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ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF CERTAIN INTEGRAL OPERATORS

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Abstract. We find exact asymptotic behavior of positive and negative eigenvalues of the operator $\int_\Omega k(x-y)a(y)\cdot dy$ where k is a real radial nonhomogenous function (satisfying some aditional condition) and a is a continuous function changing sign on $\Omega \subset R^m$.

1. Introduction. In this paper we study the asymptotic eigenvalue behavior of integral operators defined by kernels of the form

$$(*) \qquad k(x-y)a(y), \qquad x,y \in \Omega.$$

Here $\Omega \subset R^m$ is a bounded open set, a is a continuous function on Ω and k is a sufficiently regular function. There are many results concerning asymptotic behavior of eigenvalues of the integral operator with the kernel of the form $(*)$. For $a \equiv 1$, $m=1$ Widom [**16**] obtained exact asymptotic behavior of eigenvalues of the operator with kernel of the form $(*)$ if the function $K(\xi)=\int_R e^{it\xi}k(t)dt$ is bounded, nonnegative and has sufficiently regular behavior when $\xi \to +\infty$ (k is not necessary homogeneous). There is a similar result in [**12**].

In [**15**] Widom found the exact asymptotic behavior of eigenvalues of the operator with the kernel $(*)$ if a is a nonnegative bounded function when k satisfies some aditional conditions. In the case $k(x)=|x|^{-\alpha}$, a continuous strong positive function, Kac [**7**] obtained the exact asymptotic eigenvalue behavior by a probabilistic method using Karamata Tauberian Theorem. Cobos and Kühn in [**3**] found an upper bound on the eigenvalues of an operator with the kernel of the form

$$K(x,y)=L(x,y)\frac{(1+|\ln\|x-y\||)^\gamma}{\|x-y\|^{N(1-\alpha)}}, \qquad x,y\in\Omega\subset R^N$$

AMS Subject Classification (1991): Primary 47B10 where $L \in L^{\infty}(\Omega)$. Birman, Solomjak and Kostometov in [**1**], [**2**], [**8**] found he exact asymptotic eigenvalues behavior of the operators with kernel of the form $(*)$ (and more general form) but with assumption that k is a homogeneous function. Some estimation of singular values of the integral operator with the kernel $(*)$ are given in [**10**]. In this paper we find exact asymptotic eigenvalue behavior of the operator generated by the kernel $(*)$, where a is continuous (and sign changing) and k is not a homogeneous function. The method is new and is based on a construction of a normal operator (whose spectrum is easily detrmined) and its connection with the starting convolution operator. As an application of this method we give the asymptotic formula for positive and negative eigenvalues of the following boundary problem

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = \lambda a u$$
$$u|_{\partial D} = 0$$

where $a \in C(\bar{\Omega})$ (and a is a sign changing function) and $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$.

1. Preliminaries. Suppose $\mathcal{H}$ is a complex Hilbert space and T is a compact operator on $\mathcal{H}$. The singular values of $T(s_n(T))$ are the eigenvalues of $(T^*T)^{1/2}$ (or $(TT^*)^{1/2}$). The eigenvalues of $(T^*T)^{1/2}$ arranged in a decreasing order and repeated according to their multiplicity, form a sequence $s_1, s_2, s_3, \dots$ tending to zero. Denote the set of compact operators on $\mathcal{H}$ by C_∞. The operator T is a Hilbert Schmidt one ($T \in C_2$) if $\Big(\sum_{n=1}^{\infty} s_n^2(T)\Big)^{1/2} = |T|_2 < \infty$. If T is an integral operator on $L^2(\Omega)$ defined by $Tf(x) = \int_\Omega M(x,y)f(y)dy$, $x \in \Omega \subset R^m$ and $T \in C_2$ then [**6**] $|T|_2^2 = \int_\Omega \int_\Omega |M(x,y)|^2 dxdy$. Denote by $\int_\Omega K(x,y)\cdot dy$ the integral operator on $L^2(\Omega)$ with the kernel K. Let $\mathcal{N}_t(T)$ be the singular value distribution function

$$\mathcal{N}_t(T) = \sum_{s_n(T)\geq t} 1 \qquad (t > 0)$$

A positive function L is a slowly varying function on $[b, \infty)$ if it is measurable and for each $\lambda > 0$ the equality

$$\lim_{x\to+\infty} \frac{L(\lambda x)}{L(x)} = 1$$

holds. It is well known [**13**] that for every $\gamma > 0$ we have

$$\lim_{x\to+\infty} x^{\gamma}L(x) = +\infty, \qquad \lim_{x\to+\infty} x^{-\gamma}L(x) = 0.$$

In what follows we need some lemmas.

Lemma 1. *Suppose L is a continuous slowly varying function such that $\varphi(x) = x^{-r}L(x)$ and $\psi(x) = x^r L(x)$ $(r > 0)$ are monotone for $x \geq x_0$, and*

$$\lim_{x\to+\infty} \frac{L(x(L(x))^{\pm 1/r})}{L(x)} = 1. \tag{0}$$

Then

$$\varphi^{-1}(y) \sim \Big(\frac{L(y^{-1/r})}{y}\Big)^{1/r} \quad y \to 0+,$$
$$\psi^{-1}(y) \sim \Big(\frac{y}{L(y^{1/r})}\Big)^{1/r} \quad y \to +\infty,$$

where φ^{-1}, ψ^{-1} *are the inverses of* φ, ψ.

Proof. Directly follows from (0) by substitution. We observe the the function $L(x) = \prod_{i=1}^{s} (\ln_{m_i}(x))^{d_i}$ $(\ln_{m_i}(x) = \underbrace{\ln\ln\ldots\ln}_{m_i} x)$ satisfies the conditions of Lemma 1.

Lemma 2. *Suppose the operator* $H \in C_\infty$ *is such that for every* $\varepsilon > 0$ *there exists a decomposition* $H = H'_\varepsilon + H''_\varepsilon$ $(H'_\varepsilon, H''_\varepsilon \in C_\infty)$ *with following properties:*

1° *There exists* $\lim_{t\to 0+}(t/L(t^{-1/r}))^{1/r} N_t(H'_\varepsilon) = c(H'_\varepsilon)$

2° $\overline{\lim}_{n\to\infty} \frac{n^r}{L(n)} s_n(H''_\varepsilon) < \varepsilon$.

