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3-TYPE CURVES ON HYPERBOLOIDS OF REVOLUTION AND CONES OF REVOLUTION

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Abstract. We give a complete classification of all the closed 3-type curves which lie on any hyperboloid of revolution of one sheet or on a cone of revolution.

1. Introduction. The notion of submanifolds of finite type was introduced by Chen [C] around 1980. In particular, for closed curves γ in a Euclidean space E^n, the property of having finite type is equivalent to the fact that the *Fourier series expansion of* γ with respect to an arclength parameter is finite. It is shown in [C] that circles are the only closed curves of finite type in E^2. As an affirmative answer to a conjecture of Blair it is shown in [CDDVV] that the only closed finite type curves which lie on a sphere in E^3 are its great and small circles. Moreover, in [DDV], closed finite type curves on quadrics in E^3 were also studied; in particular, it was shown that the only quadrics which eventually admit closed finite type curves besides the circles are ellipsoids of revolution, hyperboloids of revolution and cones of revolution.

From the complete classification of all the closed 2–type curves in Euclidean spaces given in [CDV], it is easy to see that there are no closed 2–type curves in E^3 which lie on an ellipsoid of revolution.

A complete classification of all the closed 3-type curves in E^3 which lie on an ellipsoid of revolution is given in [PVV].

We give a complete classification of all the closed 3-type curves in E^3 which lie on hyperboloids of revolution of one sheet and on cones of revolution. More precisely, in Sections 3 and 4, we proved the following results.

THEOREM 1. *Let γ be a closed 3-type curve in E^3 which lies on a hyperboloid of revolution of one sheet $\mathcal{H}$. Assume that the eigenvalues of γ are $p_1 < p_2 < p_3$. Then:*

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(i) $2p_1 = p_3 - p_2$, *and eventually by carrying out an isometry of* E^3, *the equation of* $\mathcal{H}$ *can be turned into the form:*

$$x^2 + y^2 - \frac{p_1^2}{p_2 p_3}\, z^2 = (u - w)^2\,,$$

and the curve γ *can be parametrized by an arclength parameter* s, *such that*

$$\gamma(s) =$$
$$\Big(u\cos\frac{p_3 s}{r} + w\cos\Big(\frac{p_2 s}{r} + \theta\Big), u\sin\frac{p_3 s}{r} + w\sin\Big(\frac{p_2 s}{r} + \theta\Big), \frac{2}{p_1}\sqrt{p_2 p_3 u w}\cos\Big(\frac{p_1 s}{r} - \frac{\theta}{2}\Big)\Big),$$

where $r = p_3 u + p_2 w;\ u, w \in R_0^+;\ \theta \in [0, 2\pi]$ *;*

(ii) $2p_2 = p_3 - p_1$, *and by eventually carrying out an isometry of* E^3, *the equation of* $\mathcal{H}$ *can be turned into the form:*

$$x^2 + y^2 - \frac{p_2^2}{p_1 p_3}\, z^2 = (u - w)^2\,,$$

and the curve γ *can be parametrized by an arclength parameter* s, *such that*

$$\gamma(s) = \Big(u\cos\frac{p_3 s}{r} + w\cos\Big(\frac{p_1 s}{r} + \theta\Big), u\sin\frac{p_3 s}{r} + w\sin\Big(\frac{p_1 s}{r} + \theta\Big),$$
$$\frac{2}{p_2}\sqrt{p_1 p_3 u w}\,\cos\Big(\frac{p_2 s}{r} - \frac{\theta}{2}\Big)\Big),$$

where $r = p_3 u + p_1 w;\ u, w \in R_0^+;\ \theta \in [0, 2\pi]$. $\square$

THEOREM 2. *Let* γ *be a closed* 3*-type curve in* E^3 *which lies on a cone of revolution* $\mathcal{C}$. *Assume that* $p_1 < p_2 < p_3$. *Then:*

(i) $2p_1 = p_3 - p_2$, *and by carrying out an isometry of* E^3, *the equation of* $\mathcal{C}$ *can be turned into the form*

$$x^2 + y^2 - \frac{p_1^2}{p_2 p_3}\, z^2 = 0\,,$$

and the curve γ *can be parametrized by an arclength parameter* s, *such that*

$$\gamma(s) = \left(u\left(\cos\frac{p_3 s}{r} + w\cos\Big(\frac{p_2 s}{r} + \theta\Big)\right), u\left(\sin\frac{p_3 s}{r} + \sin\Big(\frac{p_2 s}{r} + \theta\Big)\right),\right.$$
$$\left.\frac{2u}{p_1}\sqrt{p_2 p_3}\,\cos\Big(\frac{p_1 s}{r} - \frac{\theta}{2}\Big)\right),$$

where $r = u(p_2 + p_3);\ u \in R_0^+;\ \theta \in [0, 2\pi]$ *;*

(ii) $2p_2 = p_3 - p_1$, *and up to a Euclidean motion in* E^3, *the equation of* $\mathcal{C}$ *is of the form*

$$x^2 + y^2 - \frac{p_2^2}{p_1 p_3}\, z^2 = 0\,,$$

and the curve γ can be parametrized by an arclength parameter s, such that

$$\gamma(s)=\Big(u\Big(\cos\frac{p_3s}{r}+w\cos\Big(\frac{p_1s}{r}+\theta\Big)\Big),u\Big(\sin\frac{p_3s}{r}+\sin\Big(\frac{p_1s}{r}+\theta\Big)\Big),$$
$$\frac{2u}{p_2}\sqrt{p_1p_3}\,\cos\Big(\frac{p_2s}{r}-\frac{\theta}{2}\Big)\Big),$$

where $r=u(p_1+p_3)$; $u\in R_0^+$; $\theta\in[0,2\pi]$. □

In particular, these results imply the following geometrical facts.

