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# PRIMITIVITY OF GENERALIZED DIRECT PRODUCT OF DIGRAPHS

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**Abstract**. Using spectral techniques we prove a theorem giving a necessary and sufficient condition for a generalized direct product (GDP) of strongly connected digraphs (with some additional restrictions) to be a primitive digraph.

The necessary background and terminology can be found in [**2**]. We will limit ourselves to defining only lesser known terms and those which may cause confusion.

By a digraph we mean an ordered pair $G = (V, E)$, where $V(G) = V$ is a finite nonempty set and $E(G) = E$ is a family of ordered pairs of $V$ (multiplicity of which can exceed 1). A (undirected) graph is a symmetric digraph. A digraph $G$ is called *complete* if each ordered pair of vertices $u, v$ of $G$ (if loops are not allowed, then $u \neq v$) belongs to $E(G)$ with the same multiplicity. A digraph is *regular* of degree $r$ if each indegree and each outdegree is equal to $r$. The *cycle* (directed) is a (strongly) connected regular digraph of degree 1. A strongly connected digraph $G$ is called *bipartite* if it has no odd cycles, or equivalently if the vertex set $V$ of $G$ can be partitioned into two subsets $V_1$ and $V_2$ such that every arc of $G$ joins a vertex of $V_i$ to a vertex of $V_j$, $i \neq j$. A bipartite digraph $G$ (with partite sets $V_1$ and $V_2$) having additional property that each ordered pair $(u, v)$, $(u \in V_i,\ v \in V_j,\ i \neq j)$, belongs to $E(G)$, with the same multiplicity, is called *bicomplete*. By $\vec{C}_p$ ($C_p$) we denote the directed (undirected) cycle with $p$ vertices, all arcs of which have the same multiplicity.

The *spectrum* of a digraph $G$ is the spectrum of its adjacency matrix $A(G) = [a_{ij}]_1^p$, where $|V(G)| = p$ and $a_{ij} \geq 0$ is the number of arcs leading from the vertex corresponding to $i$-th row to the vertex corresponding to $j$-th column of $A$. The *index* $r$ of a strongly connected digraph $G$ is its the greatest real eigenvalue. By a theorem of Frobenius it follows that $|\lambda_i| \leq r$ holds, for all eigenvalues $\lambda_i$ of $G$.

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Let $G$ be a digraph with at most $\nu$ parallel arcs between any two vertices or loops of a vertex in $G$ (if there are no parallel arcs, then $\nu = 1$), then complement $\bar{G}$ of $G$ is the digraph which has the same set of vertices as $G$ and for any ordered pair $(u, v)$ of vertices $u$ and $v$ of $\bar{G}$ (if loops are not allowed, then $u \neq v$) from $u$ to $v$ lead $\nu - a$ arcs, where $a$ is the number of arcs leading from $u$ to $v$ in $G$.

*Definition* 1. Let $B \subseteq \{1, 0, -1\}^n \setminus \{(0, 0, ..., 0)\}$. The generalized direct product with basis $B$ of digraphs $G_1, G_2, ..., G_n$ is the digraph $G = GDP(B; G_1, G_2, ..., G_n)$ whose vertex set is the Cartesian product of the vertex sets of digraphs $G_1, G_2, ..., G_n$. For two vertices say $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ of $G$ construct all the possible arc selections of the following type. For each $n$-tuple $(\beta_1, \beta_2, ..., \beta_n) \in B$, such that $u_k = v_k$ holds whenever $\beta_k = 0$, select an arc going from $u_i$ to $v_i$ in $G_i$ whenever $\beta_i = 1$ and an arc going from $u_i$ to $v_i$ in $\bar{G}_i$ whenever $\beta_i = -1$. The number of arcs going from $u$ to $v$ in $G$ is equal to the number of such selections.

