PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 59 (73), 1996, 89–94

ON ISOMORPHISMS BY ORTHOGONALITY OF A NORMED SPACE AND AN INNER PRODUCT SPACE

Pavle M. Miličić

Communicated by Stevan Pilipović

Abstract. Using a functional g, defined by (2), we introduce tree kinds of orthogonality in normed spaces and, using them, we prove three theorems on isomorphisms of a normed space and an inner product space. Certain new characterizations of inner product spaces are obtained using functional g.

There are many known conditions on a normed space X which ensure that X is isometrically isomorphic to an inner product space [**1**]. However, much less is konwn if we ask when X is isomorphic to an inner product space. By using the notion of orthogonality in normed spaces, Partington proved the following important result, which will be used below.

Lemma 1. [**8**, Theorem 4]. *Let X be a real normed space and let $\perp$ be an orthogonality relation in X which satisfies:*

1) *if $x \perp x$, then $x = 0$,*
2) *if $x \perp y$, then $y \perp x$,*
3) *if $x \perp y$, then $ax \perp by$ for all $a, b \in R$,*
4) *if $x \perp y$ and $x \perp z$, then $x \perp (y + z)$,*
5) *if $x, y \in X$, then there is an $a \in R$ such that $x \perp (ax + y)$,*
6) *if $x_n \perp y_n$ for all $n \in N$, and if $x_n \to x$, $y_n \to y$ $(n \to \infty)$, then $x \perp y$,*
7) *there is a constant $C > 0$ such that $\|x\| \leq \|ax + y\|$ whenever $x \perp y$ and $|a| \geq C$.*

Then X is isomorphic to an inner product space.

For a relation $\perp$, satisfying 1)–6), Partington defined a functional $x \to f_x \in X^*$ by $f_x(y) = a$ where $y = ax + z$, $x \in S(X)$, $x \perp z$ and $f_{\lambda x} = \lambda f_x$ (X^* denotes

---

AMS Subject Classification (1991): Primary 46B20, 46C

the topological dual of X and $S(X)$ is the unit sphere of X). In the same paper he proved the following: Assuming 1) through 6), the condition 7) is equivalent to

7$'$) *There is a constant $C>0$ such that $\|f_x\| \le C$ for all $x \in S(X)$.*

We are going to replace 7) with a more natural condition.

LEMMA 2. *If the relation $\perp$ satisfies 3), then 7) is equivalent to:*

7$''$) *There is a constant $C>0$ such that $x \perp y \Rightarrow \|x\| \le C\|x+y\|$.*

Proof. Assume 7$''$). Then, by 3), $x \perp y$ implies $\frac{a}{C}x \perp \frac{1}{C}y$ ($|a| \ge C$). Now 7$''$) implies $\left|\frac{a}{C}\right| \|x\| \le C \left\|\frac{a}{C}x + \frac{1}{C}y\right\|$ and, using $|a| \ge C$, we get $\|x\| \le \|ax+y\|$. Conversely, assume 7) and let $x \perp y$. Then there is a $C>0$ such that $\|x\| \le \|Cx+y\|$. Using 3) we get $x \perp Cy$ and 7) gives $\|x\| \le \|Cx+Cy\| = C\|x+y\|$.

It is clear, that in a general normed space, an orthogonality relation cannot be difined with all the properties 1), 2), 3), 4), 5), 6) and 7). Therefore we define certain orthogonality relations which, under some additional conditions, satisfy the conditions of Lemma 1.

Let $(X, \|x\|)$ be a real normed space. On X^2 the following functionals always exist:

$$\tau_{\pm}(x,y) := \lim_{t \to \pm 0} (\|x+ty\| - \|x\|)/t \ tag1$$

$$g(x,y) := 1/2 \cdot \|x\| (\tau_-(x,y) + \tau_+(x,y)), \quad x,y \in X. \tag{2}$$

The functional g is a natural generalization of inner product $(.,.)$ on X^2. It has the following properties:

$$g(x,x) = \|x\|^2, \tag{3}$$

$$g(\alpha x, \beta y) = \alpha\beta g(x,y) \quad (\alpha, \beta \in R), \tag{4}$$

$$g(x, x+y) = \|x\|^2 + g(x,y), \tag{5}$$

$$|g(x,y)| \le \|x\|\|y\| \qquad (\text{see } [\mathbf{3}]). \tag{6}$$

If X is smooth, then g is linear in the second argument, and in this case $[y,x] := g(x,y)$ defines a semi-inner product in the sense of Lumer. However, g can be linear in the second argument even if X is not smooth. For example on $l^1 \times l^1$ the functional g is defined by

$$g(x,y) = \|x\| \sum_k (\operatorname{sgn} x_k) y_k \quad (x = (x_k), y = (j_k) \in l^1)$$

and is clearly linear in y.