Then there exists $\lim_{\varepsilon\to 0+} C(H'_\varepsilon) = c(H)$ *and* $\lim_{t\to 0+}(t/L(t^{-1/r}))^{1/r}\mathcal{N}_t(H) = c(H)$. *(L is a slowly varying function satisfying the conditions from Lemma 1).*

Suppose $T \in C_\infty$ is a selfadjoint operator on $\mathcal{H}$, $\{\lambda_n^+(T)\}$, $\{\lambda_n^-(T)\}$ are the sequences of its positive and negative eigenvalues. Denote by $\mathcal{N}_t^\pm(T)$ $(t > 0)$ the corresponding eigenvalue distribution functions $\mathcal{N}_t^\pm(T) = \sum_{\pm\lambda_n^\pm(T)\geq t} 1$.

Lemma 3. *Let* H', $H'' \in C_\infty$ *and* $H = H' + H''$. *If*

$$\lim_{t\to 0+}(t/L(t^{-1/r}))^{1/r}\mathcal{N}_t^\pm(H') = C_\pm(H')$$
$$\lambda_n^\pm(H'') = o(L(n)/n^r)$$

then $\lim_{t\to 0+}(t/L(t^{-1/r}))^{1/r}\mathcal{N}_t^\pm(H) = C_\pm(H')$.

Proof. Lemma 2 and Lemma 3 can be proved by a slight modification of the proof of Ky-Fan theorem [**6**].

Lemma 4. *If* H *and* K *are positive compact operators such that*

$$s_n(H-K) = o(L(n)/n^r) \tag{1}$$

then for $\theta \in (0,1)$

$$s_n(H^\theta - K^\theta) = o((L(n)/n^r)^\theta). \tag{2}$$

Proof. In [**11**] the following inequality

$$\sum_{k=1}^{n} s_k^\beta(H^\theta - K^\theta) \leq C(\theta,\beta)\cdot n^{1-\theta}\Big(\sum_{k=1}^{n} s_k^\beta(H-K)\Big)^\theta \quad (\theta,\beta \in (0,1)) \tag{3}$$

is proved.

From (1) it follows that for any $\varepsilon > 0$ there exist $n_\varepsilon \in N$ such that

$$s_n(H-K) < \varepsilon L(n)/n^r \quad \text{for any} \quad n > n_\varepsilon.$$

Then from (3) we obtain

$$ns_n^\beta(H^\theta - K^\theta) \le C(\theta,\beta)n^{1-\theta}\Big(\sum_{k=1}^{n_\varepsilon} s_k^\beta(H-K) + \varepsilon^\beta \sum_{k=n_\varepsilon+1}^{n} \Big(\frac{L(k)}{k^r}\Big)^\beta\Big)^\theta.$$

As a consequence of the last inequality we have

$$(4) \quad \Big(\frac{n^r}{L(n)}\Big)^\theta s_n(H^\theta - K^\theta) \le$$
$$(C(\theta,\beta))^{1/\beta}\Big[n_\varepsilon s_1^\beta(H-K)\frac{n^{\beta r-1}}{(L(n))^\beta} + \varepsilon^\beta \frac{n^{\beta r-1}}{(L(n))^\beta}\sum_{k=n_\varepsilon+1}^{n}\Big(\frac{L(k)}{k^r}\Big)^\beta\Big]^{\theta/\beta}$$

Let β be a fixed number such that $0 < \beta < \min\{1, 1/r\}$. From (4) we get

$$\overline{\lim_{n\to\infty}}\Big(\frac{n^r}{L(n)}\Big)^\theta s_n(H^\theta - K^\theta) \le (C(\theta,\beta))^{1/\beta}\Big[\varepsilon^\beta \overline{\lim_{n\to\infty}} \frac{n^{\beta r-1}}{(L(n))^\beta}\sum_{k=2}^{n}\Big(\frac{L(k)}{k^r}\Big)^\beta\Big]^{\theta/\beta}.$$

Since

$$\sum_{k=2}^{n}\Big(\frac{L(k)}{k^r}\Big)^\beta \le \int_1^n \Big(\frac{L(x)}{x^r}\Big)^\beta dx \le D\frac{(L(n))^\beta}{n^{1-\beta r}}$$

(here D depends only on β and r) we obtain

$$\overline{\lim_{n\to\infty}}(n^r/L(n))^\theta s_n(H^\theta - K^\theta) \le (C(\theta,\beta))^{1/\beta}(\varepsilon^\beta D)^{\theta/\beta}$$

Because $\varepsilon > 0$ is arbitrary, we get

$$\lim_{n\to\infty}(n^r/L(n))^\theta s_n(H^\theta - K^\theta) = 0.$$

The relation (2) is proved.

2. Main result. Suppose k and k_1 are even realvalued functions from $C^\infty(R\backslash\{0\})$ sufficiently rapidly decreasing to infinity (for example k and k_1 have compact support) satisfying the following conditions

1° $k(x) = k_1 * k_1(x)(= \int_R k_1(t)k_1(x-t)dt)$, $k(x) \to 0$, $k_1(x) \to 0$ $x \to +\infty$.

2° The function $\mathcal{K}(\xi) = \int_R e^{it\xi}k(t)dt$ is a positive decreasing on $(0,\infty)$ and

$$\mathcal{K}(\xi) = \xi^{-r}L(\xi)(1+o(1)) \qquad (r > 0).$$

Let $m \in C[-1,1]$ ($m(x) \in R$, for $x \in [-1,1]$). Consider the operator A on $L^2(-1,1)$ defined by

$$Af(x) = \int_{-1}^{1} k(x-y)m(y)f(y)dy.$$

The operator $\int_{-1}^{1} k(x-y)\cdot dy$ is positive (by property 2°) and the operator $f \to mf$ is selfadjoint. Hence their product (i.e. operator A) has only real eigenvalues. If $m \geq 0$ on $[-1,1]$ then all eigenvalues of A are positive. In that case denote by $\{\lambda_n\}$ the eigenvalues sequence of A. Here the eigenvalues are repeated according to their multiplicity.