If $B$ consists of $n$-tuples of symbols 1 and 0 only, then the resulting operation is called the noncomplete extended $p$-sum (NEPS). The $p$-sum is obtained if $B$ consists of all the possible $n$-tuples with exactly $p$ 1's. If $p = n$, the $p$-sum is called the product. The 1-sum is also called the sum. The NEPS, basis of which contains all possible $n$-tuples (of course without $n$-tuple (0,0,...,0)) is called the strong product.

In order to investigate primitivity of the GDP we need the following results from [**7**] and [**8**].

THEOREM 1. *Let $G$ be a regular digraph with $p$ vertices, degree $r$, and maximum number of parallel arcs between any two vertices or loops of a vertex equal to $\nu$ and let $\lambda_1 = r, \lambda_2, ..., \lambda_p$ be the spectrum of $G$. The complement $\bar{G}$ of $G$ has the spectrum given by: $\bar{\lambda}_1 = \nu p - \nu - r, \bar{\lambda}_2 = -\nu - \lambda_2, ..., \bar{\lambda}_p = -\nu - \lambda_p$, if loops are not allowed, and $\bar{\lambda}_1 = \nu p - r, \bar{\lambda}_2 = -\lambda_2, ..., \bar{\lambda}_p = -\lambda_p$, if loops are allowed in $G$ ($\bar{G}$).*

*The eigenvectors belonging to $\lambda_i$ and $\bar{\lambda}_i$ are the same and the eigenvector belonging to the eigenvalue $\lambda$, distinct from $r$, is orthogonal to the eigenvector $(1, 1, ..., 1)$ belonging to $r$.*

Let $\otimes$ denote the Kronecker product of matrices. The following theorem [**8**] is a slight generalization of Theorem 5 from [**7**]. Its proof coincides with the proof of Theorem 2.23 in [**2**].

THEOREM 2. *If, for $i = 1, 2, ..., n$, $\lambda_{ij_i}$ $(\bar{\lambda}_{ij_i})$, $j_i = 1, 2, ..., p_i$, is the spectrum of a digraph $G_i$ (complement $\bar{G}_i$ of $G_i$, determined by Theorem 1 in the case of regularity of $G_i$) ($p_i$ being its number of vertices), then the spectrum of $GDP(B; G_1, G_2, ..., G_n)$, in which $G_i$ is a regular digraph whenever there exists $\beta \in B$ such that $\beta_i = -1$, consists of all possible values $\Lambda_{j_1, j_2, \ldots, j_n}$, where*

$$\Lambda_{j_1, j_2, \ldots, j_n} = \sum_{\beta \in B} \lambda_{1j_1}^{[\beta_1]} \cdot \lambda_{2j_2}^{[\beta_2]} \cdots \lambda_{nj_n}^{[\beta_n]},$$

$$\lambda_{ij_i}^{[1]} = \lambda_{ij_i}, \lambda_{ij_i}^{[0]} = 1, \lambda_{ij_i}^{[-1]} = \bar{\lambda}_{ij_i}, (j_i = 1, 2, ..., p_i; i = 1, 2, ..., n).$$

*The eigenvector $x_{j_1,j_2,...,j_n} = x_{1j_1} \otimes x_{2j_2} \otimes ... \otimes x_{nj_n}$ belongs to the eigenvalue $\Lambda_{j_1,j_2,...,j_n}$, where $x_{ij_i}$ is an eigenvector belonging to the eigenvalue $\lambda_{ij_i}$ of $G_i$.*

We shall consider the GDP, basis $B$ of which has property (D): for each $j \in \{1, 2, ..., n\}$ the set $\{\beta_j | \beta \in B\}$ is not a subset of $\{0, -1\}$. This condition implies that the GDP, effectively depends on each $G_i$. However, this condition does not represent an essential restriction in investigation of primitivity of a GDP. Namely, if $\{\beta_j | \beta \in B\} \subseteq \{0, -1\}$ for some $j$, then we can replace $G_j$ by its complement $\bar{G}_j$ provided in each $n$-tuple $\beta \in B$ the $j$-th coordinate $-1$ is replaced by 1 [**7**]. (The case when all $\beta_j$ are equal to 0 is not interesting and is excluded from consideration.)

