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ESTIMATE FOR GRADIENT, BMO AND LINDELÖF THEOREM

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Abstract. We give sharp inequalities between BMO and Bloch norms for functions defined on a proper subdomain D of R. In connection with this we establish a version of Lindelöf theorem for bounded harmonic function (more general for Bloch function). Some application of those results and sufficient and necessary condition for a function to belong BMO and Bloch space are given.

Throughout this paper R^n denotes Euclidean n-space and D a proper subdomain of R^n. Next for $a \in R^n$ and $0 < r < \infty$ $B_r(a)$ denotes n-ball with center a and radius r and $S_r(a)$ its boundary. If there is no possibility of confusion we will write B (respectively S) instead of $B_r(a)$ (respectively $S_r(a)$).

For a real function u defined on D and $a \in D$ we define u_a by $u(x) = u(x) - u(a)$, $x \in D$ and by ∇u we denote the gradient of u. By m we denote the standard Lebesgue measure on R^n normalized so that the measure of unit ball is 1.

LEMMA 1. *If u is a harmonic function in $\overline{B_r(a)}$, then*

$$r^n |\nabla u(a)| \leq n \int\limits_{S(a)} |u| dS$$

where dS is the volume $(n-1)$-form of the oriented Riemannian manifold $S_r(a)$, normalized such that $(n-1)$-volume of unit sphere is 1.

Proof. Since ∇u is a harmonic function, by the mean-value property we have

$$r^n \nabla u(a) = \int\limits_{B(a)} \nabla u_a dm$$

Hence, by divergence theorem

$$r^n \nabla u(a) = n \int\limits_{S(a)} u_a \vec{n} dS$$

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where $\vec{n} = n(x)$ is the unit vector normal to the boundary of $B(a)$ and directed outwards with respect to $B(a)$, which proves Lemma 1.

LEMMA 2. *Let u be a harmonic function on $\overline{B_r(a)}$. Then*

$$r^{n+1}|\nabla u(a)| \leq (n+1) \int_{B(a)} |u_a| dm \tag{1}$$

Proof. By polar coordinates and Lemma 1 we get

$$\int_{B(a)} |u_a| dm = n \int_0^r d\rho \int_{S(a)} |u_a| dS \geq (n+1)^{-1} |\nabla u(a)| r^{n+1},$$

which proves the lemma.

Let $p \geq 1$. The BMO_p norm of any locally integrable function u, defined on an arbitrary domain $D \subset R^n$, is given by

$$|u|_p^* = \sup \left\{ \frac{1}{m(B)_B} \int_B |u - u_B|^p dm \right\}^{1/p},$$

where the supremum is taken over all closed balls $\overline{B} \subset D$ and where u_B denotes the mean-value of u taken over B. We say that u belongs to $BMO_p = BMO_p(D)$ if $|u|_p^* < \infty$. Also we shall use the notation $|u|_* = |u|_1^*$ and $BMO = BMO_1$.

Let u be a differentiable function on a proper subdomain D of R^n. We say that u belongs to the Bloch space $\beta = \beta(D)$ on D if

$$|u|_\beta = \sup_{a \in D} (\operatorname{dist}(a, \partial D) |\nabla u(a)|) < \infty.$$

From this definitions and Lemma 2, it is easy to get the following result.

THEOREM 1. *Let u be a harmonic function in a proper subdomain D of R^n. If $u \in BMO(D)$, then $u \in \beta(D)$ and*

$$|u|_\beta \leq (n+1)|u|_*. \tag{2}$$

Before we state Theorem 2, which gives a result in other direction, we will make a few comments concerning previously obtained results in this area. Using Jensen's inequality and Lemma 1 we get

LEMMA 3. (Muramoto [**1**]) *if u is a harmonic function on the closed unit ball $\overline{B}$, then*

$$|\nabla u(0)|^2 \leq n^2 \int_{\partial B} |u|^2 dS.$$

Using this result Muramoto in [**1**] proved that if $u \in BMO(B)$, then $u \in \beta(B)$ and $|u|_\beta \leq a_n |u|_2^*$, where $a_n = \sqrt{n(n+2)}$.

Pavlović in [**3**] improved this inequality and showed that a_n can be replaced by $\sqrt{n+2}$. Since $|u|_* \le |u|_2^*$ the inequality (2) in Theorem 1 gives better estimates in some cases. Note that our proof of Lemma 2 is simple and use only the mean-value theorem and the divergence theorem.

THEOREM 2. *let D be a proper subdomain of R^n $(n \ge 2)$ and let u be a C^1-function on D. If $u \in \beta(D)$, then $u \in BMO(D)$ and*

$$|u|_* \le b_n |u|_\beta, \text{ where } b_n = 1 + 1/2 + \cdots + 1/n. \tag{3}$$

Remark. Pavlović in [**3**] proved that if u is a harmonic function, then (3) holds with the constant $\sqrt{2}$ of b_n. Although the constant b_n is the best possible in our setting, it seems that the main contributions of Theorem 2 lies in the emphasizing that the inclusion $BMO \supset \beta$ holds at least for C^1-function.

Proof. Let $a \in D$, $0 \le r < \operatorname{dist}(a, \partial D)$ and $x \in B_r(a)$. Then $\operatorname{dist}(x, \partial D) \ge r - |x - a|$. Without loss of generality we can suppose that $a = 0$. Next let $v(t) = u(tx)$, $\varphi(t) = tx$, $0 \le t \le 1$, and $\vec{x} = \varphi'(t) \in T_{tx}R^n$. By the chain rule $v'(t) = u'(tx)(\vec{x})$.