THEOREM 1. *If $m \in C[-1,1]$, $m(x) \geq 0$ for $x \in [-1,1]$ and*

$$s_n\Big(\int_0^2 k(x+y)\cdot dy\Big) = o\Big(\frac{L(n)}{n^r}\Big) \tag{5}$$

$$s_n\Big(\int_0^2 k_1(x+y)\cdot dy\Big) = o\Big(\sqrt{\frac{L(n)}{n^r}}\Big) \tag{6}$$

then

$$\lambda_n(A) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int_{-1}^{1}(m(x))^{1/r}dx\Big)^r. \tag{7}$$

Prior to the proof of Theorem 1, we prove the following lemma.

LEMMA 5. *If $m \in C[-1,1]$, $m(x) \geq 0$ for $x \in [-1,1]$ and (5) is valid, then*

$$s_n\Big(\int_{-1}^{1} k(x-y)m(y)\cdot dy\Big) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int_{-1}^{1}(m(x))^{1/r}dx\Big)^r.$$

Proof. From [**4**, Theorem 1] it follows

$$s_n\Big(\int_\Delta k(x-y)\cdot dy\Big) \sim L(n)\Big(\frac{n\pi}{|\Delta|}\Big)^{-r}$$

where Δ is an interval and $|\Delta|$ is its lenght. Divide interval $[-1,1]$ into N intervals

$$\Delta_i = \Big[-1 + \frac{2}{N}(i-1), -1 + \frac{2i}{N}\Big], \quad i = 1,2,\dots N$$

and denote by x_i the midpoint of Δ_i. Suppose $\varepsilon > 0$ is given and N is large enough such that $|m(y) - m(x_i)| < \varepsilon$ for $y \in \Delta_i$. Then

$$\Big|\sum_{j=1}^{N}(m(y) - m(x_j))\chi_{\Delta_j}(y)\Big| < \varepsilon \tag{8}$$

for each $y \in [-1, 1]$ (χ_{Δ_j} is the characteristic function of the interval Δ_j).

The operator A might be represented in the form $A = B_N + D_N + E_N$ where

$$B_N f(x) = \int_{-1}^{1} k(x-y)\Big[\sum_{j=1}^{N}(m(y) - m(x_j))\chi_{\Delta_j}(y)\Big] f(y)dy$$

$$D_N f(x) = \sum_{j=1}^{N}(m(x_j)\chi_{\Delta_j}(x)\int_{\Delta_j} k(x-y)f(y)dy$$

$$E_N f(x) = \sum_{\substack{i\neq j\\ i,j=1}}^{N}(m(x_j)\chi_{\Delta_j}(x)\int_{\Delta_j} k(x-y)f(y)dy$$

Since

$$m(x_j)\chi_{\Delta_i}(x)\int_{\Delta_j} k(x-y)\cdot dy : L^2(\Delta_j) \to L^2(\Delta_i)$$

then by (5) for $i \neq j$ it follows

$$s_n\Big(m(x_j)\chi_{\Delta_i}(x)\int_{\Delta_j} k(x-y)\cdot dy\Big) = o\Big(\frac{L(n)}{n^r}\Big)$$

and hence

$$s_n(E_N) = o(L(n)/n^r). \tag{9}$$

From (8) we have

$$s_n(B_N) < C\cdot \varepsilon L(n)/n^r \tag{10}$$

where the constant C does not depend on n and ε. Let $A_j^N\colon L^2(\Delta_j) \to L^2(\Delta_j)$ ($j = 1, 2, 3, \ldots, N$) be a linear operator defined by

$$A_j^N f(x) = m(x_j)\int_{\Delta_j} k(x-y)f(y)dy.$$

Then D_N is a direct sum of the operators A_j^N; hence

$$\mathcal{N}_t(D_N)=\sum_{j=1}^{N}\mathcal{N}_t(A_j^N) \tag{11}$$

Since

$$s_n(A_j^N)\sim m(x_j)L(n)\Big(\frac{n\pi}{|\Delta_j|}\Big)^{-r} \qquad (n\to\infty)$$

then

$$\mathcal{N}_t(A_j^N)\sim (m(x_j))^{1/r}\frac{|\Delta_j|}{\pi}\Big(\frac{L(t^{-1/r})}{t}\Big)^{1/r} \qquad (t\to 0^+). \tag{12}$$

From (11) and (12) it follows

$$\lim_{t\to 0^+}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t(D_N)=\frac{1}{\pi}\sum_{j=1}^{N}(m(x_j))^{1/r}|\Delta_j|. \tag{13}$$

From (9), (10), (13) and Lemma 2 we obtain

(14)

$$\lim_{t\to 0^+}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t(A)=\int_{-1}^{1}\frac{1}{\pi}(m(x))^{1/r}dx\Big(=\lim_{N\to\infty}\frac{1}{\pi}\sum_{j=1}^{N}(m(x_j))^{1/r}|\Delta_j|\Big).$$

Puting $t=s_n(A)$ in (14) we get

$$n\Big(\frac{s_n(A)}{L(1/(s_n(A))^{1/r})}\Big)^{1/r}\to\frac{1}{\pi}\int_{-1}^{1}(m(x))^{1/r}dx.$$

Let $\mu_n=(s_n(A))^{-1/r}$, then we have

$$\mu_n^rL(\mu_n)\sim n^r\Big(\frac{1}{\pi}\int_{-1}^{1}(m(x))^{1/r}dx\Big)^{-r}.$$

Applying Lemma 1 we get

$$\mu_n\sim\frac{n}{(L(n))^{1/r}}\cdot\Big(\frac{1}{\pi}\int_{-1}^{1}(m(x))^{1/r}dx\Big)^{-1}\text{ i.e. }\ s_n(A)\sim\frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int_{-1}^{1}(m(x))^{1/r}dx\Big)^{r}.$$

Lemma 5 is proved.

Observe that the system $\{\varphi_n\}_{n=1}^{\infty}$, where $\varphi_n(x) = \sin n\pi(1+x)/2$, is an orthonormal basis of $L^2(-1,1)$. Let

$$H(x,y) = \sum_{n=-\infty}^{\infty} [k(x-y+4n) - k(x+y+4n+2)] \tag{15}$$

$$H_1(x,y) = \sum_{n=-\infty}^{\infty} [k_1(x-y+4n) - k_1(x+y+4n+2)] \tag{16}$$

Cosider the operators B and B_1 acting on $L^2(-1,1)$, defined by

$$Bf(x) = \int_{-1}^{1} H(x,y)f(y)dy, \qquad B_1 f(x) = \int_{-1}^{1} H_1(x,y)f(y)dy.$$

By a direct computation we get

$$B\varphi_n = K(n\pi/2)\varphi, \qquad B_1\varphi_n = \sqrt{K(n\pi/2)}\varphi_n \quad n = 1,2,3\ldots$$

and therefore $B = B_1^2$.

