PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 60 (74), 1996, 1–4

A LOGIC WITH HIGHER ORDER PROBABILITIES

Zoran Ognjanović and **Miodrag Rašković**

Communicated by Žarko Mijajlović

Abstract. An extension of the propositional probability logic LPP given in [**3**] that allows higher order probabilities is introduced. The corresponding completeness and decidability theorems are proved.

1. Introduction. The propositional probabilistic logic LPP was given in [**3**]. LPP is a conservative extension of the classical propositional logic. Its language allows making formulas such as $P_r(A)$, with the intended meaning "the probability of A is greater or equal to r". Formulas in the scope of a probabilistic operator P_r are restricted to be propositional. In this paper we present an extension of LPP, denoted by $\mathrm{LPP}_{\mathrm{ext}}$. In this logic statements about higher order probabilities can be expressed using formulas with nested probabilistic operators. A possible-world approach is used to give semantics to probabilistic formulas of $\mathrm{LPP}_{\mathrm{ext}}$.

The first order probabilistic logic LP was also presented in [**3**]. LP can be extended in the same way as was done with LPP. Another probabilistic logics were given in [**1, 2**]. In these logics one can use linear inequalities involving probabilities. In [**1, 2**] the authors proved only the simple completeness theorems, while here we give the extended completeness theorem.

2. Syntax. The language of $\mathrm{LPP}_{\mathrm{ext}}$ consists of propositional letters, logical connectives $\wedge$ and $\vee$, and a probabilistic operator P_r, for each $r \in \mathrm{Index} \subset [0,1]$, where $\{0,1\} \in \mathrm{Index}$, and Index is finite. If $r \in \mathrm{Index}$ and $r < 1$, then $r^+ = \min\{s \in \mathrm{Index}\colon r < s\}$. If $r \in \mathrm{Index}$ and $r > 1$, then $r^- = \max\{s \in \mathrm{Index}\colon s < r\}$.

The set of $\mathrm{LPP}_{\mathrm{ext}}$-formulas is the smallest set containing propositional letters, and closed under formation rules: if A and B are formulas, then $P_r(A)$, $\neg A$ and $A \wedge B$ are formulas.

AMS Subject Classification (1991): Primary 03B80; Secondary 68T27

A preliminary version of this paper was presented at the X Seminar primenjene matematike PRIM '95, Budva 1995

3. Semantics. *Definition.* An $\mathrm{LPP}_{\mathrm{ext}}$-model is a triple $\langle W, \mathrm{Prob}, \pi\rangle$, where W is a set of worlds, $\pi(w)$ is a truth assignment to the propositional letters for every $w \in W$, and Prob is a probability assignment which assigns to every $w \in W$ a probability space. So, for every $w \in W$, $\mathrm{Prob}(w)$ is a triple $\langle V(w), H(w), \mu(w)\rangle$, where $V(w) \subset W$, $H(w)$ is an algebra of subsets of $V(w)$, and $\mu(w)\colon H(w) \to \mathrm{Index}$ such that for every $w \in W$:

a) $\mu(w)(\theta) \geq 0$, for all θ, b) $\mu(w)(V(w)) = 1$,
c) $\mu(\theta_1 \cup \theta_2) = \mu(\theta_1) + \mu(\theta_2)$, for all disjoint θ_1 and θ_2.

As it can be seen, $\mu(w)$'s are finite additive probabilistic measures with a fixed, finite range.

Definition. Let $M = \langle W, \mathrm{Prob}, \pi\rangle$ be an arbitrary model. A satisfaction relation $\models$ over the set of worlds and the set of formulas satisfies the following properties ($\forall w \in W$):

a) if p is a propositional letter, then $w\| - p$ iff $\pi(w)(p) = true$,
b) $w \Vdash P_r(A)$ iff $\mu(w)(\{u \in V(w)\colon u \Vdash A\}) \geq r$,
c) $w \Vdash \neg A$ iff it is not $w \Vdash A$ and
d) $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$.

We suppose that to every formula there is associated a well-defined probability, i.e., that formulas are satisfied by measurable sets of worlds. In the sequel $[A]$ denotes $\{w\colon w \Vdash A\}$.

4. Complete Axiomatization. The axiom system $\mathrm{AX}_{LPP_{ext}}$ involves eight axiom schemas:

$A1.$ $A \to (B \to A)$
$A2.$ $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
$A3.$ $(\neg B \to \neg A) \to (A \to B)$
$A4.$ $P_0(A)$
$A5.$ $P_s(A) \to P_r(A),\ \ s \geq r$
$A6.$ $(P_s(A) \wedge P_r(B) \wedge P_1(\neg A \vee \neg B)) \to P_{\min(1,s+r)}(A \vee B)$
$A7.$ $(P_{1-s}(\neg A) \wedge P_{1-r}(\neg B)) \to P_{\max(0,1-(s+r))}(\neg A \wedge \neg B)$
$A8.$ $\neg P_{1-s}(\neg A) \leftrightarrow P_{s^+}(A)$

and two rules of inference (θA means that A is provable):

$R1.$ From θA and $\theta A \to B$ infer θB (modus ponens).
$R2.$ From θA infer $\theta P_1(A)$.