If g is linear in the second argument, then we say that X has property (G).

The orthogonality of a vector x to a vector y in X can be defined in several ways (see [**5**]). The most commonly used definition of orthogonality is the Birkhoff orthogonality:

$$x \perp_B y \Leftrightarrow (\forall \lambda \in R) \|x\| \le \|x + \lambda y\|.$$

By use of the functional g the orthogonality $\perp$ can be defined in several ways:

$$\begin{aligned}
x \perp_g y &\Leftrightarrow g(x,y) = 0,\\
x \underset{g}{\perp} y &\Leftrightarrow g(x,y) = g(y,x) = 0,\\
x \perp^g y &\Leftrightarrow g(x,y)g(y,x) = 0,\\
x \overset{g}{\perp} y &\Leftrightarrow g(x,y) + g(y,x) = 0,\\
x_g \perp y &\Leftrightarrow g(x + t_0 y, x) = \|x\|^2,
\end{aligned}$$

where $\min_t \|x + ty\| = \|x + t_0 y\|$ (Such a t_0 always exists since $\varphi(t) = \|x + ty\|$ is convex). All of these notions of orthogonality are generalizations of the classical one, defined by a scalar product.

In [**2**] it was proved that X is smooth if and only if, $x \perp_g y \Leftrightarrow x \perp_B y$ for every $x, y \in X\backslash\{0\}$.

Let us introduce some definitions and establish the notations. The sequence (e_n) in X is g-orthonormal if

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

A g-orthonormal sequence (e_n) is a total sequence in X if

$$(\forall k \in N)\ e_k \perp_g x \Rightarrow x = 0.$$

$x_y := \dfrac{g(y,x)}{\|y\|^2} y$ (the projection of the vector x on the vector y (see [**4**])).

We define, using a g-orthonormal sequence (e_n), the following

$$x \perp^e y \Leftrightarrow \sum_n \frac{g(e_n, x)g(e_n, y)}{n^2} = 0. \tag{7}$$

We quote, for reference, the following result:

Lemma 3. [**9**, Theorem 3.6] *If $[\cdot,\cdot]$ is a semi-inner product on X^2 and if X is smooth, then $\lim_{n\to\infty}[x_k, y_k] = [x, y]$ whenever $\|x_k - x\| \to 0$ and $\|y_k - x\| \to 0$ $(n \to \infty)$.*

Next we state the main results of this paper.

Theorem 1. *Assume X is smooth and assume the functional g has the property*

$$g(x,y)g(y,x) = g(x,z)g(z,x) = 0 \Rightarrow g(x, y+z)g(y+z, x) = 0. \tag{8}$$

Then X is isomorphic to an inner product space.

Proof. Since X is smooth, using properties (3)–(6) of g, the properties (8) and Lemma 3, it is easy to check that $\perp^g$ satisfies the conditions 1), 2), 3), 4) and 6) of Lemma 1. Let us check that the conditions 5) and 7) are also satisfied.

5) Choose $x, y \in X$, $x \neq 0$. Then, with $a = g(x,y)/\|x\|^2$ we have that $g(x, ax+y)\, g(ax+y, x) = 0$ which gives $x \perp^g ax+y$. Therefore 5) holds.

7) Assume $x \perp^g y$ i.e. $g(x,y)g(y,x) = 0$. If $g(x,y) = 0$ then $g(x, x+y) = \|x\|^2 \leq \|x\|\,\|x+y\|$ and hence $\|x\| \leq \|x+y\|$. If $g(y,x) = 0$ then $\|y\| \leq \|x+y\|$ and by inequality $\|x\| - \|y\| \leq \|x+y\|$ we have $\|x\| \leq 2\|x+y\|$. From the Lemma 2 we conclude that 7) holds.