Let K and K_1 be linear operators on $L^2(-1,1)$ defined by

$$Kf(x) = \int_{-1}^{1} k(x-y)f(y)dy, \qquad K_1 f(x) = \int_{-1}^{1} k_1(x-y)f(y)dy$$

(K and K_1 are positive operators because $\hat{k} > 0$ and $\hat{k}_1 > 0$, [**16**]).

Lemma 6. *If the functions k and k_1 satisfy the conditions of Theorem 1, then*

$$s_n(K - K_1^2) = o(L(n)/n^r) \tag{17}$$

Proof. From (5), (6), (15), (16) and the definition of the operators B and B_1 it follows $B = K + R_1$ and $B_1 = K_1 + R_2$ where R_1 and R_2 are compact operators such that

$$s_n(R_i) = o(L(n)/n^r) \qquad i = 1,2. \tag{18}$$

As we have $K - K_1^2 = -R_1 + R_2B_1 + B_1R_2 - R_2^2$ (because $B = B_1^2$), then using (18), equalities

$$s_n(B_1) = O\Big(\sqrt{L(n)/n^r}\Big), \qquad s_n(B) = O\Big(\sqrt{L(n)/n^r}\Big)$$

and the properties of singular values from [**6**], we get

$$s_n(K - K_1^2) = o(L(n)/n^r),$$

proving (17).

Now, we prove Theorem 1. Let $M\colon L^2(-1,1) \to L^2(-1,1)$ be a linear operator defined by $Mf(x) = m(x)f(x)$ (Since $m \geq 0$ we have $M \geq 0$). Obviously, we have $A = KM$. Then

$$\lambda_n(A) = \lambda_n(KM) = \lambda_n(M^{1/2}KM^{1/2}) = s_n(M^{1/2}KM^{1/2})$$

(because $M^{1/2}KM^{1/2} \geq 0$). By Lemma 6 and Ky Fan Theorem [**6**] we conclude

$$s_n(M^{1/2}KM^{1/2}) \sim s_n(M^{1/2}K_1^2M^{1/2}) = \lambda_n((K_1M^{1/2})^*K_1M^{1/2}) = s_n^2(K_1M^{1/2}).$$

So

$$\lambda_n(A) \sim s_n^2(K_1M^{1/2}). \tag{19}$$

Let $L_1(\xi) = \sqrt{L(\xi)}$, $m_1(\xi) = \sqrt{m(\xi)}$ and $r_1 = r/2$. It can be easily verified that if the function $L(x)x^{-r_1}$ satisfies the same conditions.

From Lemma 5 we obtain

$$s_n(K_1M^{1/2}) \sim \frac{L_1(n)}{n^{r_1}}\Big(\frac{1}{\pi}\int\limits_{-1}^{1}(m_1(x))^{1/r_1}dx\Big)^{r_1} = \sqrt{\frac{L(n)}{n^r}}\Big(\frac{1}{\pi}\int\limits_{-1}^{1}(m(x))^{1/r}dx\Big)^{r/2}.$$

Combining this with (19) we get

$$\lambda_n(A) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int\limits_{-1}^{1}(m(x))^{1/r}dx\Big)^{r}. \tag{20}$$

The theorem is proved.

Remark 1. By substituing a variable in the eigenvalue relation $A\varphi = \lambda\varphi$, from (20), we get ($m \geq 0$ on $[a,b]$, $m \in C[a,b]$):

$$\lambda_n\Big(\int\limits_a^b k(x-y)m(y)\cdot dy\Big) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int\limits_a^b(m(x))^{1/r}dx\Big)^{r}$$

Remark 2. A similar asymptotic formula is valid in the case when the integration domain is some Jordan measurable set Ω. From Lemma 2, Lemma 5 and the asymptotic relation (19) we get

$$\lambda_n\Big(\int\limits_\Omega k(x-y)m(y)\cdot dy\Big) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int\limits_\Omega(m(x))^{1/r}dx\Big)^{r}. \tag{21}$$

(see [**4**, Theorem 4]), where $m \in c(\Omega)$, $m \geq 0$ on Ω and k satisfies conditions of the Theorem 1.

Let k and k_1 satisfy conditions 1° and 2° from the begining of this section.

THEOREM 2. *Let $m \in c[-1,1]$ be a real valued function changing sign on $[-1,1]$ and let the functions k and k_1 satisfy conditions* (5) *and* (6) *of Theorem* 1. *Denote by $\lambda_1^+ \geq \lambda_2^+ \geq \ldots > 0$ the positive and by $-\lambda_1^- \leq -\lambda_2^- \leq \ldots < 0$ the negative eigenvalues of A (repeated according to their multiplicity). Then we have*

$$\lambda_n^+(A) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int\limits_{\Omega^+}(m(x))^{1/r}dx\Big)^r, \quad \lambda_n^-(A) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int\limits_{\Omega^-}(-m(x))^{1/r}dx\Big)^r,$$

where $\Omega^+ = \{x \in [-1,1]\colon m(x) > 0\}$, $\Omega^- = \{x \in [-1,1]\colon m(x) \leq 0\}$.

Before the proof of Theorem 2, we prove the following Lemma.