Note that $AX_{LPP_{ext}}$ is the same as the axiom system for LPP, but formulas A and B in the axioms and rules can be arbitrary $\mathrm{LPP}_{\mathrm{ext}}$-formulas.

A formula A is said to be consistent with respect to $AX_{LPP_{ext}}$, if $\neg A$ is not provable; otherwise A is inconsistent. A finite set of formulas $T = \{A_1, A_2, \ldots, A_n\}$

is consistent if $\neg(A_1 \wedge \ldots \wedge A_n)$ is not provable. An infinite set of formulas is consistent if every its finite subset is consistent.

LEMMA. *For every consistent set T of LPP_{ext}-formulas there is a maximal consistent set that contains T.*

Proof. Let $A_1, A_2, \ldots$ be an enumeration of all $\mathrm{LPP}_{\mathrm{ext}}$-formulas. We define a sequence $G_0, G_1, \ldots$ of sets of the formulas in the following way: $G_0 = T$, and if $G_i \cup \{A_{i+1}\}$ is consistent, then $G_{i+1} = G_i \cup \{A_{i+1}\}$; otherwise $G_{i+1} = G_i \cup \{\neg A_{i+1}\}$. By the hypothesis G_0 is consistent. Let us suppose that for some $i > 0$, the set G_i is not consistent. That means that there are formulas $B_1, \ldots, B_m$ and $C_1, \ldots, C_n$ from G_{i-1}, so that $\theta\neg(B_1 \wedge \ldots \wedge B_m \wedge A_i)$ and $\theta\neg(C_1 \wedge \ldots \wedge C_n \wedge \neg A_i)$. By the propositional reasoning it follows that $\theta\neg(B_1 \wedge \ldots \wedge B_m \wedge C_1 \wedge \ldots \wedge C_n)$, i.e., that G_{i-1} is not consistent, a contradiction. Now, it is easy to show that $G = \cup_n G_n$ is a maximal consistent set of formulas, and that $T \subset G$.

EXTENDED COMPLETENESS THEOREM. *A set of formulas is consistent with respect to $AX_{LPP_{ext}}$ iff it has an LPP_{ext} model.*

Proof. ($\rightarrow$) Since $AX_{LPP_{ext}}$ is sound, a satisfiable set of formulas is consistent.

($\leftarrow$) Suppose that a set T of formulas is consistent. We construct a probabilistic model so that T is satisfiable in it. This model $M = \langle W, \mathrm{Prob}, \pi\rangle$ is defined as follows: $W = \{w\colon w \text{is a maximal consistent set of formulas}\}$, $\pi(w)(p) = \text{true}$ iff $p \in w$ and $\mathrm{Prob}(w) = (W, H(w), \mu(w))$, where $H(w)$ is an algebra of sets of worlds of the pattern $[A]$, and $\mu(w)[A] = \max_r\{P_r(A) \in w\}$. The axioms of probability (A4–A8) guarantee that everything is well defined.

For example, let us suppose that $[A] \subset [B]$, but $\mu(w)([A]) > \mu(w)([B])$, i.e., that there is no $w \in W$ such that $A \wedge \neg B \in w$, but $P_r(A) \in w$ and $P_r(B) \in w$ for some r. Since $A \wedge \neg B$ is not consistent, $A \rightarrow B$, and $P_1(A \rightarrow B)$ are theorems. So, $P_1(A \rightarrow B) \wedge P_r(A) \wedge \neg P_r(B) \in w$. It follows that $\neg(P_1(A \rightarrow B) \wedge P_r(A) \wedge \neg P_r(B))$ is not provable. By A8 it can be rewritten as $(P_{1-(1-r)}(A) \wedge P_{1-(r^-)}(\neg B)) \rightarrow P_{0^+}(A \wedge \neg B)$. But, this formula is an instance of the axiom A7, a contradiction. Hence, $\mu(w)$'s are well defined. In a similar way we can prove that $\mu(w)$'s are finite additive measures, and that their ranges are subsets of the set Index.

It follows that M is a $\mathrm{LPP}_{\mathrm{ext}}$-model satisfying $(\forall w \in W)(w \Vdash A \text{ iff } A \in w)$. For example, let $w \Vdash P_r(A)$. Hence, $\mu(w)([A]) = \max\{s\colon P_s(A) \in w\} \geq r$. By the axiom A5, the formula $P_r(A) \in w$. On the other hand, if $P_r(A) \in w$, then $\max\{s\colon P_s(A) \in w\} = \mu(w)([A]) \geq r$, and $w\theta P_r(A)$.