COROLLARY 1. *All the closed 3-type curves in E^3 which lie on a hyperboloid of revolution of one sheet or on a cone of revolution are mass–symmetric, i.e. the center of mass of these curves coincides with the center of mass of the hyperboloid or of the cone.* □

Remark. As a partial answer to the question stated in Problem 1 of [DDV], we proved that there are no 3-type curves on hyperboloids of revolution of two sheets.

We also mention here, that there are no 3-type curves on paraboloids; in fact the only finite type curves on paraboloids are circles [DDV].

2. Preliminaries. Let M be a compact submanifold of a Euclidean space E^n and Δ the Laplacian of the induced metric. Then, following Chen [C], M is said to be of finite type if the position vector x of M has a decomposition of the form:

$$x=x_0+x_{j_1}+x_{j_2}+\cdots+x_{j_k}\,,$$

where x_0 is a constant vector in E^n, and $\Delta x_{j_t}=\lambda_{j_t}x_t$, $(t\in\{1,2,\ldots,k\})$, for k distinct eigenvalues $\lambda_{j_1},\lambda_{j_2},\ldots,\lambda_{j_k}$ of Δ. We also say that the submanifold M is of finite Chen type, in fact of k–type, for some natural number k.

Every closed curve $\gamma\colon[0,2\pi]\to E^n$ of length $2\pi r$ in E^n can be seen as an isometric immersion $S^1(r)\to E^n$ of a circle of radius r into E^n. Denote by s the arclength parameter. Then the Laplace operator Δ of $S^1(r)$ is given by $\Delta=-\,\mathrm{d}^2/\mathrm{d}s^2$. The eigenvalues of Δ on $S^1(r)$ are given by $\lambda_t=(t/r)^2$ and corresponding eigenspaces V_t are spanned by $\cos(ts/r)$ and $\sin(ts/r)$. Hence, a closed curve $\gamma\colon S^1(r)\to E^n$ has a spectral decomposition of the form

$$(*)\qquad \gamma(s)=A_0+\sum_{t=1}^{\infty}\Big(A_t\cos\frac{ts}{r}+B_t\sin\frac{ts}{r}\Big)\,,$$

for some fixed vectors A_t, $B_t\in E^n$, where A_0 is the center of gravity or of mass of γ. This means, that, as periodic functions of s, the coordinate functions $(\gamma_1,\gamma_2,\ldots,\gamma_n)$ of γ, in any Cartesian coordinate system, have a Fourier series expansion with respect to s.

According to a definition of Chen, γ is of finite type if and only if $(*)$ contains only a finite number of non–zero terms in the sum $\sum$. More precisely, γ is of k–type if and only if $(*)$ contains exactly k non–zero terms in the sum $\sum$. Thus, a closed curve γ, of length $2\pi r$ is of k–type if and only if it can be written in the form

$$\gamma(s) = A_0 + \sum_{i=1}^{k} \left(A_i \cos \frac{p_i s}{r} + B_i \sin \frac{p_i s}{r} \right),$$

where $A_0 \in E^n$, $p_1 < p_2 < \cdots < p_k$ are natural numbers and $A_1, A_2, \ldots, A_k$, $B_1, B_2, \ldots, B_k$ are vectors in E^n such that for each $i \in \{1, 2, \ldots, k\}$, A_i, B_i are not simultaneously zero.

It is proved in [C] that this last condition is equivalent to

$$\sum_{i=1}^{k} p_i^2 D_{ii} = 2r^2 \,, \tag{2.1}$$

$$\sum_{\substack{i=1 \\ 2p_i=\ell}}^{k} p_i^2 A_{ii} + 2 \sum_{\substack{i,j=1 \\ i>j \\ p_i+p_j=\ell}}^{k} p_i p_j A_{ij} - \sum_{\substack{i,j=1 \\ i>j \\ p_i-p_j=\ell}}^{k} p_i p_j D_{ij} = 0 \,, \tag{$I(\ell)$}$$

$$\sum_{\substack{i=1 \\ 2p_i=\ell}}^{k} p_i^2 \overline{A}_{ii} + 2 \sum_{\substack{i,j=1 \\ i>j \\ p_i+p_j=\ell}}^{k} p_i p_j \overline{A}_{ij} - \sum_{\substack{i,j=1 \\ i>j \\ p_i-p_j=\ell}}^{k} p_i p_j \overline{D}_{ij} = 0 \,, \tag{$\overline{I}(\ell)$}$$

where

$$\begin{aligned} A_{ij} &= \langle A_i, A_j \rangle - \langle B_i, B_j \rangle, \quad & D_{ij} &= \langle A_i, A_j \rangle + \langle B_i, B_j \rangle \,, \\ \overline{A}_{ij} &= \langle A_i, B_j \rangle + \langle A_j, B_i \rangle, \quad & \overline{D}_{ij} &= \langle A_i, B_j \rangle - \langle A_j, B_i \rangle \,. \end{aligned}$$

From now on, we will assume that γ is a 3-type curve which lies on a hyperboloid of revolution of one sheet, or on a cone of revolution. If necessary, after applying an isometry of E^3, we may assume that the hyperboloid of revolution is given by

$$a(x^2 + y^2) - cz^2 = 1 \qquad (a, c > 0) \,, \tag{2.2}$$

and the equation of the cone of revolution is given by

$$a(x^2 + y^2) - cz^2 = 0 \qquad (a, c > 0) \,, \tag{2.3}$$

We call a finite type curve mass–symmetric if the center of mass of γ coincides with the center of mass of the cone. From the equation of the hyperboloid of revolution or of the cone of revolution it follows that γ is mass-symmetric if and only if $A_0 = 0$.

