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ON GENERALIZATION OF FUNCTIONS $n!$ AND $!n$

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Abstract. A sequence y_n is defined, and its relation to Đuro Kurepa's left factorial hypothesis is discussed. Also, a generalization of functions $n!$, $!n$ and y_n, a sequence $u_{n,m}$ is defined, and a number of its properties is proved.

1. Introduction. Kurepa in [**6**] defined $!n$ (left factorial) by:

$$!n = 0! + 1! + 2! + \cdots + (n-1)!, \qquad n \in N \tag{1.1}$$

and stated the hypothesis that

$$(!n, n!) = 2, \qquad \text{for} \qquad n > 1, \tag{KH}$$

where (a,b) denotes the greatest common divisor of integers a and b. In [**6**] was proved that (KH) is equivalent to assertion that

$$!p \not\equiv 0 \pmod{p}, \quad \text{for all primes} \quad p > 2 \tag{1.2}$$

and this is the usual form of KH.

In [**8**], [**9**], [**11**], [**12**] and [**13**] there are several statements equivalent to KH, which are all exposed in [**5**]. Here we cite, for example, the assertion proved in [**14**], that KH is equivalent to

$$\sum_{k=1}^{p-2} (k+1)^{p-k} k^{k-1} \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2 \tag{1.3}$$

KH is verified in [**2**] for $n < 10^6$. In this paper we will try to open some new possibilities for considering KH.

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2. The sequence y_n. Let $f(x) = \dfrac{e^{-x}}{1-x}$ and $n \in N \cup \{0\}$. We define a sequence y_n by:

$$y_n = f^{(n)}(0). \tag{2.1}$$

It is easy to see that

$$y_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!. \tag{2.2}$$

The first few members, are: $y_0 = 1$, $y_1 = 0$, $y_2 = 1$, $y_3 = 2$, $y_4 = 9$, $y_5 = 44, \ldots$. Let us notice that the sequence y_n has a combinatorial meaning to. Namely, y_n, $n > 0$, is the number of derangements of the set of n elements, i.e. number of permutations of an n-element set, in which no element is fixed. Let us establish some properties of the sequence y_n.

Proposition 2.1. *For every $n \in N$ we have*

$$y_n = ny_{n-1} + (-1)^n, \quad (2.3) \qquad \sum_{k=0}^{n} \binom{n}{k} y_k = n!, \quad (2.4)$$

$$\sum_{k=1}^{n} \binom{n}{k} y_{k-1} = !n, \quad (2.5) \qquad \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} (!k) = y_{n-1}. \quad (2.6)$$

Proof. Since $(1-x)f(x) = e^{-x}$, it follows that

$$(-1)^n = [(1-x)f(x)]_{x=0}^{(n)} = f^{(n)}(0) - nf^{(n-1)}(0) = y_n - ny_{n-1},$$

so the equality (2.3) is correct. Further we have:

$$\frac{1}{1-x} = e^x f(x) \Rightarrow n! = [e^x f(x)]_{x=0}^{(n)} = \sum_{k=0}^{n} \binom{n}{k} y_k,$$

$$!n = \sum_{k=0}^{n-1} k! = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{k}{i} y_i = \sum_{i=0}^{n-1} y_i \sum_{k=0}^{n-1} \binom{k}{i} = \sum_{i=0}^{n-1} \binom{n}{i+1} y_i,$$

i.e. the equalities (2.4) and (2.5) are correct. From (2.5) it follows that:

$$!n = u_1^{(n)}(0), \quad u_1(x) = e^x \int_0^x f(t)dt, \quad (!0 = 0) \tag{2.7}$$

and further $u_1(x)e^{-x} = \int_0^x f(t)dt \Rightarrow f^{(n-1)}(0) = y_{n-1} = [u_1(x)e^{-x}]_{x=0}^{(n)}$, i.e. the equality (2.6) also holds.

Proposition 2.2. *For every $n \in N$, and every $m \in N \cup \{0\}$ the following holds:*

$$\sum_{k=0}^{n} \binom{n}{k} y_{k+m} = n! \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \binom{n+i}{i} (i!). \tag{2.8}$$

The proof is easy and can be omitted.