Let $h(G) = h$ be the greatest common divisor of the lengths of all the cycles in a digraph $G$. The digraph $G$ is called *primitive* if it is strongly connected and $h = 1$ and *imprimitive* if it is strongly connected and $h > 1$. In the second case $h$ is called the *index of imprimitivity* ($h$ is the index of imprimitivity of the adjacency matrix of the digraph $G$ as well [**1**, p. 183]). A nonconnected digraph is primitive if all its components are primitive.

A *maximal eigenvalue* of a digraph $G$ is an eigenvalue of $G$ modulus of which is equal to the index of $G$. According to theorem of Frobenius and Theorems 0.4 and 0.5 in [**2**] if a digraph $G$ has $N$ maximal eigenvalues and $K$ strong components which are its components too, and each component have the same index of imprimitivity $h$, then $N = Kh$. Then, the digraph $G$ is primitive if and only if $N = K$.

A regular, connected digraph $G$ has property (M) if there exists a maximal eigenvalue of $G$, different from the index, such that by Theorem 1 corresponding eigenvalue of $\bar{G}$ is maximal too. The following lemma [**8**] describes which regular digraphs have the property (M).

LEMMA 1. *Only regular, bicomplete digraphs have the property* ($M$) *and, in the case when loops are not allowed, regular, bicomplete digraphs and the cycle of length* 3, *with the same multiplicity of all arcs* (*in both cases*).

*In the case of bicomplete digraphs the argument of the corresponding eigenvalue of $\bar{G}$ is equal to zero and in the case of the cycle of length* 3 *it is twice greater than the argument of the corresponding eigenvalue of $G$.*

If $h_1, h_2, ..., h_n$ ($n > 0$) are natural numbers, $\dfrac{x_1}{h_1} + \dfrac{x_2}{h_2} + \cdots + \frac{x_n}{h_n} = y$ equation on $n+1$ variables in integers $x_1, x_2, ..., x_n, y$ and $x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}, y^{(0)}$ satisfy this equation, then the classes $x_1 = x_1^{(0)} (\operatorname{mod} h_1), x_2 = x_2^{(0)} (\operatorname{mod} h_2), ..., x_n = x_n^{(0)} (\operatorname{mod} h_n)$ are called a solution of this equation.

A subset $\{i_1, i_2, ..., i_s\}$ of $\{1, 2, ..., n\}$ is consistent with digraphs $G_1$, $G_2$, ..., $G_n$ with respect to a basis $B$, if for each $k \in \{i_1, i_2, ..., i_s\} \cap \{\nu | \exists \beta \in B \wedge \beta_\nu = -1\}$, the digraph $G_k$ is bicomplete (which is indicated by $e(G_k) = 0$) or, in the case when loops are forbidden, bicomplete (again $e(G_k) = 0$) or isomorphic to $\vec{C}_3$ (indicated by $e(G_k) = 1$).

THEOREM 3. *Let $GDP(B; G_1, ..., G_n)$ satisfy the following conditions: $(i)$ Basis $B$ ($|B| \geq 2$) has property $(D)$; $(ii)$ For $i = 1, 2, ..., n$, $G_i$ is a strongly connected digraph with at least two vertices; $(iii)$ For $i \in K = \{k | \exists\beta \in B \wedge \beta_k = -1\} \subset \{1, 2, ..., n\}$, $G_i$ is a regular noncomplete digraph; $(iv)$ For $j \in L \subset \{1, 2, ..., n\}$, $G_j$ is imprimitive with the index of imprimitivity $h_j$, otherwise it is primitive. Then $GDP(B; G_1, ..., G_n)$ is a primitive digraph if and only if the following systems of equations* (1) *and* (2)