Hence

$$|v'(t)| \le |u|_\beta |x| \quad (r - |tx|)^{-1}. \tag{4}$$

Let $K = |u|_\beta$. It follows from (4) that

$$|u(x) - u(0)| \le \int_0^1 |v'(t) dt \le -K \ln(1 - |x|/r). \tag{5}$$

Hence, using polar coordinates,

$$\int_B |u_0| dm \le rK \left(\sum_{k=1}^{\infty} (1/k - 1/(k+n)) \right)$$

which proves the theorem.

Lindelöf theorem (see, for example, Theorem 8.4.1 of [**4**]) states: If bounded holomorphic function f on the unit disc U has a limit along a continuous curve c which terminates at a point $a \in \partial U$, then f has non-tangential boundary value at a.

Let $h : U \to (-\pi/2, \pi/2)$ be a harmonic function defined by $h(z) = \arg((1+z)(1-z)^{-1})$. This example shows that Lindelöf theorem (for bounded harmonic function) does not hold at 1.

Using the estimate obtained in the proof of Theorem 2, we can give a version of Lindelöf–Chirka theorem (see Theorem 8.4.4 of [**4**]) for bounded harmonic functions.

Let D be a proper subdomain of R^n. We shall use the notation $d(x) = \operatorname{dist}(x, \partial D)$ for $x \in D$.

THEOREM 3. *Let u be bonded harmonic, or more generally $u \in \beta(D)$, and let $\gamma,\ \Gamma : [0,1) \to D$ be two continuous curves in D terminating at $\zeta \in \partial D$. If $u(\gamma(t)) \to L$ when $t \to 1$ and*

$$d(\gamma(t))^{-1}\, dist(\Gamma(t), \gamma) \to 0$$

when $t \to 1$, then u has also limit L along the curve Γ.

Proof. For given $t \in [0,1)$ choose a $\tilde{t} \in [0,1)$ suchh that $\operatorname{dist}(\Gamma(t), \gamma) = |\Gamma(t) - \gamma(\tilde{t})|$ and let $d(\gamma(t)) = R(t)$. By Lemma 1, $u \in \beta$ and as in the proof of Theorem 2, we conclude that

$$|u(\gamma(\tilde{t}) - u(\Gamma(t))| \leq 2|u|_\infty |\gamma(t) - \Gamma(t)| [R(t)]^{-1} n \tag{6}$$

if t is enough close to 1. Let t_n be arbitrary sequence such that $t_n \to 1$. We can verify that $\tilde{t}_n \to 1$. Hence, by (6), $u(\Gamma(t_n)) \to L$ as $n \to \infty$.

Using Lemma 1 and the chain rule, one can prove the following result.

PROPOSITION 1. *Let D be a proper subdomain of R^n and $g(x) = \ln u(x)$, where u is a positive harmonic function on D. Then $g \in BMO(D)$.*

Recall if u is a positive harmonic function on $R^n (n \geq 2)$, then u is a constant. In the rest of the paper we shall use the notation from [**2**].

PROPOSITION 2. *Let u be C^1-function on D satisfying the condition (h_K^+). If $u \in BMO(D)$, then $u \in \beta(D)$.*

Proof. Let $a \in D,\ B = B_r(a) \subset D$ and $2\tau = r$. Then the function v satisfies (h_{2k}), where $v(x) = u(x) - u_B,\ x \in D$. Hence

$$r|\nabla u(a)| \leq 4K|v(x_0)|,$$

where $x_0 \in B_\tau(a)$. Now, using an interesting result of Pavlović (see Theorem 2 of [**2**] we conclude that ν satisfies (sh_C) on $B_\tau(x_0)$ for a constant C depending only on K and n. Since $B_\tau(x_0) \subset B_r(a)$, we get

$$r^{n+1}|\nabla u(a)| \leq 4KC\, 2^n \int\limits_{B(a)} |u - u_B|\, dm \quad \text{i.e.} \quad u \in \beta.$$

COROLLARY. *If a C^1-function u on D satisfies the condition (h_K^+),then $u \in BMO(D)$ if and only if $u \in \beta(D)$.*

Since every harmonic function u on D satisfies the condition (h_K^+), this result can be viewed as a generalization of the corresponding results of Pavlović [**3**] and Muramoto [**1**].

We will make a few comments concerning further investigation.

It seems natural to consider the above estimates and results for hyperbolic - harmonic (and more general) functions as well as Lindelöf type theorem for quasi-regular functions.

A version of the inequality (5), which was crucial in the proof of Theorem 2, can be used to give the following generalization of Chirka's theorem (see Theorem 8.4.4 of [**4**]):

If u is a bounded function on the unit ball $B \subset R^n$, having radial limit L at 1 and if u is harmonic with respect to $x_2, \dots, x_n$ for every fixed x_1 close enough to 1, then u has radial limit L along any special 1-curve in B. For definition of special curve see [**4**].

The inequality (5) can be improved by using quasihyperbolic metric.

For applications of estimates of gradient on boundary behavior of holomorphic function of several complex variables we refer to Stein's book [5]

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REFERENCES

[1] K. Muramoto, *Harmonic Bloch and BMO function on the unit ball in several variables*, Tokyo J. Math. **11** (1988), 381–386.

[2] M. Pavlović, *On subharmonic behavior and oscillation of functions on balls in R^n*, Publ. Inst. Math. (Beograd) **55** (**69**) (1994), 18–22.

[3] M. Pavlović, *A remark on BMO and Bloch functions on the unit ball of R^n*, Tokyo J. Math. **16** (1993), 291–294.

[4] W. Rudin, *Function Theory in the Unit Ball of C^n*, Springer –Verlag, New York, 1980.

[5] E. Stein, Boundary behavior of holomorphic function of several complex variables, Princeton, New Jersey, 1972.

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