It is easy to see that γ lies on a hyperboloid of revolution if and only if

$$(2.4)\qquad M' + \sum_{i=1}^{3} D'_{ii} = 2\,,$$

$$(H(\ell))\qquad \sum_{\substack{i=1\\ p_i=\ell}}^{3} M'_i + \sum_{\substack{i=1\\ 2p_i=\ell}}^{3} A'_{ii} + 2\sum_{\substack{i,j=1\\ i>j\\ p_i+p_j=\ell}}^{3} A'_{ij} + 2\sum_{\substack{i,j=1\\ i>j\\ p_i-p_j=\ell}}^{3} D'_{ij} = 0\,,$$

$$(\overline{H}(\ell))\qquad \sum_{\substack{i=1\\ p_i=\ell}}^{3} M'_i + \sum_{\substack{i=1\\ 2p_i=\ell}}^{3} \overline{A}'_{ii} + 2\sum_{\substack{i,j=1\\ i>j\\ p_i+p_j=\ell}}^{3} \overline{A}'_{ij} - 2\sum_{\substack{i,j=1\\ i>j\\ p_i-p_j=\ell}}^{3} \overline{D}'_{ij} = 0\,,$$

($\ell = 1, 2, \ldots, 2p_3$; $i, j = 1, 2, 3$; $i > j$), where

$$\begin{aligned}
&M'_i = 4[A_i, A_0], \quad \overline{M}'_i = 4[B_i, A_0], \quad M' = 2[A_0, A_0],\\
&A'_{ij} = [A_i, A_j] - [B_i, B_j], \quad D'_{ij} = [A_i, A_j] + [B_i, B_j],\\
&\overline{A}'_{ij} = [A_i, B_j] + [A_j, B_i], \quad \overline{D}'_{ij} = [A_i, B_j] - [A_j, B_i],
\end{aligned}$$

and $[\,.\,,.\,]$ is defined by $[u, v] = au_1v_1 + au_2v_2 - cu_3v_3$, and $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. Throughout the rest of the paper, we will put $A_i = (u_i, v_i, w_i), B = (u'_i, v'_i, w'_i), A_0 = (u_0, v_0, w_0)$. Finally, we recall some examples of 3-type curves on hyperboloids of revolution from [DDV]. Let γ_1 be the curve in E^3 given by

$$\begin{aligned}
\gamma_1 = \Big(&u\cos\frac{p_3 s}{r} + w\cos\Big(\frac{p_1 s}{r} + \theta\Big),\; u\sin\frac{p_3 s}{r} + w\sin\Big(\frac{p_1 s}{r} + \theta\Big),\\
&\frac{2}{p_2}\sqrt{p_1p_3uw}\,\cos\Big(\frac{p_2 s}{r} - \frac{\theta}{2}\Big)\Big),
\end{aligned}$$

where $p_1 < p_2 < p_3$ are three natural numbers such that $2p_2 = p_3 - p_1$, $r = up_3 + wp_1$ ($u > 0, w > 0$). Then γ_1 is a 3-type curve which lies on the hyperboloid of revolution

$$x^2 + y^2 - \frac{p_2^2}{p_1p_3}\,z^2 = (u - w)^2\,.$$

Let γ_2 be the curve in E^3 given by

$$\begin{aligned}
\gamma_2 = \Big(&u\cos\frac{p_3 s}{r} + w\cos\Big(\frac{p_2 s}{r} + \theta\Big),\; u\sin\frac{p_3 s}{r} + w\sin\Big(\frac{p_2 s}{r} + \theta\Big),\\
&\frac{2}{p_1}\sqrt{p_2p_3uw}\,\cos\Big(\frac{p_1 s}{r} - \frac{\theta}{2}\Big)\Big),
\end{aligned}$$

where $p_1 < p_2 < p_3$ are three natural numbers such that $2p_1 = p_3 - p_2$, $r = up_3 + wp_2$ ($u > 0, w > 0$). Then γ_2 is a 3-type curve which lies on the hyperboloid of revolution

$$x^2 + y^2 - \frac{p_1^2}{p_2p_3}\,z^2 = (u - w)^2\,.$$

In the next section, we will show that these examples are essential in the sense that they represent all 3-type curves lying on a hyperboloid of revolution.

Concerning parameters $p_1 < p_2 < p_3$, we will distinguish the following 22 cases, which give a complete classification of all cases when some of indices p_1, p_2, p_3, $2p_1$, $2p_2, 2p_3$, p_1+p_2, p_1+p_3, p_2+p_3, p_2-p_1, p_3-p_1, p_3-p_2 coincide or they are all distinct. The distinction of these cases seems to be essential, since the system we have to solve is different in each case.

