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ON INDEPENDENT VERTICES AND EDGES OF BELT GRAPHS

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Abstract. Let $m(G,k)$ and $n(G,k)$ be the number of distinct k-element sets of independent edges and vertices, respectively, of a graph G. Let $h, p_1, p_2, \ldots, p_h$ be positive integers. For each selection of $h, p_1, p_2, \ldots, p_h$ we construct two graphs $N = N_h(p_1, p_2, \ldots, p_h)$ and $M = M_h(p_1, p_2, \ldots, p_h)$, such that $m(N,k) = m(M,k)$ and $n(N,k) = n(M,k)$ for all but one value of k. The graphs N and M correspond respectively to a normal and a Möbius-type belt.

Introduction

In this paper we are concerned with finite graphs without loops, multiple and directed edges. If G is such a graph, then $m(G,k)$ and $n(G,k)$ denote the number of its distinct k-element independent edge and vertex sets, respectively, $k = 0, 1, 2, \ldots$. Recall that $m(G,O) = n(G,O) = 1$, $m(G,1) =$ number of edges of G, $n(G,1) =$ number of vertices of G. Independent edge and vertex sets of graphs and, in particular, the invariants $m(G,k)$ and $n(G,k)$ were much investigated in graph theory [**1**–**16**].

Nonisomorphic graphs G_1 and G_2 for which the equalities

$$m(G_1,k) = m(G_2,k) \tag{1}$$

and

$$n(G_1,k) = n(G_2,k) \tag{2}$$

hold for all $k \geq 0$ are easily constructed; a pair of such graphs is depicted in Fig. 1.

Nonisomorphic graphs for which the equalities (1) and (2) are obeyed for all but one value of k are found even easier. The simplest examples of this kind are the two 2-vertex graphs and the two connected 3-vertex graphs. However, all such examples known until now consist of graphs with small number of vertices.

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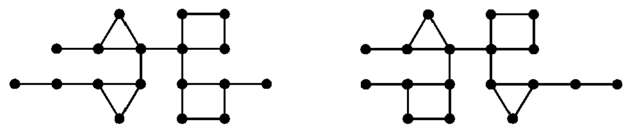

G_1 G_2

Fig. 1

The invariants $m(G, k)$, $k = 1, 2, \dots$, are highly intercorrelated and a number of their "collective" properties is known: for instance, they form a unimodal sequence [**16**] and their distribution is asymptotically normal [**3**]. All the zeros of the matching polynomial (see below) are real-valued [**1**, **4**, **5**]. Numerous pairs of graphs were found for which if the relation $m(G_1, k) > m(G_2, k)$ holds for some k, then $m(G_1, k) \geq m(G_2, k)$ holds for all k [**6**, **13**, **14**].

Similar, but not fully analogous, properties of the numbers $n(G, k)$ have also been deduced [**7–12**].

In view of the above, one may ask if, for graphs G with sufficiently large number of vertices, $m(G, k^*)$ is determined by $m(G, k)$, $k \geq 1$, $k \neq k^*$. We show here that speculations in such a direction are not very promising. We namely have the following result.

THEOREM 1. *There exist graphs G_1 and G_2 with arbitrarily many vertices, for which both equalities* (1) *and* (2) *hold for all k, $k \geq 0$, $k \neq k^*$, but are both violated for a certain $k = k^*$.*

Theorem 1 is an immediate consequence of our main result, namely Theorem 2. In order to be able to formulate it we have to define the belt graphs.

Belt graphs and statement of the main result

Let h be a positive integer. Let $p_1, p_2, \dots, p_h$ be positive integers. Define $r_0 = 0$, $r_1 = p_1$, $r_2 = p_1 + p_2, \dots, r_h = p_1 + p_2 + \dots + p_h$. For brevity, instead of r_h we shall write r.

Let P_{r+1} be the $(r+1)$-vertex path whose vertices are labeled consecutively by $u_0, u_1, \dots, u_r$. Let P'_{r+1} be another copy of the $(r+1)$-vertex path, whose vertices are labeled by $v_0, v_1, \dots, v_r$.

Definition 1. The graph $L = L_h(p_1, p_2, \dots, p_h)$ is obtained from P_{r+1} and P'_{r+1} by joining (by means of a new edge) the vertices u_{r_i} and v_{r_i}, $i = 0, 1, \dots, h$.