It only remains to apply Lemma 1, and it follows that X is isomorphic to an inner product space.

THEOREM 2. *Let X be a smooth normed space and assume g satisfies the following condition*

$$g(x,y) + g(y,x) = g(x,z) + g(z,x) = 0 \Rightarrow g(y+z, x) = g(y,x) + g(z,x). \tag{9}$$

Then X is isomorphic to an inner product space.

Proof. Using properties (3)–(6) of g, Lemma 3 and Theorem 1 [**7**], it is easy to check that $\overset{g}{\perp}$ satisfies conditions 1), 2), 3), 5) and 6) of Lemma 1. Let us verify conditions 4) and 7).

4) Assume $x \overset{g}{\perp} y$ and $x \overset{g}{\perp} z$ i.e. $g(x,y) + g(y,x) = 0$ and $g(x,z) + g(z,x) = 0$. Since X is smooth, g is linear in the second variable and this gives

$$g(x, y+z) + g(y,x) + g(z,x) = 0.$$

Now, using (9), we conclude

$$g(x, y+z) + g(y+z, x) = 0,$$

that is $x \overset{g}{\perp} y+z$.

7) Assume $x \overset{g}{\perp} y$ i.e. $g(x,y) + g(y,x) = 0$. This implies

$$\|x\|^2 + g(x,y) + \|y\|^2 + g(y,x) = \|x\|^2 + \|y\|^2.$$

Using (5) we get $g(x, k+y) + g(y, x+y) = \|x\|^2 + \|y\|^2$. This and condition (6) gives $\|x\|^2 + \|y\|^2 \leq (\|x\| + \|y\|)\|x+y\|$. Set $\|x\| = \varrho \cos\theta$, $\|y\| = \varrho \sin\theta$, the last inequality becomes $\varrho \leq (\cos\theta + \sin\theta)\|x+y\|$ and therefore $\|x\| \leq 2\|x+y\|$. Using Lemma 2 we conclude that $\overset{g}{\perp}$ satisfies 7).

The result now following from Lemma 1.

Problem 1. If the condition (8) (or the condition (9)) holds, will it follow that X is isometrically isomorphic to an inner product space?

THEOREM 3. *Let X be a normed space satisfying condition (G). Let (e_n) be at most countable total sequence in X such that*

$$\inf_{x\in S(X)} \sum_k \frac{g^2(e_k,x)}{k^2} > 0. \tag{10}$$

Then X is isomorphic to an inner product space.

Proof. Clearly, $\perp^e$ satisfies 1), 2), 3), 4) and 5). In veiw of Lemma 1, it suffices to check conditions 6) and 7′).

6) Assume $x_n \to x$, $y_n \to y$ $(n\to\infty)$, $x_n \perp^e y_n$, $\|x_n\| \le C_1\|y_k\| \le C_2$. Since $g(e_k,.) \in S(X^*)$ (properties (G) and (6)), from

$$0 = \sum_n \frac{g(e_k,x_k)g(e_n,y_k)}{n^2} \le \sum_n \frac{C_1C_2}{n^2},$$

we get

$$0 = \lim_{k\to\infty} \sum_n \frac{g(e_k,x_k)g(e_k,y_k)}{n^2} = \sum_k = \frac{g(e_k,x)g(e_k,y)}{n^2}.$$

This means $x \perp^e y$.

7′) Assume $x \perp^e z$, $y = ax + z$ and $x \in S(X)$. Then $g(e_k,y) = ag(e_k,x) + g(e_k,z)$ for each k, that is

$$\frac{g(e_k,x)g(e_k,y)}{k^2} = a\frac{g^2(e_k,x)}{k^2} + \frac{g(e_k,x)g(e_k,z)}{k^2}.$$

Summing over k, we get

$$\sum_k \frac{g(e_k,x)g(e_k,y)}{k^2} = a\sum_k \frac{g^2(e_k,x)}{k^2} + \sum_k \frac{g(e_k,x)g(e_k,z)}{k^2}.$$

The last term is equal to zero, so

$$a = \sum_k \frac{g(e_k,x)g(e_k,y)}{k^2} \Big/ \sum_k \frac{g^2(e_k,x)}{k^2}$$

For such an a, we have

$$|a| \le \sum_k \frac{1}{k^2} \Big/ \inf_{x\in S(X)} \sum_k \frac{g^2(e_k,x)}{k^2}.$$

Using (10) we deduce that there is a $C > 0$ such that $|a| \le C$ for all $x,y \in S(X)$. Since $f_x(y) = a$ we are done.