(1) $p_2 \neq 2p_1, 3p_1$; $p_3 \neq 2p_1, 3p_1, 2p_2, 3p_2, p_1+p_2, p_1+2p_2, 2p_1+p_2, 2p_2-p_1$.
(2) $p_2 = 2p_1$, $p_3 \neq 3p_1, 4p_1, 5p_1, 6p_1$.
(3) $p_2 = 2p_1$, $p_3 = 3p_1$.
(4) $p_2 = 2p_1$, $p_3 = 4p_1$.
(5) $p_2 = 2p_1$, $p_3 = 5p_1$.
(6) $p_2 = 2p_1$, $p_3 = 6p_1$.
(7) $p_2 = 3p_1$, $p_3 \neq 4p_1, 5p_1, 6p_1, 7p_1, 9p_1$.
(8) $p_2 = 3p_1$, $p_3 = 4p_1$.
(9) $p_2 = 3p_1$, $p_3 = 5p_1$.
(10) $p_2 = 3p_1$, $p_3 = 6p_1$.
(11) $p_2 = 3p_1$, $p_3 = 7p_1$.
(12) $p_2 = 3p_1$, $p_3 = 9p_1$.
(13) $p_2 = 3p_1/2 \quad (p_1 = 2m) \quad (\Longrightarrow p_2 \neq 2p_1, 3p_1)$, $p_3 = 2p_1$.
(14) $p_2 \neq 3p_1/2, 2p_1, 3p_1$; $p_3 = 2p_1$.
(15) $p_2 = 3p_1/2 \quad (\neq 2p_1, 3p_1)$; $p_3 = 3p_1$.
(16) $p_2 \neq 3p_1/2, 2p_1, 3p_1$; $p_3 = 2p_2$.
(17) $p_1 \neq 3p_1/2, 2p_1, 3p_1$; $p_3 = 3p_1$.
(18) $p_2 \neq 2p_1, 3p_1$; $p_3 = 3p_2$.
(19) $p_2 \neq 2p_1, 3p_1$; $p_3 = 2p_2 + p_1$.
(20) $p_2 \neq 2p_1, 3p_1$; $p_3 = p_2 + 2p_1$.
(21) $p_2 \neq 3p_1/2, 2p_1, 3p_1$; $p_3 = 2p_2 - p_1$.
(22) $p_2 \neq 2p_1, 3p_1$; $p_3 = p_1 + p_2$.

3. The classification of 3-type curves on hyperboloids of revolution. In this section, we study 3-type curves which lie on a hyperboloid of revolution of one sheet that, after a suitable Euclidean transformation, satisfies the equation $ax^2 + ay^2 - cz^2 = 1$, with $a > 0$, $c > 0$. First, we give some lemmas. Their proofs are more or less long, but we omit almost all the proofs, trying to make this paper maximally short.

LEMMA 3.1. *We always have* $A_3 = (u_3, v_3, 0)$, $\quad B_3 = \epsilon(v_3, -u_3, 0) \quad (\epsilon = \pm 1)$, *thus* $A_3, B_3 \subset Oxy$, $||A_3|| = ||B_3||$ *and* $A_3 \perp B_3$. *If we denote* $A_2 = mA_3 + nB_3 + w_2 e$ $(e = (0,0,1))$, *then* $B_2 = -nA_3 + mB_3 + w_2' e$.

Remark. In the sequel, we shall always denote $\delta = ||A_3|| = ||B_3||$, hence $\delta = |u_3| > 0$.

LEMMA 3.2. *If $A_{31} = \overline{A}_{31} = 0$, then $A_1 = (u_1, v_1, w_1)$, $B_1 = (\epsilon v_1, -\epsilon u_1, w_1')$, where ϵ is from B_3.*

LEMMA 3.3. *If $A_{11} = \overline{A}_{11} = A_{31} = \overline{A}_{31} = 0$, then $A_1 = (u_1, v_1, 0)$, $B_1 = \epsilon(v_1, -u_1, 0)$, where ϵ is related to B_3.*

Remark. If $A_{22} = \overline{A}_{22} = 0$, then $A_2 = (u_2, v_2, 0)$, $B_2 = (\epsilon v_2, -\epsilon u_2, 0)$. In fact, we can get $A_{32} = \overline{A}_{32} = 0$, and next we can use a similar procedure as in Lemma 3.3.

LEMMA 3.4. *If $A_{11} = \overline{A}_{11} = A_{11}' = \overline{A}_{11}' = 0$, then*

$$A_1 = (u_1, v_1, 0), \quad B_1 = (\phi v_1, -\phi u_1, 0) \qquad (\phi = \pm 1).$$

LEMMA 3.5. *The case*

$$A_{11} = \overline{A}_{11} = A_{31} = \overline{A}_{31} = D_{31} = \overline{D}_{31} = 0 \tag{3.1}$$

is impossible.

COROLLARY 3.6. *The case*

$$A_{22} = \overline{A}_{22} = D_{32} = \overline{D}_{32} = 0 \tag{3.2}$$

is impossible.

LEMMA 3.7. *The cases* (1), (2), (6), (7), (8), (10), (14), (15), (16), (17), (18) *and* (22) *are impossible.*

Proof. In the cases (1), (2), (7), (8), (14), (17) and (22), we find by $I(2p_2)$, $\overline{I}(2p_2)$, $I(p_1)$ and $\overline{I}(p_1)$ that

$$A_{22} = \overline{A}_{22} = D_{32} = \overline{D}_{32} = 0,$$

which is impossible by Corollary 3.6.

In the cases (10), (15) and (16), we also find (3.1) by the relations $I(2p_2)$, $\overline{I}(2p_2)$, $I(p_2)$, and $\overline{I}(p_2)$.

Finally, in the cases (6) and (18), by the relations $I(2p_1)$, $\overline{I}(2p_1)$, $I(p_1+p_3)$, $\overline{I}(p_1+p_3)$, $I(p_3-p_1)$ and $\overline{I}(p_3-p_1)$ we find

$$A_{11} = \overline{A}_{11} = A_{31} = \overline{A}_{31} = D_{31} = \overline{D}_{31} = 0,$$

which is impossible by Lemma 3.5. $\square$

LEMMA 3.8. *If $M_3' = \overline{M}_3' = 0$, then $A_0 = (0, 0, \gamma_0)$. Moreover, if also $M_1' = \overline{M}_1' = M_2' = \overline{M}_2' = 0$, then $A_0 = 0$ or $A_1, B_1, A_2, B_2 \in Oxy$.*