Definition 2. The graph $N = N_h\ (p_1, p_2, \dots, p_h)$ is obtained from $L_h(p_1, p_2, \dots, p_h)$ by identifying the vertices u_0 and v_0 with u_r and v_r, respectively. We say that N is a *normal belt graph.*

Definition 3. The graph $M = M_h(p_1, p_2, \dots, p_h)$ is obtained from $L_h(p_1, p_2, \dots, p_h)$ by identifying the vertices u_0 and v_0 with v_r and u_r, respectively. We say that M is a *Möbius-type belt graph.*

In Fig. 2 is depicted a graph L as well as two pairs of belt graphs. From these examples the meaning of the parameters $h, p_1, p_2, \ldots, p_h$ is clear: The graph $L = L_h(p_1, p_2, \ldots, p_h)$ is composed of h linearly arranged circuits of sizes $(2p_i + 2)$, $i = 1, 2, \ldots, h$. The cyclomatic number of L is h. The graphs $N = N_h(p_1, p_2, \ldots, p_h)$ and $M = M_h(p_1, p_2, \ldots, p_h)$ are formed of h cyclically arranged circuits of sizes $(2p_i + 2)$, $i = 1, 2, \ldots, h$. Their cyclomatic numbers are $h + 1$. Their number of vertices is $2r$.

The graphs N can be viewed as representing normal (two-sided) belts. The graphs M, on the other hand, correspond to Möbius-type (one-sided) belts.

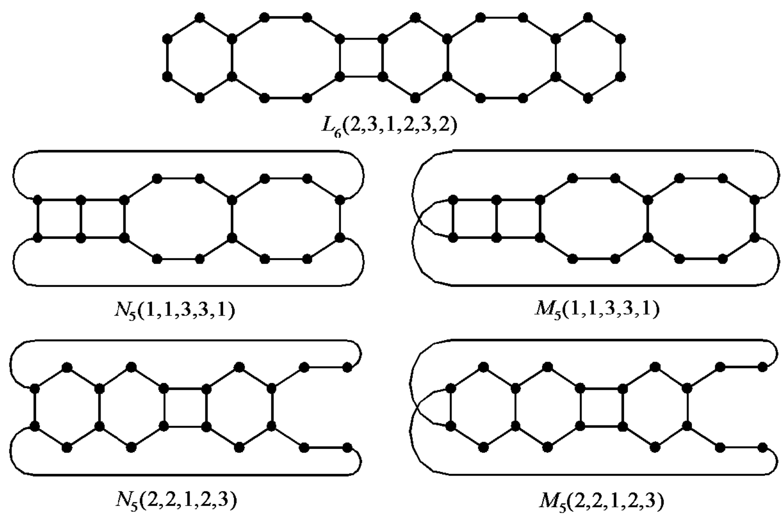

Fig. 2

If r is even, then N is a bipartite graph whereas M is non-bipartite. If r is odd, then N is non-bipartite and M is bipartite. Recall that r is odd if and only if among the parameters $p_1, p_2, \ldots, p_h$ odd values occur odd number of times. In other words: r is odd if and only if among the circuits forming N and M there is an odd number of circuits whose size is divisible by 4.

Our main result reads now as follows.

THEOREM 2. *Let $h, p_1, p_2, \ldots, p_h$ be arbitrary positive integers. Let $r = p_1 + p_2 + \cdots + p_h$. Let $N = N_h(p_1, p_2, \ldots, p_h)$ and $M = M_h(p_1, p_2, \ldots, p_h)$ be belt graphs in the sense of Definitions* 2 *and* 3. *Then:*

(a) $m(N, k) = m(M, k)$ *and* $n(N, k) = n(M, k)$ *for* $0 \leq k \leq r - 1$,

(b) $m(N, r) = m(M, r) + (-1)^r 2$ *and* $n(N, r) = n(M, r) + (-1)^r 2$

(c) $m(N, k) = m(M, k) = n(N, k) = n(M, k) = 0$ *for* $k > r$.

The proof of Theorem 2 requires some preparations.