LEMMA 7. *If Ω_1 and Ω_2 are measurable sets ($\Omega_i \subset [-1,1]$, $i = 1,2$) such that $\Omega_1 \cap \Omega_2 = \emptyset$, then the singular values of the operator $C\colon L^2(\Omega_1) \to L^2(\Omega_2)$ defined by*

$$Cf(x) = \int\limits_{\Omega_1} k_1(x-y)f(y)dy$$

have the following property

$$\lim_{n\to\infty}\sqrt{n^r/L(n)}\,s_n(C) = 0 \tag{22}$$

Proof. It is enough to prove (22) in the case when Ω_1 and Ω_2 are intervals. Then (22) follows from $\Omega_1 \cap \Omega_2 = \emptyset$ and (6). Denote by P and Q orthoprojectors on $L^2(-1,1)$ defined by

$$Pf(x) = \chi_\Omega^+(x)f(x), \quad Qf(x) = \chi_\Omega^-(x)f(x), \quad P + Q = I$$

and

$$m_+(x) = \begin{cases} m(x); & x \in \Omega^+ \\ 0; & x \in \Omega^- \end{cases} \qquad m_-(x) = \begin{cases} 0; & x \in \Omega^+ \\ -m(x); & x \in \Omega^-. \end{cases}$$

The functions m_+ and m_- are continuous on $[-1,1]$ and $m = m^+ - m^-$. Denote by M_+, M_- the operators acting on $L^2(-1,1)$ defined by $M_\pm f(x) = m_\pm(x)f(x)$. The operators $M_\pm$ are positive and $M = M_+ - M_-$. Clearly $M_+ = PM$ and $M_- = -QM$.

LEMMA 8.

a) $s_n(QK^{1/2}M_+K^{1/2}P) = o(L(n)/n^r)$ c) $s_n(QK^{1/2}M_-K^{1/2}P) = o(L(n)/n^r)$

b) $s_n(QK^{1/2}M_-K^{1/2}Q) = o(L(n)/n^r)$ d) $s_n(PK^{1/2}M_-K^{1/2}P) = o(L(n)/n^r)$.

Proof. We prove here a). Other equalities can be proved similarly. From $QK^{1/2}M_+K^{1/2}P = QK^{1/2}PMK^{1/2}P$ we get

$$s_{2n}(QK^{1/2}M_+K^{1/2}P) \le s_n(QK^{1/2}P) \cdot s_n(MK^{1/2}P).$$

Since

$$s_n(K^{1/2}) \sim \sqrt{\frac{L(n)}{(n\pi/2)^r}}$$

[**4**, Theorem 1] it is enough to prove that

$$s_n(QK^{1/2}P) = o\Big(\sqrt{L(n)/n^r}\Big). \tag{23}$$

From Lemma 6 and Lemma 4 (for $\theta = 1/2$) it follows

$$s_n(K^{1/2} - K_1) = o\Big(\sqrt{L(n)/n^r}\Big). \tag{24}$$

Now, consider the operator

$$QK_1Pf(x) = \chi_{\Omega^-}(x) \int_{-1}^{1} k_1(x-y)\chi_{\Omega^+}(y)f(y)dy \qquad (: L^2(\Omega^+) \to L^2(\Omega^-)).$$

Since $\Omega^+ \cap \Omega^- = \emptyset$, by Lemma 7 we have

$$s_n(QK_1P) = o\Big(\sqrt{L(n)/n^r}\Big). \tag{25}$$

Then (23) follows from (24), (25) and the properties of the singular values [**6**].

By Lemma 8, a) and c) we get

$$\begin{aligned} &|\lambda_n(QK^{1/2}M_+K^{1/2}P + (QK^{1/2}M_-K^{1/2}P)^*)| = o(L(n)/n^r) \\ &|\lambda_n(QK^{1/2}M_-K^{1/2}P + (QK^{1/2}M_-K^{1/2}P)^*)| = o(L(n)/n^r). \end{aligned} \tag{26}$$

Let

$$\begin{aligned} S = {} & QK^{1/2}M_+K^{1/2}P + (QK^{1/2}M_+K^{1/2}P)^* + QK^{1/2}M_+K^{1/2}Q \\ & - PK^{1/2}M_-K^{1/2}P - (QK^{1/2}M_-K^{1/2}P)^* - QK^{1/2}M_-K^{1/2}P. \end{aligned}$$

The operator S is selfadjoint. From (26) and Lemma 8 b) and d) it follows

$$|\lambda_n(S)| = o(L(n)/n^r) \tag{27}$$

If $\{\lambda_n^+(S)\}$ and $\{-\lambda_n^-(S)\}$ are sequences of positive and negative eigenvalues of S ($\lambda_1^+ \ge \lambda_2^+ \ge \ldots > 0, \lambda_1^- \ge \lambda_2^- \ge \ldots > 0$) then from (27) it follows

$$|\lambda_n^{\pm}(S)| = o(L(n)/n^r). \tag{28}$$

Proof of Theorem 2. The operator $A = KM$ and the selfadjoint operator $A_1 = K^{1/2}MK^{1/2}$ have the same positive and negative eigenvalues. Since $P+Q = I$, then

$$\begin{aligned}A_1 &= K^{1/2}M_+K^{1/2} - K^{1/2}M_-K^{1/2}\\ &= (P+Q)K^{1/2}M_+K^{1/2}(P+Q) - (P+Q)K^{1/2}M_-K^{1/2}M_-K^{1/2}(P+Q).\end{aligned}$$

After simplification we get

$$A_1 = PK^{1/2}M_+K^{1/2}P - QK^{1/2}M_-K^{1/2}Q + S. \tag{29}$$

From (28) and (29) and Lemma 3 we get

$$\begin{aligned}&\lim_{t\to 0^+} \mathcal{N}_t^{\pm}(A_1)\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\\ &\qquad = \lim_{t\to 0^+}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t^{\pm}(PK^{1/2}M_+K^{1/2}P - QK^{1/2}M_-K^{1/2}Q).\end{aligned}$$

Since $\mathcal{N}_t^{\pm}(A) = \mathcal{N}_t^{\pm}(A_1)$ and the operators $PK^{1/2}M_+K^{1/2}P$ and $QK^{1/2}M_-K^{1/2}Q$ are positive and orthogonal, we get

$$\begin{aligned}\lim_{t\to 0^+}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t^{+}(A) &= \lim_{t\to 0^+}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t(PK^{1/2}M_+K^{1/2}P)\\ \lim_{t\to 0^+}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t^{-}(A) &= \lim_{t\to 0}\Big(\frac{t}{L(t^{-1/r})}\Big)^{1/r}\mathcal{N}_t(QK^{1/2}M_-K^{1/2}Q).\end{aligned}$$