LEMMA 3.9. *The cases* (12) ($p_2 = 3p_1, p_3 = 9p_1$), (13) ($p_2 = 3p_1/2, p_1 = 2m, p_3 = 2p_1$), (21) ($p_2 \neq 1.5p_1, 2p_1, 3p_1;\ p_3 = 2p_2 - p_1$), *and* (3) ($p_2 = 2p_1, p_3 = 3p_1$), *are impossible.*

LEMMA 3.10. *In the case* (4) ($p_2 = 2p_1;\ p_3 = 4p_1,\ p_3 = 2p_1 + p_2$), *we have*

$$\begin{aligned}
&A_0 = (0,0,0),\ a/c = 8,\\
&A_1 = w_1 e, \quad B_1 = w_1' e,\\
&A_2 = mA_3 + nB_3,\ B_2 = -nA_3 + mB_3,\\
&m = \frac{w_1^2 - w_1'^2}{32\delta^2}, \quad n = \frac{2w_1 w_1'}{32\delta^2},\\
&S = w_1^2 + w_1'^2 = 32\delta(\delta \pm 1/\sqrt{a}).
\end{aligned}$$

LEMMA 3.11. *In the case* (9) ($p_2 = 3p_1;\ p_3 = 5p_1,\ (p_3 = 2p_1 + p_2)$), *we have*

$$\begin{aligned}
&A_1 = w_1 e, \quad B_1 = w_1' e,\\
&A_2 = mA_3 + nB_3, \quad B_2 = -nA_3 + mB_3,\\
&m = \frac{w_1^2 - w_1'^2}{60\delta^2}, \quad n = \frac{2w_1 w_1'}{60\delta^2}, A_0 = (0,0,0),\\
&S = w_1^2 + w_1'^2 = 60\delta(\delta \pm 1/\sqrt{a}),\ a/c = 15.
\end{aligned}$$

LEMMA 3.12. *In the case* (20) ($p_2 \neq 2p_1, 3p_1;\ p_3 = 2p_1 + p_2$), *we have*

$$\begin{aligned}
&A_0 = (0,0,0),\ \frac{a}{c} = \frac{\Delta}{2},\ \text{where } \Delta = \frac{2p_2(2p_1 + p_2)}{p_1^2},\\
&A_1 = w_1 e, \quad B_1 = w_1' e,\\
&A_2 = mA_3 + nB_3, \quad B_2 = -nA_3 + mB_3,\\
&m = \frac{w_1^2 - w_1'^2}{2\Delta\delta^2}, \quad n = \frac{w_1 w_1'}{\Delta\delta^2},\\
&S = w_1^2 + w_1'^2 = 2\Delta\delta(\delta \pm 1/\sqrt{a}).
\end{aligned}$$

LEMMA 3.13. *In the case* (5) ($p_2 = 2p_1;\ p_3 = 5p_1;\ p_3 = p_1 + 2p_2$), *we have*

$$\begin{aligned}
&A_0 = (0,0,0),\\
&A_1 = pA_3 + qB_3, \quad B_1 = -qA_3 + pB_3,\\
&A_2 = w_2 e, \quad B_2 = w_2' e,\ a/c = 5/4,\\
&p = \frac{w_2^2 - w_2'^2}{10\delta^2}, \quad q = \frac{2w_2 w_2'}{10\delta^2},\\
&S = 5\delta(\delta \pm 1/\sqrt{a}) = w_2^2 + w_2'^2.
\end{aligned}$$

Lemma 3.14. *In the case* (11) ($p_2 = 3p_1, p_3 = 7p_1$; $p_3 = p_1 + 2p_2$), *we have*

$$\begin{aligned}
&A_0 = (0,0,0),\ a/c = 7/9,\\
&A_1 = pA_3 + qB_3, \quad B_1 = -qA_3 + pB_3,\\
&A_2 = w_2 e, \quad B_2 = w_2' e,\\
&p = \frac{w_2^2 - w_2'^2}{28\,\delta^2/9}, \quad q = \frac{2w_2 w_2'}{14\,\delta^2/9},\\
&w_2^2 + w_2'^2 = \frac{28}{9}\,\delta(\delta \pm 1/\sqrt{a})\,.
\end{aligned}$$

Proof. By equations $H(p_1)$, $\overline{H}(p_1)$, $H(p_2)$, $\overline{H}(p_2)$, $H(p_3)$ and $\overline{H}(p_3)$, we have

$$M_1' = \overline{M}_1' = M_2' = \overline{M}_2' = M_3' = \overline{M}_3' = 0\,,$$

whence $A_0 = (0,0,\gamma_0)$, and $\gamma_0 = 0$ or $A_1, B_1, A_2, B_2 \in Oxy$.

Next, by the equations $I(p_1+p_3)$, $\overline{I}(p_1+p_3)$, $H(p_1+p_3)$, $\overline{H}(p_1+p_3)$, we have $A_{31} = \overline{A}_{31} = A_{31}' = \overline{A}_{31}' = 0$, whence, if $A_1 = pA_3 + qB_3 + w_1 e$, $B_1 = p'A_3 + q'B_3 + w_1' e$, we easily get $p' = -q$ and $q' = p$, thus $B_1 = -qA_3 + pB_3 + w_1' e$.

Next, by the equations $I(2p_2)$, $\overline{I}(2p_2)$, $H(2p_2)$, $\overline{H}(2p_2)$, we get

$$A_{22} = -\frac{10}{9}A_{31}, \quad \overline{A}_{22} = -\frac{10}{9}\overline{A}_{31}, \quad A_{22}' = -2D_{31}', \quad \overline{A}_{22}' = 2\overline{D}_{31}'.$$

Since we easily find that $A_{31} = \overline{A}_{31} = 0$, from above relations we get $w_2 = w_2' = 0$ and $p = q = 0$. Hence $A_1 = w_1 e$, $B_1 = w_1' e$, $A_2 = mA_3 + nB_3$, $B_2 = -nA_3 + mB_3$.