Finally, let us state two additional characterizations of inner product spaces in terms of the functional g (see [**4**] and [**6**]).

THEOREM 4. *Let X be a normed space. Then the following conditions are equivalent:*

(i) *X is an inner product space,*

(ii) $\sup_{f,\varphi\in S(X^*)}\{f(x)\varphi(y)-f(y)\varphi(x)\}=\sqrt{\|x\|^2\|y\|^2-g^2(x,y)}$,

(iii) *X is smooth and $g(y-y_x,x)=0$ for all $x,y\in S(X)$.*

Proof. (i) $\Leftrightarrow$ (ii). If X is an inner product space, then

$$D(x,y):=\sup_{f,\varphi\in S(X^*)}\{f(x)\varphi(y)-f(y)\varphi(x)\}$$

is the area of a parallelogram with vertices in 0, x, y and $x+y$, so $D(x,y)=\|x\|\,\|y\||\sin(x,y)|$, where $\cos(x,y)=(x,y)/\|x\|\,\|y\|$. Thus,

$$D(x,y)=\sqrt{\|x\|^2\|y\|^2-(x,y)^2}.$$

Also, in an inner product space we have $g(x,y)=(x,y)$. We proved (i) $\Rightarrow$ (ii). Now assume (ii). Then $D(x,y)=D(y,x)$ implies $g(x,y)=g(y,x)$ $(x,y\in X)$. Using Theorem 4 of [**6**], this gives $g(x,y)=(x,y)$ $(x,y\in X)$.

(i) $\Leftrightarrow$ (iii). Clearly (i) implies (iii). Let us prove the converse. Assume X is smooth and $g(y-y_x,x)=0$ for all $x,y\in S(X)$. By definition of y_x we get: $g(y-y_x,x)=0\Leftrightarrow g(y-g(x,y)x,x)=0\Rightarrow g(y-g(x,y)x,g(x,y)x)=0\Leftrightarrow g(y-g(x,y)x,y-g(x,y)x-y)=0\Rightarrow \|y-g(x,y)x\|^2=g(y-g(x,y)x,y)\Rightarrow \|y-g(x,y)x\|\le 1$ (because of $|g(y-g(x,y)x,y)|\le\|y-g(x,y)x\|$). Therefore $g(y-y_x,x)=0$ implies $\|y-g(x,y)x\|\le 1$. Now we refer to Proposition 18.17 from [**1**]: If $\|y-\varphi'_+(x,y)x\|\le 1$ for all $x,y\in S(X)$, then X is an inner product space. (Note that in a smooth space we have $\varphi'_+(x,y)=g(x,y)$).

REFERENCES

[1] Dan Amir, *Characterizations of inner product spaces*, Birkhäuser Verlag, 1986.

[2] P.M. Miličić, *Sur la g-orthogonalité dans des espaces normés*, Mat. Vesnik **39** (1987), 325–334.

[3] P.M. Miličić, *Une généralisation naturelle du produit scalaire dans un espace normé et son utilisation*, Publ. Inst. Math. Belgrade **42** (56) (1987), 63–70.

[4] P.M. Miličić, *On the Gram-Schmidt projection in normed spaces*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. **4** (1993), 85–92.

[5] P.M. Miličić, *On orthogonalities in normed spaces*, Mathematica Montisnigri **3** (1994), 69–77.

[6] P.M. Miličić, *Sur le g-angle dans un espace normé*, Mat. Vesnik **45** (1993), 43–48.

[7] P.M. Miličić, *On resolvability of g-orthogonality in normed space* (to appear).

[8] J.R. Partington, *Orthogonality in normed spaces*, Bull. Austral. Math. Soc. **33** (1986), 449–455.

[9] M. Pavlović, *Magistarski rad*, Beograd 1978.

Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 06 09 1994)
(Revised 05 07 1995)