$$\sum_{i \in L} \left( \frac{1}{2}(\beta_i^2 + \beta_i)\frac{x_i}{h_i} + \frac{x_i}{3} e(G_i)(\beta_i^2 - \beta_i) \right) = y_\beta, \ \beta \in B; \tag{1}$$

$$\sum_{i \in L} \left( \frac{1}{2}(\beta_i - \alpha_i)(\beta_i + \alpha_i + 1)\frac{x_i}{h_i} + \frac{x_i}{3} e(G_i)(\beta_i - \alpha_i)(\beta_i + \alpha_i - 1) \right) = z_\beta,$$

$$\beta \in B \ (\beta \neq \alpha); \tag{2}$$

*have the same number of solutions $x_i, y_\beta, z_\beta$ such that, if for any $i \in K \cap L$, $G_i$ is not bicomplete or, in the case when loops are forbidden, neither bicomplete nor isomorphic to $\vec{C}_3$, then $x_i = 0 \pmod{h_i}$.*

*Proof.* Let $A(G) = A$ be the adjacency matrix [**7**] of $G = GDP(B; G_1, G_2, ..., G_n)$. If for $i = 1, 2, ..., n$, $G_i$ is strongly connected, a positive eigenvector belongs to the index $r_i$ both in $A(G_i)$ and $A^T(G_i)$. By Theorem 2, it also follows that the positive eigenvectors belong to the index both in $A$ and $A^T$. Then, according to Theorem 0.5 from [**2**] the number of strong components of the GDP (which are its components too [**8**]) is equal to the multiplicity of its index. Further, the GDP is a primitive digraph if each its component is primitive. All components of a GDP of strongly connected digraphs have the same index (Theorem 0.5 from [**2**] and Theorem 2). Therefore, a necessary and sufficient condition for primitivity of a GDP is that each its component contains the index in the spectrum (with multiplicity 1) as the unique maximal eigenvalue. Thus, a necessary and sufficient condition for primitivity of GDP is that the number of maximal eigenvalues of the GDP is equal to the multiplicity of its index.

Multiplicity of the index of GDP, under considered conditions, according to Theorem 5 in [**8**], is given by the number of solutions of the system of equations (1), satisfying given conditions. We should determine the number of maximal eigenvalues of the GDP.

By Theorem 3 the index $\Lambda$ of $G = GDP(B; G_1, ..., G_n)$ is given by

$$\Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \cdots r_n^{[\beta_n]}, \ r_i^{[1]} = r_i, \ r_i^{[0]} = 1, \ r_i^{[-1]} = \nu_i p_i - r_i - \nu_i l(G_i),$$

where $l(G_i) = 1$ if loops are forbidden and 0 otherwise, and $\nu_i$ is the maximum number of parallel arcs between any two vertices or loops of a vertex of $G_i$.

By the same theorem, if none of $G_j$, for $j \in K$, is complete, a maximal eigenvalue of $G$ is obtainable only from those eigenvalues of the digraphs $G_j$ $(\bar{G}_j)$, $j = 1, 2, ..., n$, which have a modulus equal to $r_j$ $(\nu_j p_j - r_j - \nu_j \cdot l(G_j))$. All these eigenvalues of $G_j$ can be written in the form $r_j \exp(\frac{\ell_j}{h_j} 2\pi i)$, $0 \leq \ell_j \leq h_j - 1$, $(i^2 = -1)$ (theorem of Frobenius). Therefore by Theorem 3 we have

$$(3) \quad \Lambda = \sum_{\beta \in B} \prod_{j=1}^{n} \Big( \frac{1}{2}(\beta_j^2 + \beta_j) r_j \exp\Big(\frac{\ell_j}{h_j} 2\pi i\Big) + (1 - \beta_j^2) + \frac{1}{2}(\beta_j^2 - \beta_j)\Big( \bar{sg}(\ell_j)\nu_j p_j - \nu_j \cdot l(G_j) - r_j \exp\Big(\frac{\ell_j}{h_j} 2\pi i\Big)\Big)\Big),$$

where $\bar{sg}(0) = 1$ and $\bar{sg}(x) = 0$ for $x > 0$.