Since every w is a maximal consistent set, and T can be extended to a maximal consistent set, there is a world $w \in W$ satisfying T.

5. Decidability It is well known that there is a decision procedure to answer whether a classical propositional formula is satisfiable. We can show that the same holds for $\mathrm{LPP}_{\mathrm{ext}}$.

LEMMA. *If a LPP_{ext}-formula A is satisfiable, then it is satisfiable in a finite LPP_{ext} model.*

Proof. Suppose A holds in a world of an $\mathrm{LPP}_{\mathrm{ext}}$ model $M = \langle W, \mathrm{Prob}, \pi\rangle$. Let Φ_A be the set of all subformulas of A, and let $\approx$ be an equivalence relation over W^2, such that $w \approx u$ iff $(\forall B \in \Phi_A)(w \Vdash B$ iff $u \Vdash B)$. The quotient set $W/\approx$ is finite. From every class C_i we choose an element and denote it by w_i. We consider a model $M^* = \langle W^*, \mathrm{Prob}^*, \pi^*\rangle$, where $W^* = \{w_i\}$, $\pi^*(w_i)(p) = \pi(w_i)(p)$, for every propositional letter, and Prob^* is defined as follows: $V^*(w_i) = \{u\colon (\exists v \in C_u) v \in V(w_i)\}$ and $H^*(w_i)$ is the power set of $V^*(w_i)$. Let u be a world such that in the model M all the formulas of Φ_A, satisfied in u, are $B_1, \dots, B_k$. Then, we define $\mu^*(w_i)(u) = \mu(w_i)([B_1 \wedge \dots \wedge B_k]) = \mu(w_i)(C_u)$, and for a set $D \in H^*(w_i)$, the measure $\mu^*(w_i)(D) = \sum_{u \in D} \mu^*(w_i)(u)$. Since

$$\mu^*(w_i)(V^*(w_i)) = \sum_{u \in V^*(w_i)} \mu^*(w_i)(u) = \sum_{C_u \in W/\approx} \mu^*(w_i)(C_u) = 1$$

μ^* is a finite additive probability measure, and M^* is an $\mathrm{LPP}_{\mathrm{ext}}$ model.

Now, every formula $B \in \Phi_A$ is satisfiable in M iff it is satisfiable in M^*. If B is a propositional letter, and $\langle M, w\rangle \Vdash B$, then $\langle M, w_i\rangle \Vdash B$ holds for $w_i \in C_w$. Obviously, $\langle M, w_i\rangle \Vdash B$ iff $\langle M^*, w_i\rangle \Vdash B$. If $B = B_1 \wedge B_2$, $\langle M, w\rangle \Vdash B$, and $w_i \in C_w$, then $\langle M, w_i\rangle \Vdash B$ iff $\langle M, w_i\rangle \Vdash B_1$ and $\langle M, w_i\rangle \Vdash B_2$ iff $\langle M^*, w_i\rangle \Vdash B_1$ and $\langle M^*, w_i\rangle \Vdash B_2$ iff $\langle M^*, w_i\rangle \Vdash B$. The case when $B = \neg C$ follows similarly. Finally, if $B = P_r(B_1)$ and $\langle M, w\rangle \Vdash B$, then $\langle M, w_i\rangle \Vdash B$ holds for $w_i \in C_w$, and

$$\begin{aligned}&\langle M, w_i\rangle \Vdash B \quad \text{iff}\\ &\quad r \leq \mu(w_i)([B_1]) = \sum_{C_u \Vdash B_1} \mu(w_i)(C_u) = \sum_{C_u \Vdash B_1} \mu^*(w_i)(C_u) = \mu^*(w_i)([B_1])\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{iff} \quad \langle M^*, w_i\rangle \Vdash B.\end{aligned}$$

The model M^* from the lemma has no more than 2^N worlds, where N is the number of subformulas of the considered formula A. Since there is a finite number of such $\mathrm{LPP}_{\mathrm{ext}}$-models, the following theorem holds:

THEOREM. *LPP_{ext}-logic is decidable.*

References

1. R. Fagin, J. Halpern, and N. Megiddo, *A logic for reasoning about probabilities*, Information and Computation **87** (1990), 78–128
2. R. Fagin, J. Halpern, *Reasoning about knowledge and probability*, J. Assoc. Comput. Mach. **41** (1994), 340–367
3. M. Rašković, *Classical logic with some probability operators*, Publ. Inst. Math. (Beograd)(N.S.) **53(67)** (1993), 1–3

Zoran Ognjanović
Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367
Yugoslavia
zorano@mi.sanu.ac.yu

Miodrag Rašković
Prirodno-matematički fakultet
Radoja Domanovića 12
34000 Kragujevac
Yugoslavia
miodragr@mi.sanu.ac.yu

(Received 08 12 1995)