Applying Lemma 1, we see that positive (negative) eigenvalues of A have the same asymptotic behavior as the eigenvalues of the operator

$$PK^{1/2}M_+K^{1/2}P \qquad (QK^{1/2}M_-K^{1/2}Q).$$

Eigenvalues of the operator $PK^{1/2}M_+K^{1/2}P$ are equal to eigenvalues of the operator $K^{1/2}M_+K^{1/2}\colon L^2(\Omega^+) \to L^2(\Omega^+)$. The operator $K^{1/2}M_+K^{1/2}$ is positive and its eigenvalues are equal to the eigenvalues of the operator $KM_+\colon L^2(\Omega^+) \to L^2(\Omega^+)$. So, $\lambda_n^+(A) \sim \lambda_n(KM_+)$ and from Theorem 1 (Remark 2) we obtain

$$\lambda_n(KM_+) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int_{\Omega_+}(m+(x))^{1/r}\,dx\Big)^r.$$

Therefore

$$\lambda_n^+(A) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int_{\Omega^+}(m(x))^{1/r}\,dx\Big)^r.$$

Similarly

$$\lambda_n^-(A) \sim \frac{L(n)}{n^r}\Big(\frac{1}{\pi}\int\limits_{\Omega_-}(-m(x))^{1/r}dx\Big)^r.$$

Theorem 2 is proved.

Multidimensional case. Let k_0 and k_0^1 be functions from $C^\infty(0,\infty)$ rapidly enough decreasing on infinity and tending to zero (or having compact support). Let $k(t) = k_0(\|t\|)$, $h(t) = k_0^1(\|t\|)$ and $k = h * h$ $(t \in R^m)$. It is well known [**14**] that

$$\int\limits_{R^m} e^{ixy}k(y)dy = \frac{(2\pi)^{m/2}}{\|x\|^{(m-2)/2}}\int\limits_0^\infty k_0(\varrho)\cdot\varrho^{m/2}J_{m/2-1}(\varrho\|x\|)d\varrho,$$

where J_λ is the Bessel function. Suppose that the function

$$K_0(\xi) = \frac{(2\pi)^{m/2}}{\xi^{(m-2)/2}}\int\limits_0^\infty k_0(\varrho)\varrho^{m/2}J_{m/2-1}(\varrho\xi)d\varrho$$

is positive decreasing and

$$K_0(\xi) = \xi^{-r}\frac{L(\xi)}{\xi}(1+o(1)), \qquad \xi\to\infty,$$

where L is a slowly varying function from Lemma 1. Then

$$\int\limits_{R^m} e^{it\xi}h(t)dt = \sqrt{K_0(\xi)} \qquad (\text{because } k = h*h).$$

Let $I = [-1,1]$ and $a \in C(I^m)$. Now, consider the operator $A\colon L^2(I^m) \to L^2(I^m)$ defined by

$$Af(x) = \int\limits_{I^m} k(x-y)a(y)f(y)dy$$

Theorem 3. *If*

(30)

(a) $$s_n\Big(\int\limits_{[0,2]^m} k_0\big(\sqrt{(x_1\pm y_1)^2+\cdots+(x_m\pm y_m)^2}\big)\cdot dy\Big) = o\Big(\frac{L(n^{1/m})}{n^{r/m}}\Big)$$

(b) $$s_n\Big(\int\limits_{[0,2]^m} k_0^1\big(\sqrt{(x_1\pm y_1)^2+\cdots+(x_m\pm y_m)^2}\big)\cdot dy\Big) = o\Big(\sqrt{\frac{L(n^{1/m})}{n^{r/m}}}\Big)$$

holds for all the combinations of $+$ *and* $-$ *except for the one with all signs* $-$, *then*

$$s_n\Big(\int\limits_{I_m} k(x-y)\cdot dy\Big) \sim \Big(\frac{2}{\pi d_0}\Big)^r \frac{L(n^{1/m})}{n^{r/m}}.$$

Furthermore, if $a(x)\geq 0$ $(x\in I^m)$, *then*

$$\lambda_n\Big(\int\limits_{I^m} k(x-y)a(y)\cdot dy\Big) \sim \frac{1}{(\pi d_0)^r}\Big(\int\limits_{I^m}(a(x))^{m/r}dx\Big)^{r/m}\frac{L(n^{1/m})}{n^{r/m}}$$

where $d_0=\frac{2}{\sqrt{\pi}}\Big(\Gamma\Big(1+\frac{m}{2}\Big)\Big)^{1/m}$.

Before the proof of Theorem 3, we prove the following Lemma.

Lemma 9. *Consider all possible numbers* $\sum_{k=1}^m n_k^2$, *where* $n_k\in\mathbf{N}\cup\{0\}$, $k=1,2,\ldots m$. *If we arrange these numbers in a nondecreasing order* $\lambda_1'\leq\lambda_2'\leq\lambda_3'\leq\ldots$ *then*

$$\lambda_n' \sim \frac{n^{2/m}}{C_m^{2/m}}, \qquad \textit{where} \quad C_m=\frac{\pi^{m/2}}{2^m\Gamma(1+m/2)}$$

Proof of Lemma 9. Let N be some fixed positive integer. Denote by r_1 and r_2 the smallest and the largest values n such that $\lambda_n'=N^2$. It is known [**9**] that

$$r_1=\frac{\pi^{m/2}N^m}{2^{m-1}m\Gamma(m/2)}+o(N^m), \qquad r_2=\frac{\pi^{m/2}N^m}{2^{m-1}m\Gamma(m/2)}+o(N^m).$$

Since $r_1\leq n\leq r_2$, we get

$$n=\frac{\pi^{m/2}N^m}{2^{m-1}m\Gamma(m/2)}+o(N^m),$$

and therefore $N\sim C_m^{-1/m}n^{1/m}$. The statement of Lemma 9 then follows from $\lambda_n'=N^2$.