Let $J = \{1, 2, ..., n\}$. For any choice of integers $\ell_{j_1}, \ell_{j_2}, ..., \ell_{j_s}$, $1 \leq \ell_{j_t} \leq h_{j_t} - 1$, $\{j_1, j_2, ..., j_s\} \subset L$ and any $\beta \in B$ let $J_\beta = \{j_1, j_2, ..., j_s\} \cap \{k | \beta_k \neq 0\}$. Then from (3) we have:

$$\Lambda = \sum_{\beta \in B} \Big( \prod_{j \in J \setminus J_\beta} r_j^{[\beta_j]} \Big) \Big( \prod_{j \in J_\beta} \Big( \frac{1}{2}(1 + \beta_j) r_j \exp\Big(\frac{\ell_j}{h_j} 2\pi i\Big) + \frac{1}{2}(1 - \beta_j)\Big( - \nu_j \cdot l(G_j) - r_j \exp\Big(\frac{\ell_j}{h_j} 2\pi i\Big)\Big)\Big)\Big),$$

or

(4)

$$\Lambda = \sum_{\beta \in B} \Big( \prod_{j \in J \setminus J_\beta} r_j^{[\beta_j]} \Big) \Big( \prod_{j \in J_\beta} \Big( r_j^2 + \frac{1}{2}(1 - \beta_j)\Big(2 r_j \nu_j \cos\frac{\ell_j}{h_j} 2\pi + \nu_j^2\Big) l(G_j)\Big)^{1/2} \times \exp\Big(\frac{1}{2}(1 + \beta_j)\frac{\ell_j}{h_j} 2\pi i + \frac{1}{2} i (1 - \beta_j)\Theta_j\Big)\Big),$$

$$\Big(\Theta_j = \arg\Big( - \nu_j \cdot l(G_j) - r_j \exp\Big(\frac{\ell_j}{h_j} 2\pi i\Big)\Big)\Big).$$

According to Lemma 1, a maximal eigenvalue of $G$ is obtainable only from those choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_s}$, $\{j_1, j_2, \ldots, j_s\} \subset L$, $1 \leq \ell_{j_t} \leq h_{j_t} - 1$, $t = 1, 2, \ldots, s$, for which $G_{j_t}$ is bicomplete or, in the case when loops are forbidden, bicomplete or isomorphic to $\vec{C}_3$ whenever $j_t \in K \cap L$ holds. Then (4) can be written in the form:

$$(5) \quad \Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \cdots r_n^{[\beta_n]} \times \exp\Big( i \sum_{k \in \{j_1, \ldots, j_s\}} \Big( \frac{1}{2}(\beta_k^2 + \beta_k)\frac{\ell_k}{h_k} 2\pi + \frac{1}{2}(\beta_k^2 - \beta_k)\frac{\ell_k}{3} 4\pi e(G_k)\Big)\Big).$$

Now, a maximal eigenvalue of GDP is given whenever the arguments of the operator exp in summands (5) are differ by exactly $2z_\beta\pi$, where $z_\beta$ is an integer. Equalizing the differences of the arguments of arbitrary summand $\alpha \in B$ and remaining summands to $2z_\beta\pi$, $(z_\beta \in Z)$ we give the system of equations (2).

This completes the proof of the theorem.

The solution $x_i = 0(\mathrm{mod}\, h_i)$, $(i = 1, 2, ..., n)$, $y_\beta = 0$, $z_\beta = 0$ of the systems of equations (1) and (2) is called the trivial solution.

The special cases of this theorem are described in the next few theorems and examples.

