequality we get

$$|a_{n-k}| \leq C_{n,n-k} \|Q\|_{d\hat{\sigma}}$$

where $C_{n,n-k} = \|W_{n-m}\|_{d\hat{\sigma}} = \left(\sum_{j=0}^{k} \Big(\sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}_{n-m-i}^{(n-m-j)} \Big)^2 \right)^{1/2}$. Since

$$\|Q\|_{d\hat{\sigma}}^2 = \int_{\mathbb{R}} |Q(x)|^2 d\hat{\sigma}(x) = \int_{\mathbb{R}} |P(x)|^2 d\sigma(x) = \|P\|_{d\sigma}^2$$

we obtain inequality (2.5).

The extremal polynomial is $x \mapsto W_{n-m}(x) \prod_{k=1}^{m} (x - \xi_k)$. $\square$

Remark 2.1. For $k = 0$ and $k = 1$ Theorem 2.1 gives the results obtained by Milovanović and Guessab [**4**] (see also [**6**, pp. 432–439]).

Consider now the generalized Laguerre measure $d\sigma(x) = x^\alpha e^{-x} dx$, $\alpha > -1$, on $(0, +\infty)$. With $\tilde{L}_n^{(\alpha)}(x)$ we denote the generalized orthonormal Laguerre polynomial. The coefficient $b_k^{(n)}$ of x^k in $\tilde{L}_n^{(\alpha)}(x)$ is given by

$$b_k^{(n)} = (-1)^{n-k} \binom{n}{k} \frac{(\alpha + k + 1)_{n-k}}{\sqrt{n! \Gamma(n + \alpha + 1)}}.$$

As a direct corollary of Theorem 2.1, we have:

COROLLARY 2.2. *Under restriction* $P^{(i)}(0)=0$, $i=0,1,\dots,m-1$, *we have that*

$$|a_{n-k}| \le \sqrt{A_{n,k}}\,\|P\|_{d\sigma},$$

where

$$A_{n,k}=\frac{1}{(n-m-k)!\Gamma(n+m-k+\alpha+1)}\sum_{j=0}^{k}\binom{n+m-j+\alpha}{k-j}\binom{n-m-j}{k-j}$$

for $n-k\ge m$, *and* $A_{n,k}=0$ *for* $n-k<m$. *The equality is attained if and only if* $P(x)$ *is a constant multiple of the polynomial*

$$x^m\sum_{j=0}^{k}\hat{b}_{n-m-k}^{(n-m-j)}\tilde{L}_{n-m-j}^{(\alpha+2m)}(x).$$

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