Proof of Theorem 3. Introduce the function $H(x,y)$ $(x=(x_1,\ldots,x_m)$, $y=(y_1,\ldots,y_m))$ by the following system of recurent relations

$$\begin{aligned}
k_1(t_1,\ldots,t_{m-1}) &= \sum_{n_m\in\mathbf{Z}}[k(t_1,t_2,\ldots,t_{m-1},x_m-y_m+4n_m)\\
&\qquad\qquad - k(t_1,t_2,\ldots,t_{m-1},x_m+y_m+4n_m+2)]\\
k_2(t_1,\ldots,t_{m-1}) &= \sum_{n_{m-1}\in\mathbf{Z}}[k_1(t_1,\ldots,t_{m-2},x_{m-1}-y_{m-1}+4n_{m-1})\\
&\qquad\qquad - k_1(t_1,\ldots,t_{m-2},x_{m-1},y_{m-1}+4n_{m-1}+2)]\\
&\ \vdots\\
k_{m-1}(t_1) &= \sum_{n_2\in\mathbf{Z}}[k_{m-2}(t_1,x_2-y_2+4n_2)-k_{m-2}(t_1,x_2+y_2+4n_2+2)]
\end{aligned}$$

$$H(x,y)=\sum_{n_1\in\mathbf{Z}}[k_{m-1}(x_1-y_1+4n_1)-k_{m-1}(x_1+y_1+4n_1+2)].$$

The functions k_0 and k_0^1 are chosen so that all these series converge. By a direct computation we get

$$\int_{I^m} H(x,y)\varphi_{n_1n_2\dots n_m}(y)dy=K_0\Big(\frac{\pi}{2}\sqrt{n_1^2+n_2^2+\dots n_m^2}\Big)\varphi_{n_1n_2\dots n_m}(x)$$

where

$$\varphi_{n_1n_2\dots n_m}(x)=\prod_{i=1}^m \sin\frac{n_i\pi(1+x_i)}{2}.$$

From (30) a) by Ky Fan theorem [**6**] it follows

$$s_n\Big(\int_{I^m} k(x-y)\cdot dy\Big)\sim s_n\Big(\int_{I^m} H(x,y)\cdot dy\Big) \tag{31}$$

Singular values of the operator $\int_{I^m} H(x,y)dy$ are

$$K_0\Big(\frac{\pi}{2}\sqrt{n_1^2+n_2^2+\dots+n_m^2}\Big)=s_{n_1\dots n_m}$$

and therefore

$$n_1^2+\cdots n_m^2=\Big(\frac{2}{\pi}K_0^{-1}(s_{n_1\dots n_m})\Big)^2$$

(K_0^{-1} is the inverse function of K_0).

Let the sequence $\{s_{n_1n_2\dots n_m}\}$ be arranged in a nonincreasing order $s_1\geq s_2\geq s_3\geq\dots$. Put $\lambda_n'=\Big(\frac{2}{\pi}K_0^{-1}(s_n)\Big)^2$. From Lemma 9 we obtain $\lambda_n'\sim c_m^{-2/m}n^{2/m}$ i.e.

$$K_0^{-1}(s_n)\sim\frac{\pi}{2}C_m^{-1/m}n^{1/m}.$$

Since $K_0(\xi)\sim L(\xi)\cdot\xi^{-r}$ ($\xi\to+\infty$) we get $s_n\sim\Big(\frac{2}{\pi d_0}\Big)^r\frac{L(n^{1/m})}{n^{r/m}}$. From (31) it follows

$$s_n\Big(\int_{I^m} k(x-y)\cdot dy\Big)\sim\Big(\frac{2}{\pi d_0}\Big)^r\frac{L(n^{1/m})}{n^{r/m}}.$$

The first part of Theorem 3 is proved.

Remark 3. By substitution as in the one-dimensional case we get

$$s_n\Big(\int_{[-a,a]^m} k(x-y)\cdot dy\Big)\sim\Big(\frac{2a}{\pi d_0}\Big)^r\frac{L(n^{1/m})}{n^{r/m}}.$$

If Δ is some cube in R^m then

$$s_n\Big(\int\limits_{\Delta} k(x-y)\cdot dy\Big) \sim \frac{|\Delta|^{r/m}}{(\pi d_0)^r}\frac{L(n^{1/m})}{n^{r/m}},$$

where $|\Delta|$ denotes the measure of Δ. Analogously as in Theorem 1 (see [**4**, Theorem 4]) we get (for $a \in C(\Delta)$, $a(x) \geq 0$ on Δ):

$$\lambda_n\Big(\int\limits_{\Delta} k(x-y)a(y)\cdot dy\Big) \sim \frac{1}{(\pi d_0)^r}\Big(\int\limits_{\Delta}(a(x))^{m/r}dx\Big)^{r/m}\frac{L(n^{1/m})}{n^{r/m}}$$

and more generaly

$$\lambda_n\Big(\int\limits_{\Omega} k(x-y)a(y)\cdot dy\Big) \sim \frac{1}{(\pi d_0)^r}\Big(\int\limits_{\Omega}(a(x))^{m/r}dx\Big)^{r/m}\frac{L(n^{1/m})}{n^{r/m}},$$

where Ω is a Jordan measurable set in R^m ($a \in C(\bar{\Omega})$, $a \geq 0$ on Ω).

We formulate now a multidimensional variant of Theorem 2. (The proof is carried out in the same way as in Theorem 2). Assume that $a \in C(I^m)$ and that a changes sign.

Theorem 4. *The following*

$$s_n\Big(\int\limits_{[0,2]^m} k_0\big(\sqrt{(x_1 \pm y_1)^2 + \cdots + (x_m \pm y_m)^2}\big)\cdot dy\Big) = o\Big(\frac{L(n^{1/m})}{n^{r/m}}\Big)$$

$$s_n\Big(\int\limits_{[0,2]^m} k_0^1\big(\sqrt{(x_1 \pm y_1)^2 + \cdots + (x_m \pm y_m)^2}\big)\cdot dy\Big) = o\Big(\sqrt{\frac{L(n^{1/m})}{n^{r/m}}}\Big)$$

holds for all the combinations of $+$ *and* $-$ *except for the one with all signs* $-$, *when*

$$\lambda_n^+(A) \sim \frac{1}{(\pi d_0)^r}\Big(\int\limits_{\Omega^+}(a(x))^{m/r}dx\Big)^{r/m}\frac{L(n^{1/m})}{n^{r/m}}$$

$$\lambda_n^-(A) \sim \frac{1}{(\pi d_0)^r}\Big(\int\limits_{\Omega^-}(-a(x))^{m/r}dx\Big)^{r/m}\frac{L(n^{1/m})}{n^{r/m}}.$$

Here $\Omega^+ = \{x \in I^m : a(x) > 0\}$, $\Omega^- = \{x \in I^m : a(x) \leq 0\}$ and $\lambda_n^+(A)$ and $\lambda_n^-(A)$ have the same meaning as in Theorem 2.