THEOREM 4. *Let $G_1, G_2, ..., G_n$ be strongly connected digraphs each containing at least two vertices. Suppose also that $G_{i_1}, G_{i_2}, ..., G_{i_s}$ $(\{i_1, i_2, ..., i_s\} \subset \{1, 2, ..., n\})$ are imprimitive with the indices of imprimitivity $h_{i_1}, h_{i_2}, .., h_{i_s}$, respectively, while others are primitive. The NEPS with the basis $B$ $(|B| \geq 2)$ satisfying condition $(D)$, of digraphs $G_1, G_2, ..., G_n$ is a primitive digraph if and only if the following systems of equations $(\alpha \in B)$*

$$\frac{x_{i_1}}{h_{i_1}}\beta_{i_1} + \frac{x_{i_2}}{h_{i_2}}\beta_{i_2} + \cdots + \frac{x_{i_s}}{h_{i_s}}\beta_{i_s} = y_\beta, \beta \in B,$$

*and*

$$\frac{x_{i_1}}{h_{i_1}}(\beta_{i_1} - \alpha_{i_1}) + \frac{x_{i_2}}{h_{i_2}}(\beta_{i_2} - \alpha_{i_2}) + \cdots + \frac{x_{i_s}}{h_{i_s}}(\beta_{i_s} - \alpha_{i_s}) = z_\beta, \beta \in B, \beta \neq \alpha,$$

*have the same number of solutions $x_i$, $y_\beta$, $z_\beta$.*

If $|B| = 1$ in Theorem 3, then we in fact come get to the product of digraphs. As the number of components of the product ([**6**], [**4**]) is[1] $\dfrac{h_1 h_2 \cdots h_n}{\mathrm{lcm}(h_1, h_2, \cdots, h_n)}$, and the number of maximal eigenvalues is obviously $h_1 h_2 \cdots h_n$, we have a new proof of the statement [**6**] in the following example.

*Example* 1. The product of strongly connected digraphs $G_1, G_2, ..., G_n$, each containing at least two vertices is a primitive digraph if and only if all digraphs $G_1, G_2, ..., G_n$ are primitive.

THEOREM 5. *Let the digraphs $G_1, G_2, ..., G_n$ satisfy conditions of Theorem 4. The p-sum of these digraphs is a primitive digraph if and only if one of the following conditions holds:*

$1^o$ *$p$ is equal to $n$ and all digraphs $G_1, G_2, ..., G_n$ are primitive;*

$2^o$ *$p$ is less than $n$ and $n - s > p - 1$;*

$3^o$ *$p$ is less than $n$, $n - s \leq p - 1$ and the systems of equations*

$$\frac{x_{j_1}}{h_{j_1}} + \frac{x_{j_2}}{h_{j_2}} + \cdots + \frac{x_{j_p}}{h_{j_p}} = y_{j_1, j_2, ..., j_p},$$

[1] lcm denotes the lowest common multiple

*and*
$$\frac{x_{p+k}}{h_{p+k}} - \frac{x_{p-j}}{h_{p-j}} = z_{kj},\ k = 1, 2, ..., n-p;\ j = 0, 1, ..., p-1,$$
*where* $\{j_1, j_2, ..., j_p\}$ *runs over all* $p$*-subsets of* $\{1, 2, ..., n\}$*, have the same number of solutions* $x_i$, $y_{j_1, j_2, ..., j_p}$, $z_{kj}$.

*Example* 2. The sum of strongly connected digraphs $G_1, G_2, ..., G_n$, each containing at least two vertices, is a primitive digraph if and only if $h_1$, $h_2$, ..., $h_n$ are relative prime, i.e.[2] $\gcd(h_1, h_2, ..., h_n) = 1$.

*Proof.* According to Theorem 3 the sum of considered digraphs is a primitive digraph if and only if the system of equations
$$\frac{x_k}{h_k} - \frac{x_1}{h_1} = z_k, \quad k = 2, 3, ..., n,$$
has only the trivial solution $x_i$, $z_i$. This is the case if and only if $h_1, h_2, ..., h_n$ are relative prime.