Since $A_1 \neq 0$ or $B_1 \neq 0$, we necessarily have that $A_0 = (0,0,0)$. Equations $I(p_1+p_2)$, $\overline{I}(p_1+p_2)$, $H(p_1+p_2)$, $\overline{H}(p_1+p_2)$ are then satisfied identically.

Now, using equations $I(2p_1)$, $\overline{I}(2p_1)$, $H(2p_1)$, and $\overline{H}(2p_1)$, we get

$$\begin{aligned}
&A_{11} = 6D_{21} + 30D_{32}\,, \quad \overline{A}_{11} = -6\overline{D}_{21} - 30\overline{D}_{32},\\
&A_{11}' = -2D_{21}' - 2D_{32}'\,, \quad \overline{A}_{11}' = 2\overline{D}_{21}' + 2\overline{D}_{32}'\,,
\end{aligned}$$

whence we easily find that $a/c = 15$, and

$$m = \frac{w_1^2 - w_1'^2}{60\delta^2}, \qquad n = \frac{2w_1 w_1'}{60\delta^2}\,.$$

By equation $2[A_0, A_0] + \sum_{i=1}^{3}\big([A_i, A_i] + [B_i, B_i]\big) = 2$ we finally get

$$w_1^2 + w_1'^2 = 60\delta(\delta \pm 1/\sqrt{a})\,. \quad \square$$

LEMMA 3.15. *In the case* (19) $(p_2 \neq 2p_1, 3p_1;\ p_3 = p_1 + 2p_2)$, *we have*

$$A_1 = pA_3 + qB_3, \quad B_1 = -qA_3 + pB_3,$$
$$A_2 = w_2 e, \quad B_2 = w_2' e,$$
$$A_0 = (0,0,0),\ \frac{a}{c} = \frac{p_1(p_1+2p_2)}{p_2^2} = \frac{\Delta}{2},$$
$$p = \frac{w_2^2 - w_2'^2}{2\Delta\delta^2}, \quad q = \frac{w_2 w_2'}{\Delta\delta^2},$$
$$w_2^2 + w_2'^2 = 2\Delta\delta(\delta \pm \frac{1}{\sqrt{a}}).$$

Proof of Theorem 1. Collecting all previous results, we conclude that if γ is a closed 3-type curve on the hyperboloid of revolution of one sheet $a(x^2+y^2) - cz^2 = 1$, then $A_0 = (0,0,0)$, $A_3 = (u_3, v_3, 0)$, $B_3 = (\epsilon v_3, -\epsilon u_3, 0)$, $(\epsilon = \pm 1)$, $\delta = ||A_3|| = ||B_3|| > 0$ and only two cases are possible: (i) $2p_1 = p_3 - p_2$, (ii) $2p_2 = p_3 - p_1$.

In the case (i), we have:

$$\frac{a}{c} = \frac{p_2 p_3}{p_1^2},$$
$$A_1 = (0,0,w_1), \quad B_1 = (0,0,w_1'),$$
$$A_2 = mA_3 + nB_3, \quad B_2 = -nA_3 + mB_3,$$
$$m = \frac{w_1^2 - w_1'^2}{2\Delta\delta^2}, \quad n = \frac{2w_1 w_1'}{2\Delta\delta^2},$$
$$S = w_1^2 + w_1'^2 = 2\Delta\delta(\delta \pm 1/\sqrt{a}),$$

where $\Delta = \dfrac{2p_2p_3}{p_1^2} = \dfrac{2a}{c}$.

In this case, we find that $r = \delta p_3 + (\delta \pm 1/\sqrt{a})\, p_2$. If we put $\delta = u > 0$, $\delta \pm 1/\sqrt{a} = w > 0$ and $a = \dfrac{1}{(u-w)^2}$, then we obtain that $r = up_3 + wp_2$ and $\dfrac{c}{a} = \dfrac{p_1^2}{p_2p_3}$. The equation of the hyperboloid then reads:

$$\frac{1}{(u-v)^2}\,(x^2+y^2) - \frac{p_1^2}{p_2p_3(u-v)^2}\, z^2 = 1,$$

that is

$$x^2 + y^2 - \frac{p_1^2}{p_2p_3}\, z^2 = (u-w)^2.$$

Since $S = w_1^2 + w_1'^2$, there is an angle $\theta/2 \in [0,\pi]$ such that $w_1 = \sqrt{S}\,\cos(\theta/2)$, $w_1' = \sqrt{S}\,\sin(\theta/2)$. Hence

$$\sqrt{S} = 2/p_1, \quad \sqrt{p_2p_3uw} = \lambda,$$
$$A_1 = (0,0,\lambda\cos(\theta/2)), \quad B_1 = (0,0,\lambda\sin(\theta/2)),$$
$$m = \frac{w}{u}\,\cos\theta, \quad n = \frac{w}{u}\,\sin\theta.$$

Consider the system $\{e_1, e_2, e_3\}$ in which $e_1 = A_3/||A_3||$, $e_2 = B_3/||B_3||$, $e_3 = (0,0,1)$. The equation of the hyperboloid remains the same, and $A_3 = \delta e_1 = ue_1 = (u,0,0)$, $B_3 = \delta e_2 = ue_2 = (0,u,0)$. It is easily seen that the equation of the curve $\gamma(s)$ in this system reads:

$$\gamma(s) = \left(u\cos\frac{p_3 s}{r} + w\cos\left(\frac{p_2 s}{r} + \theta\right),\, u\sin\frac{p_3 s}{r} + w\sin\left(\frac{p_2 s}{r} + \theta\right),\, \lambda\cos\left(\frac{p_1 s}{r} - \frac{\theta}{2}\right)\right).$$