Remark 4. The similar asymptotic formulas are valid if I^m is substituted by some Jordan measurable set $\Omega \subset R^m$.

3. An example. The function

$$G_\alpha(x) = \frac{2^{(2-m-\alpha)/2}}{\pi^{m/2}\Gamma\left(\frac{\alpha}{2}\right)} \frac{K_{(m-\alpha)/2}(|x|)}{|x|^{(m-\alpha)/2}} \qquad (x \in R^m,\ \ \alpha > 0)$$

(K_r is McDonald function) satisfies the following conditions [**14**]

1° $$G_\alpha \in L^1(R) \quad \text{and} \quad \int\limits_{R^m} e^{itx} G_\alpha(t)dt = (1+|x|^2)^{-\alpha/2};$$

$$G_\alpha(x) \sim \begin{cases} \Gamma\left(\frac{m-\alpha}{2}\right)/2^\alpha \pi^{m/2}\Gamma\left(\frac{\alpha}{2}\right)|x|^{\alpha-m}; & 0<\alpha<m \\ 1/2^{m-1}\pi^{m/2}\Gamma\left(\frac{m}{2}\right)\ln\frac{1}{|x|}; & \alpha = n \\ \Gamma\left(\frac{\alpha-m}{2}\right)/2^m\Gamma\left(\frac{\alpha}{2}\right); & \alpha > m \end{cases} \qquad \text{for} \quad x \to 0^+$$

and

$$G_\alpha(x) \sim \frac{|x|^{(\alpha-m-1)/2}}{2^{(m+\alpha-1)/2}\pi^{(m-1)/2}\Gamma\left(\frac{\alpha}{2}\right)} e^{-|x|} \quad \text{for} \quad |x| \to +\infty.$$

Let $m = \alpha = 2$, $k(x) = G_2(x)$ $\left(= \frac{1}{2\pi}K_0(x)\right) h(x) = G_1(x)$. Since

$$\int\limits_{R^2} e^{itx} G_2(t)dt = \frac{1}{1+|x|^2}$$

then $K_0(\xi) = 1/(1+\xi^2) \sim 1/\xi^2$ $(\xi \to +\infty)$. The functions G_1 and G_2 satisfy the conditions of Theorem 4.

If $\Omega \subset R^2$ is a Jordan measurable set, $a \in c(\bar{\Omega})$ and a changes sign, then by Theorem 4 (Remark 4) we get

$$\lambda_n^+\Big(\int\limits_\Omega G_2(x-y)a(y)dy\Big) \sim \frac{1}{4\pi n}\int\limits_{\Omega^+} a(x)dx$$
$$\lambda_n^-\Big(\int\limits_\Omega G_2(x-y)a(y)dy\Big) \sim \frac{1}{4\pi n}\int\limits_{\Omega^-} (-a(x))dx$$

$(\Omega^+ = \{x \in \Omega: a(x) > 0\}$, $\Omega^- = \{x \in \Omega: a(x) \le 0\})$.

Since $K_0(x) = -\ln x + \varphi_1(x)\ln x + \varphi_2(x)$ [**14**], where φ_1 and φ_2 are even entire functions and $\varphi_1(0) = 0$, from theorem of Birman and Solomjak [**2**] and Ky Fan follows

$$\begin{aligned} &\lambda_n^+\Big(\int\limits_\Omega \frac{-\ln|x-y|}{2\pi}a(y)dy\Big) \sim \frac{1}{4\pi n}\int\limits_{\Omega^+} a(x)dx \\ &\lambda_n^-\Big(\int\limits_\Omega \frac{-\ln|x-y|}{2\pi}a(y)dy\Big) \sim \frac{1}{4\pi n}\int\limits_{\Omega^-} (-a(x))dx. \end{aligned} \tag{33}$$

Now, as an ilustration, we show how the asymptotic behavior of the positive and negative eigenvalues for an eliptic boundary problem, can be deduced from (33).

Let D be a bounded simple conected domain in R^2 with smooth boundary and $a_{ij} = a_{ji} \in C^1(\bar{D})$. Assume that the differential expresion

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)$$

is uniformly eliptic on D. Let $0 < \mu_1^+ \le \mu_2^+ \le \dots$ and $0 > -\mu_1^- \ge -\mu_2^- \ge \dots$ be the positive and negative eigenvalues for the following boundary problem

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big) = \lambda a(x)u, \quad u|_{\partial D} = 0$$

THEOREM 5. [**5**] *The following asymptotic formula is valid*

$$(\mu_n^{\pm})^{-1} \sim \frac{1}{4\pi n}\int\limits_{\Omega^{\pm}} \frac{|a(x)|}{\begin{vmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x)\end{vmatrix}} dx$$

where $\Omega^+ = \{x \in D : a(x) > 0\}$ *and* $\Omega^- = \{x \in D : a(x) \le 0\}$.

Proof. Applying the procedure for the reduction of elliptic differential expresion to the canonical form, it is obviously enough to find the asymptotic behavior of eigenvalues for the following boundary problem

$$-\Delta u = \lambda a \cdot u, \qquad u|_{\partial D} = 0. \tag{34}$$

To prove Theorem 5 it is enough to prove that the problem (34) has the following asymptotic behavior of eigenvalues

$$(\mu_n^{\pm})^{-1} \sim \frac{1}{4\pi n}\int\limits_{\Omega^{\pm}} |a(x)|dx. \tag{35}$$

Firstly, consider the case when D is the unit disc.

Then, (34) can be written in integral form

$$-u(z) = \lambda \int\limits_{D} G(z,\xi)a(\xi)u(\xi)dA(\xi) \tag{36}$$

where $G(z,\xi) = \frac{1}{2\pi}\ln\left|\frac{z-\xi}{1-z\xi}\right|$ is Green function for the Laplace operator on the unit disc; $dA(\xi) = dpdq$, $\xi = p + iq$. Since the operator

$$\int\limits_{D} -\ln|1 - z\bar{\xi}|dA(\xi) \qquad (\cdot : L^2(D, dA))$$

is a nuclear one, then

$$s_n\Big(\int\limits_D -\ln|1-z\bar{\xi}|\cdot dA(\xi)\Big)=o(n^{-1})$$

and (35) follows from (33), (34) and Ky Fan Theorem.

The general case, when D is an arbitrary domain, is reduced to the just considered one by a conformal mapping into the unit disc.

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