The next example is obvious.

*Example* 3. The strong product of strongly connected digraphs $G_1, G_2, ...,$ $G_n$, each containing at least two vertices, is always a primitive digraph.

The strong product is a subgraph of the majority of associative products [**5**] and therefore those products are, under considered conditions, primitive.

*Example* 4. The generalized direct product with the basis $B$ containing all $n$-tuples with an odd number of 1's of regular, connected digraphs $G_1, G_2, ..., G_n$, each containing at least two vertices, is a primitive digraph except in the case when all factors are bicomplete, and in this case it is bipartite [**9**].

*Proof.* It can be readily seen, in the case of completeness of any of the factors, that this product is primitive. As the basis of this product consists of all possible $n$-tuples of symbols $1, 0, -1$ with odd number of 1's, it follows that the system of equations (2) in this example has a nontrivial solutions ($x_1 = x_2 = \cdots = x_n = 1 (\operatorname{mod} 2)$) if and only if all factors $G_1, G_2, ..., G_n$ are bicomplete. Since, this product, under considered conditions, is strongly connected [**8**], the statement of the example follows.

*Example* 5. The generalized direct product with basis $B = \{(1,1),\ (1,-1),$ $(-1,1)\}$ of regular, connected digraphs $G_1, G_2$, each containing at least two vertices, is a primitive digraph.

*Proof.* The system of equations (2) in this example is
$$\frac{x_2}{h_2}(2e(G_2) - 1) = y,\ \frac{x_1}{h_1}(2e(G_1) - 1) = z,$$
where $h_i$ is the index of imprimitivity of $G_i$. This system has, obviously, only the trivial solution $x_1 = 0 (\operatorname{mod} h_1)$, $x_2 = 0 (\operatorname{mod} h_2)$, and since this product is strongly

[2] gcd denotes the greatest common divisor

connected the statement of this assertion follows. In the case of completeness of any of factors this product is isomorphic to the ordinary product of complete digraphs and is a primitive digraph.

*Example* 6. The generalized direct product with the basis $B = \{(1,1), (-1,-1)\}$ of regular, connected, noncomplete digraphs $G_1$ and $G_2$, each containing at least two vertices, is a primitive digraph. This product has two component if $G_1$ and $G_2$ are bicomplete and three components, each isomorphic to $C_3$, if loops are forbidden in the factors and each factor is isomorphic to $\vec{C_3}$. In other cases the product has one component. If any of the factors is complete then this product has one component and index of imprimitivity equal to the index of imprimitivity of the other factor.

*Proof.* The system of equations (2) in this example is an equation

$$\frac{x_1}{h_1}(2e(G_1)-1) + \frac{x_2}{h_2}(2e(G_2)-1) = z.$$

This equation, in all cases of $G_1$ and $G_2$, has the number of solutions equal to the number of components [**8**], from which the first part of the statement follows. The second part follows from the fact that, in the case of completeness of any factor, this product is reduced to the ordinary product of digraphs.

*Example* 7. The generalized direct product with basis $B = \{(1,-1), (-1,1)\}$ of regular, connected, noncomplete digraphs $G_1$ and $G_2$, each containing at least two vertices, is a primitive digraph except in the case that both digraphs $G_1$ and $G_2$ are bicomplete, in which case it is a bipartite digraph [**9**]. This product is always connected except in the case when loops are forbidden in the factors and both of them are isomorphic to $\vec{C_3}$. In this case the product has three components, each isomorphic to $C_3$.

*Proof.* System of equations (2) in this example is

$$\frac{x_1}{h_1}(2e(G_1)-1) - \frac{x_2}{h_2}(2e(G_2)-1) = z.$$

It is not hard to see that this system has the number of solutions equal to the number of components except in the case when $G_1$ and $G_2$ are bicomplete. In this case this product has one component and above equation two solutions ((0,0),(1,1)), from which the statements follows.