In the case (ii), we have:

$$\begin{aligned}
&\frac{a}{c} = \frac{p_1 p_3}{p_2^2},\\
&A_1 = pA_3 + qB_3, \quad B_1 = -qA_3 + pB_3,\\
&A_2 = (0,0,w_2), \quad B_2 = (0,0,w_2'),\\
&p = \frac{w_2^2 - w_2^{'2}}{2\Delta\delta^2}, \quad q = \frac{2w_2 w_2'}{2\Delta\delta^2},\\
&S = w_2^2 + w_2^{'2} = 2\Delta\delta(\delta \pm 1/\sqrt{a}).
\end{aligned}$$

where $\Delta = \dfrac{2p_1p_3}{p_2^2} = \dfrac{2a}{c}$. In this case, we find that $r = \delta p_3 + (\delta \pm 1/\sqrt{a})p_1$. If we put $\delta = u > 0$, $\delta \pm 1/\sqrt{a} = w > 0$, then $u \neq w$, and

$$a = \frac{1}{(u-w)^2}, \quad r = up_3 + wp_1, \quad \frac{c}{a} = \frac{p_2^2}{p_1p_3}.$$

The equation of the hyperboloid then reads:

$$x^2 + y^2 - \frac{p_2^2}{p_1p_3}z^2 = (u-w)^2.$$

Since $S = w_2^2 + w_2^{'2}$, there is an angle $\theta/2 \in [0,\pi]$ such that $w_2 = \sqrt{S}\cos(\theta/2)$, $w_2' = \sqrt{S}\sin(\theta/2)$, whence

$$\begin{aligned}
&\lambda = \sqrt{S} = \frac{2}{p_1}\sqrt{p_1p_2uw},\\
&A_2 = (0,0,\lambda\cos(\theta/2)), \quad B_2 = (0,0,\lambda\sin(\theta/2)),\\
&p = \frac{w}{u}\cos\theta, \quad q = \frac{w}{u}\sin\theta.
\end{aligned}$$

In the same system $\{e_1, e_2, e_3\}$, we find that the equation of $\gamma(s)$ reads:

$$\gamma(s) = \left(u\cos\frac{p_3 s}{r} + w\cos\left(\frac{p_1 s}{r} + \theta\right),\, u\sin\frac{p_3 s}{r} + w\sin\left(\frac{p_1 s}{r} + \theta\right),\, \lambda\cos\left(\frac{p_2 s}{r} - \frac{\theta}{2}\right)\right).$$

This completes the proof of the Theorem 1. $\square$

Remark 1. For the above curves, it is known that they are 3-type curves. Hence, these two kinds of curves which depend on parameters $u, w > 0$, $(u \neq w)$, θ, p_1, p_2, p_3 are all 3-type curves on hyperboloids of revolution of one sheet.

Remark 2. There are no 3-type curves on any hyperboloid of revolution of two sheets,

$$ax^2 + ay^2 - cz^2 = -1 \qquad (a, c > 0). \tag{3.3}$$

It can be seen from the proof of Theorem 1. We can proceed a similar proof as for a hyperboloid of revolution of one sheet. The system of equations for the hyperboloid (3.3) differs from the corresponding system for the hyperboloid of revolution of one sheet only in the equation

$$A = -2, \tag{3.4}$$

where $A = 2[A_0, A_0] + \sum_{i=1}^{3} \big([A_i, A_i] + [B_i, B_i]\big)$. We see that all Lemmas 3.1, 3.7, 3.9, remain true also in this case, since they do not use equation (3.4). Equation (3.4) is used only in the last 6 subcases (Lemmas 3.10 – 3.15).

In these subcases, we get that $A = 2a\big(\frac{S}{2\Delta\delta} - \delta\big)^2$, and the equation (3.4) reads $2a\big(\frac{S}{2\Delta\delta} - \delta\big)^2 = -2$, which is a contradiction again. $\square$

4. The classification of 3-type curves on cones of revolution. In this section, we study 3-type curves which lie on a cone of revolution that, after a suitable Euclidean transformation satisfies the equation

$$ax^2 + ay^2 - cz^2 = 0 \qquad (a, c > 0),$$

in fact the equation

$$x^2 + y^2 - dz^2 = 0 \qquad (d > 0). \tag{4.1}$$

Proof of Theorem 2. In this case we can preceed a quite similar method as in Chapter 3. The corresponding system of equations for the curve on a cone differs from the corresponding system of equations in Chapter 3, only in the equation:

$$A = 2[A_0, A_0] + \sum_{i=1}^{3} \big([A_i, A_i] + [B_i, B_i]\big) = 0. \tag{4.2}$$

All Lemmas (3.1, 3.7, 3.9) remain true for the 3-type curve on a cone, since they do not use the equation (4.2). In the last 6 subcases (Lemmas 3.10 – 3.15), we get $A = 2\big(\frac{S}{2\Delta\delta} - \delta\big)^2$, so that the equation (4.2) reads $2\big(\frac{S}{2\Delta\delta} - \delta\big)^2 = 0$, thus

$$S = 2\Delta\delta^2. \tag{4.3}$$

Collecting all these results, we get that if $\gamma(s)$ is a closed 3-type curve which lies on a cone of revolution (4.1), then $A_0 = (0,0,0)$, $A_3 = (u_3, v_3, 0)$, $B_3 = \epsilon\,(v_3, -u_3, 0)$ $(\epsilon = \pm 1)$, $\delta = ||A_3|| = ||B_3|| > 0$, and only two cases are possible:
(i) $2p_1 = p_3 - p_2$; (ii) $2p_2 = p_3 - p_1$.