Matching and independence polynomials of belt graphs

The generating functions associated with the invariants $m(G,k)$ and $n(G,k)$ are

$$\alpha(G) = \alpha(G,x) = \sum_{k\geq 0} m(G,k)x^k \quad \text{and} \quad \omega(G) = \omega(G,x) = \sum_{k\geq 0} n(G,k)x^k$$

Because G is supposed to be finite, both $\alpha(G)$ and $\omega(G)$ are polynomials; we call them matching [**1**, **2**, **4**, **5**, **12**] and independence polynomial [**1**, **10**–**12**], respectively. (The matching polynomial is usually defined [**1**, **2**, **4**, **5**, **15**] in a slightly different, but fully equivalent, way; the definition given above is convenient for the present considerations.) These polynomials obey the following recurrence relations [**1**, **4**]:

LEMMA 1. (a) *Let $e = (u,v)$ be an edge of G, connecting the vertices u and v. Let N_v be the set of vertices of G, consisting of the vertex v and the first neighbors of v. Then*

$$\alpha(G) = \alpha(G-e) + x\alpha(G-u-v); \qquad \omega(G) = \omega(G-v) + x\omega(G-N_v)$$

(b) *If u is the only neighbor of v, then*

$$\alpha(G) = \alpha(G-u) + x\alpha(G-u-v); \qquad \omega(G) = \omega(G-u) + x\omega(G-u-v)$$

LEMMA 2. *Let v' and v'' be two distinct vertices of the graph G and let $e' = (u',v')$ and $e'' = (u'',v'')$ be two distinct edges of G.*

(a) *If e' and e'' are independent, then*

$$\alpha(G) = \alpha(G-e'-e'') + x[\alpha(G-e'-u''-v'') + \alpha(G-e''-u'-v')] + x^2\alpha(G-u'-v'-u''-v'')$$

(b) *If e' and e'' are not independent, then*

$$\alpha(G) = \alpha(G-e'-e'') + x[\alpha(G-u'-v') + \alpha(G-u''-v'')]$$

(c) *If v' and v'' are independent, then*

$$\omega(G) = \omega(G-v'-v'') + x[\omega(G-v'-N_{v''}) + \omega(G-v''-N_{v'})] + x^2\omega(G-N_{v'}-N_{v''})$$

(d) *If v' and v'' are not independent, then*

$$\omega(G) = \omega(G-v'-v'') + x[\omega(G-N_{v'}) + \omega(G-N_{v''})]$$

Needless to say that Lemma 2 is obtained by a two-fold application of the recurrence relations given in Lemma 1a.

By applying Lemma 2a to the edges (u_{r-1}, u_r) and (v_{r-1}, v_r) of the belt graphs $N = N_h(p_1, p_2, \dots, p_h)$ and $M = M_h(p_1, p_2, \dots, p_h)$ and by recalling that in $N, u_r \equiv u_0$, $v_r \equiv v_0$, whereas in $M, u_r \equiv v_0$ and $v_r \equiv u_0$, we arrive at

$$\begin{aligned}\alpha(N) &= \alpha(L - u_r - v_r) + x[\alpha(L - u_0 - u_r - u_{r-1} - v_r) \\ &+ \alpha(L - v_0 - v_r - v_{r-1} - u_r)] + x^2\alpha(L - u_0 - u_r - u_{r-1} - v_0 - v_r - v_{r-1}) \quad (3)\\ \alpha(M) &= \alpha(L - u_r - v_r) + x[\alpha(L - u_0 - v_r - v_{r-1} - u_r) \\ &+ \alpha(L - v_0 - u_r - u_{r-1} - v_r)] + x^2\alpha(L - u_0 - u_r - u_{r-1} - v_0 - v_r - v_{r-1}) \quad (4)\end{aligned}$$

Here L stands for the graph $L_h(p_1, p_2, \dots, p_h)$. It is assumed that the vertices of L, N and M are labeled in accordance with Definitions 1, 2 and 3.

Bearing in mind that the graphs $L-u_0-u_r-u_{r-1}-v_r$ and $L-v_0-v_r-v_{r-1}-u_r$ as well as $L - u_0 - v_r - v_{r-1} - u_r$ and $L - v_0 - u_r - u_{r-1} - v_r$ are isomorphic, we obtain from (3) and (4):

$$\alpha(N) - \alpha(M) = 2x[\alpha(L - u_0 - u_r - u_{r-1} - v_r) - \alpha(L - u_0 - v_r - v_{r-1} - u_r)] \quad (5)$$

The vertices u_{r-1} and v_{r-1} in N and M may be independent, but need not. It is easy to see that u_{r-1} and v_{r-1} are adjacent if $p_h = 1$ and are independent if $p_h > 1$.

Suppose first that $p_h > 1$. Applying Lemma 2c to the vertices u_{r-1} and v_{r-1} and performing calculations fully analogous to those leading to eq. (5), we obtain

$$\begin{aligned}\omega(N) - \omega(M) = 2x[&\omega(L - u_0 - u_r - u_{r-1} - u_{r-2} - v_r - v_{r-1}) \\ &-\omega(L - u_0 - v_r - v_{r-1} - v_{r-2} - u_r - u_{r-1})]\end{aligned} \quad (6)$$

The precisely same formula (6) holds also in the case $p_h = 1$.