If, in the previous example, one of the factors is complete, then this product yields to the ordinary product of that factor with the complement of the other one. Then the questions of connectedness and of the index of imprimitivity have been treated in [**6**].

The following theorem is a specialization of Theorem 3 to undirected graphs. Preliminary we introduce the following function and quote Lemma 2 from [**8**].

$$(6) \qquad f(x_1, x_2, ..., x_n) = \sum_{\beta \in B} x_1^{[\beta_1]} x_2^{[\beta_2]} \cdots x_n^{[\beta_n]}, \;\; x_i^{[1]} = x_i, \; x_i^{[0]} = 1, \; x_i^{[-1]} = x_i^2.$$

LEMMA 2. *Let $f$ be arbitrary polynomial of $n$ variables $x_1, x_2, ..., x_n$, and let $\{x_{i_1}, x_{i_2}, ..., x_{i_s}\}$ ($\{i_1, i_2, ..., i_s\} \subset \{1, 2, ..., n\}$) be an arbitrary nonempty subset of variables. If there exists a nonempty subset of $\{x_{i_1}, x_{i_2}, ..., x_{i_s}\}$ with respect to which the function $f$ is odd (for definition see [**2**]), then the number of nonempty subsets of $\{x_{i_1}, x_{i_2}, ..., x_{i_s}\}$ with respect to which the function $f$ is odd, is greater by one than the number of such subsets with respect to which this function is even.*

THEOREM 6. *Let $GDP(B; G_1, ..., G_n)$ satisfy the following conditions: $(i)$ Basis $B$ has property $(D)$, $(ii)$ For $i = 1, 2, ..., n$, $G_i$ is an undirected, connected graph containing at least two vertices, $(iii)$ For $i \in K = \{k | \exists \beta \in B \wedge \beta_k = -1\} \subset \{1, 2, ..., n\}$, $G_i$ is a regular noncomplete graph, $(iv)$ For $j \in L \subset \{1, 2, ..., n\}$, $G_j$ is a bipartite graph, otherwise it is primitive. Then the $GDP(B; G_1, ..., G_n)$ is a primitive graph if and only if the function* (6) *is never odd with respect to a nonempty subset of $L$, which is consistent with graphs $G_1, G_2, ..., G_n$ with respect to the basis $B$.*

*Proof.* According to Theorem 2 it is easy to see that, in the case of undirected graphs, the GDP of $G_1, G_2, ..., G_n$, under the above conditions, is primitive if and only if the number of nonempty subsets of $L$, which are consistent with graphs $G_1, G_2, ..., G_n$ with respect to the basis $B$, with respect to which the function (6) is even is smaller by 1 than the number of such subsets with respect to which it is odd. The statement of the theorem follows now by Lemma 2.

THEOREM 7. *Let $G_1, G_2, ..., G_n$ be connected, undirected graphs each containing at least two vertices. The p-sum of these graphs is a primitive graph if and only if one of the following conditions holds:*

$1^o$ *$p$ is equal to $n$ and all the graphs $G_1, G_2, ..., G_n$ are primitive;*

$2^o$ *$p$ is even and less than $n$;*

$3^o$ *$p$ is odd, less than $n$ and at least one of the graphs $G_1, G_2, ..., G_n$ is primitive.*

Comparing conditions of our Theorem 6 and Theorem 5 in [**9**] we can see that these conditions are complementary. That means that all components of a GDP of undirected, connected graphs are, under above conditions, or primitive or bipartite (i.e., have the same index of imprimitivity). Thus, we have proved the following theorem.

THEOREM 8. *All components of a GDP of undirected connected graphs, under conditions of Theorem 6, have the same index of imprimitivity, i.e., all components are bipartite or all components are primitive graphs.*

This theorem supports the conjecture: All components of the NEPS are almost cospectral [**3**, p. 60].

In [**10**] it is proved that all components, of the NEPS of strongly connected digraphs, have the same index of imprimitivity.

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