In the case (i) , we have:

$$\begin{aligned}
&d = \frac{p_1^2}{p_2p_3},\\
&A_1 = (0,0,w_1), \quad B_1 = (0,0,w_1'),\\
&A_2 = mA_3 + nB_3, \quad B_2 = \,-nA_3 + mB_3,\\
&m = \frac{w_1^2 - w_1^{'2}}{2\Delta\delta^2}, \quad n = \frac{2w_1w_1'}{2\Delta\delta^2},\\
&S = w_1^2 + w_1^{'2} = 2\Delta\delta^2,
\end{aligned}$$

where $\Delta = \dfrac{2}{d} = \dfrac{2p_2p_3}{p_1^2}$. In this case we can find that $r = \delta p_2 + \delta p_3$, and if we put $\delta = u > 0$, we get $r = u(p_2 + p_3)$.

The equation of the cone then reads:

$$x^2 + y^2 - \frac{p_1^2}{p_2p_3}\,z^2 = 0. \tag{4.4}$$

Next, we have:

$$S = 2\Delta\delta^2 = \frac{4p_2p_3}{p_1^2}\,\delta^2 = \frac{4p_2p_3}{p_1^2}\,u,$$

whence

$$\sqrt{S} = \frac{2}{p_1}(\sqrt{p_2p_3})\,u = \lambda\,.$$

Since $S = w_1^2 + w_1^{'2}$, there is an angle $\theta/2 \in [0,\pi]$ such that $w_1 = \sqrt{S}\,\cos(\theta/2)$, $w_1' = \sqrt{S}\,\sin(\theta/2)$.

Hence,

$$A_1 = (0,0,\lambda\cos(\theta/2)), \quad B_1 = (0,0,\lambda\sin(\theta/2)), \quad m = \cos\theta, \quad n = \sin\theta.$$

In the system $\{e_1, e_2, e_3\}$, where $e_1 = A_3/||A_3||$, $e_2 = B_3/||B_3||$, $e_3 = (0,0,1)$, we have

$$A_3 = \delta e_1 = ue_1, \qquad B_1 = \delta e_2 = ue_2,$$

and the equation of the curve $\gamma(s)$ in this system reads:

$$\begin{aligned}
\gamma(s) = \Big(u\Big(&\cos\frac{p_3s}{r} + w\cos\Big(\frac{p_2s}{r} + \theta\Big)\Big),\, u\Big(\sin\frac{p_3s}{r} + w\sin\Big(\frac{p_2s}{r} + \theta\Big)\Big)\,,\\
&\frac{2}{p_1}\sqrt{p_2p_3}u\cos\Big(\frac{p_1s}{r} - \frac{\theta}{2}\Big)\Big)\,.
\end{aligned} \tag{α}$$

In the case (ii), we have:

$$\begin{aligned}
&d = \frac{p_2^2}{p_1 p_3},\\
&A_1 = pA_3 + qB_3, \quad B_1 = -qA_3 + pB_3,\\
&A_2 = (0,0,w_2), \quad B_2 = (0,0,w_2'),\\
&p = \frac{w_2^2 - w_2'^2}{2\Delta\delta^2}, \quad q = \frac{2w_2 w_2'}{2\Delta\delta^2},\\
&S = w_2^2 + w_2'^2 = 2\Delta\delta^2,
\end{aligned}$$

where $\Delta = \dfrac{2}{d} = \dfrac{2p_1p_3}{p_2^2}$. In this case, we find that $r = \delta p_1 + \delta p_3$, and putting $\delta = u > 0$, we find $r = u(p_1 + p_3)$ $(u > 0)$. The equation of the cone then reads:

$$x^2 + y^2 - \frac{p_2^2}{p_1 p_3} z^2 = 0\,. \tag{4.5}$$

Next, $S = 2\Delta\delta^2 = \dfrac{4p_1p_3}{p_2^2}\,\delta^2 = \dfrac{4p_1p_3}{p_2^2}\,u^2$, $\sqrt{S} = \dfrac{2\sqrt{p_1p_3}}{p_2}\,u = \lambda$. Since $S = w_2^2 + w_2'^2$, there is an angle $\theta/2 \in [0,\pi]$ such that $w_2 = \sqrt{S}\cos(\theta/2)$, $w_2' = \sqrt{S}\sin(\theta/2)$, whence $A_2 = (0,0,\lambda\cos(\theta/2))$, $B_2 = (0,0,\lambda\sin(\theta/2))$, $p = \cos\theta$, $\quad q = \sin\theta$.

In the same system $\{e_1, e_2, e_3\}$ we find that the equation of the curve $\gamma(s)$ reads:

$$\tag{β} \begin{aligned}\gamma(s) = \Big(&u\Big(\cos\frac{p_3 s}{r} + w\cos\Big(\frac{p_1 s}{r} + \theta\Big)\Big),\ u\Big(\sin\frac{p_3 s}{r} + w\sin\Big(\frac{p_1 s}{r} + \theta\Big)\Big),\\
&\frac{2}{p_2}\sqrt{p_1p_3}\,u\cos\Big(\frac{p_2 s}{r} - \frac{\theta}{2}\Big)\Big)\,.\end{aligned}$$

This completes the proof of Theorem 2. □

Remark. The curves of types (α) and (β) give all 3-type curves which lie on cones of revolution. They all depend on parameters $u > 0$, $\theta/2 \in [0,\pi]$, $p_1, p_2, p_3 \in N$ $(p_1 < p_2 < p_3)$.

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