At this point it is convenient to introduce the abbreviate notation:

$$L - u_0 - u_r - u_{r-1} - \dots - u_{r-i} - v_r - v_{r-1} - \dots - v_{r-i+1} \equiv L_i' \quad (7)$$
$$L - u_0 - v_r - v_{r-1} - \dots - v_{r-i} - u_r - u_{r-1} - \dots - u_{r-i+1} \equiv L_i'' \quad (8)$$

Then equations (5) and (6) can be rewritten in a more compact form:

$$\alpha(N) - \alpha(M) = 2x[\alpha(L_1') - \alpha(L_1'')] \quad (5a)$$
$$\omega(N) - \omega(M) = 2x[\omega(L_2') - \omega(L_2'')] \quad (6a)$$

Proof of Theorem 2

LEMMA 3. *For* $1 \le i \le r-2$, $\alpha(L_i') - \alpha(L_i'') = -x[\alpha(L_{i+1}') - \alpha(L_{i+1}'')]$.

Proof. Notice that in the graph L_i' the vertex v_{r-i} is of degree one. The same is true for the vertex u_{r-i} in L_i''. Then Lemma 1b is applicable, yielding

$$\alpha(L_i') = \alpha(L - u_0 - u_r - u_{r-1} - \ldots - u_{r-i} - v_r - v_{r-1} - \ldots - v_{r-i}) + x\alpha(L_{i+1}'')$$
$$\alpha(L_i'') = \alpha(L - u_0 - u_r - u_{r-1} - \ldots - u_{r-i} - v_r - v_{r-1} - \ldots - v_{r-i}) + x\alpha(L_{i+1}'')$$

From the above relations Lemma 3 is deduced straightforwardly. □

LEMMA 4. $\alpha(L_1') - \alpha(L_1'') = (-1)^r x^{r-1}$.

Proof. From Lemma 3,

$$\alpha(L_1') - \alpha(L_1'') = -x[\alpha(L_2') - \alpha(L_2'')] = \ldots = (-x)^{r-2}[\alpha(L_{r-1}') - \alpha(L_{r-1}'')]$$

From (7) is seen that L_{r-1}' consists of two vertices (v_0 and v_1), connected by an edge. Therefore, $\alpha(L_{r-1}') = 1 + x$. From (8) follows that L_{r-1}'' consists of two disconnected vertices (v_0 and u_1). Therefore, $\alpha(L_{r-1}'') = 1$. □

LEMMA 5. *For* $1 \le i \le r-2$, $\omega(L_i') - \omega(L_i'') = -x[\omega(L_{i+1}') - \omega(L_{i+1}'')]$

Proof is fully analogous to the proof of Lemma 3.

LEMMA 6. $\omega(L_2') - \omega(L_2'') = (-1)^r x^{r-1}$.

Proof. From Lemma 5,

$$\omega(L_2') - \omega(L_2'') = -x[\omega(L_3') - \omega(L_3'')] = \ldots = (-x)^{r-3}[\omega(L_{r-1}') - \omega(L_{r-1}'')]$$

Lemma 6 follows from taking into account that $\omega(L_{r-1}') = 1 + 2x$ and $\omega(L_{r-1}'') = 1 + 2x + x^2$. □

Combining the formulas (5a) and (6a) with Lemmas 4 and 6 we arrive at:

LEMMA 7. $\alpha(N) - \alpha(M) = (-1)^r 2x^r$ *and* $\omega(N) - \omega(M) = (-1)^r 2x^r$. □

Proof of Theorem 2. The statements (a) and (b) of Theorem 2 are just another way of expressing the result of Lemma 7. It thus remains only to show that also the statements (c) in Theorem 2 are valid.

A graph with $2r$ vertices has at most r independent edges. Therefore, $m(N,k)$ and $m(M,k)$ must be zero for $k > r$.

One of the graphs N and M is bipartite. Denote this has graph by G' and color its vertices in the usual manner. This graph has equal number ($= r$) of vertices of each color. Each group of equally colored vertices forms an independent vertex set of cardinality r, hence $n(G', r) = 2$. Evidently, $n(G', k) = 0$ for $k > r$. For the other graph, say G'', which is non-bepartite, it cannot be $n(G'', r) > 0$. Therefore, it must be $n(G'', k) = 0$ also for $k > r$.

By this also part (c) of Theorem 2 is verified. □

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