Let us notice, that from proven equalities, one can obtain a number of congruences by module p, when $p \in P$, and P denotes the same set as above. From (2.3) it follows

$$y_p \equiv -1 \pmod{p}, \tag{2.9}$$

$$y_{p-1} + y_{p-2} \equiv 1 \pmod{p}, \tag{2.10}$$

Also, the substitution $n = p - 1$ in (2.4) give

$$\sum_{k=0}^{p-1} (-1)^k y_k \equiv -1 \pmod{p}, \tag{2.11}$$

Finally, by substitution $n = p$ in (2.6) and (2.8) we obtain

$$y_{p-1} \equiv\, !p \pmod{p} \tag{2.12}$$

$$y_m + y_{m+p} \equiv 0 \pmod{p}, \quad m \in N \cup \{0\} \tag{2.13}$$

Bearing in mind (1.2), one can, without great effort, formulate a number of assertions equivalent to KH. Really, according to congruences (2.9)-(2.13), KH is equivalent to every one of the following statements:

$$y_{p-1} \not\equiv 0 \pmod{p}, \qquad \text{for all primes } p \geq 3, \tag{2.14}$$

$$y_{p-2} \not\equiv 1 \pmod{p}, \qquad \text{for all primes } p \geq 3, \tag{2.15}$$

$$y_p \not\equiv -1 \pmod{p^2}, \quad \text{for all primes } p \geq 3, \tag{2.16}$$

$$\sum_{k=0}^{p-2} (-1)^k y_k \not\equiv -1 \pmod{p}, \quad \text{for all primes } p \geq 3. \tag{2.17}$$

Obviously, one can formulate a number of similar statements.

The sequence y_n can be represented in another way.

Proposition 2.3. *For every $n \in N$*

$$y_n = \left[\frac{n!}{e}\right] + \frac{1 + (-1)^n}{2}, \tag{2.18}$$

where $[x]$ denotes integer part of x, i.e. $[x] \in Z$ and $[x] \leq x < [x] + 1$.

Proof. From (2.2) it follows

$$
\begin{aligned}
y_n &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\Gamma(k+1) = \int_0^{+\infty} e^{-x}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^k dx \\
&= \int_0^{+\infty}(x-1)^n e^{-x}dx = \int_0^{1}(x-1)^n e^{-x}dx + \frac{1}{e}\int_0^{+\infty} t^n e^{-t}dt \\
&= \frac{n!}{e} + \int_0^{1}(x-1)^n e^{-x}dx.
\end{aligned}
$$

Since we have

$$
\Big|\int_0^{1}(x-1)^n e^{-x}dx\Big| \le \int_0^{1}|x-1|e^{-x}dx = \frac{1}{e}.
$$

It follows that $y_n = \left[\frac{n!}{e}\right]+1$, for n even and $y_n = \left[\frac{n!}{e}\right]$, for n odd and thus, the equality (2.18) holds.

Bearing in mind properties of the sequence y_n from (2.18) one immediately obtains

$$
\left[\frac{n!}{e}\right] = \left[\frac{e^{-x}}{1-x} - \frac{e^{x}+e^{-x}}{2}\right]^{(n)}_{x=0} \tag{2.19}
$$

Also, according to Proposition 2.1, it follows that for every $n \in N$, the following equalities hold:

$$
\left[\frac{n!}{e}\right] = n\left[\frac{(n-1)!}{e}\right] + \frac{1-(-1)^n}{2}(n-1), \tag{2.20}
$$

$$
\sum_{k=0}^{n}\binom{n}{k}\left[\frac{k!}{e}\right] = n! - 2^{n-1}, \tag{2.21}
$$

$$
\sum_{k=1}^{n}\binom{n}{k}\left[\frac{(k-1)!}{e}\right] = !n - 2^{n-1}. \tag{2.22}
$$

Bearing in mind (2.9)–(2.13), it follows that for every prime $p \ge 3$, the following congruences hold:

$$
\left[\frac{p!}{e}\right] \equiv -1 \pmod{p}, \tag{2.23}
$$

$$
\left[\frac{(p-1)!}{e}\right] + \left[\frac{(p-2)!}{e}\right] \equiv 0 \pmod{p}, \tag{2.24}
$$

$$\sum_{k=0}^{p-1}(-1)^k\left[\frac{k!}{e}\right]\equiv\frac{p-3}{2}\pmod p, \tag{2.25}$$

$$\left[\frac{(p-1)!}{e}\right]\equiv\, !p-1\pmod p, \tag{2.26}$$

$$\left[\frac{m!}{e}\right]+\left[\frac{(m+p)!}{e}\right]\equiv -1\pmod p,\ \ m\in N\cup\{0\}. \tag{2.27}$$

The statements (2.14)–(2.17) all equivalent to KH, could be reformulated similarly. From the discussion above, it is clear, that the sequence y_n is closely related to the functions $n!$ and $!n$.

3. The sequence $u_{n,m}$. Let $f(x)=\dfrac{e^{-x}}{1-x}$ and let $u_m(x)$, $m\in Z$, be the sequence of functions defined by:

$$u_m(x)=\begin{cases} e^x\int\limits_0^x dt_1\int\limits_0^{t_1}dt_2\cdots\int\limits_0^{t_{m-1}}f(t_m)dt_m, & m>0\\ e^x f^{(-m)}(x), & m\le 0\end{cases}. \tag{3.1}$$

The sequence of numbers $u_{n,m}$, $n\in N\cup\{0\}$, is defined by

$$u_{n,m}=u_m^{(n)}(0). \tag{3.2}$$

Let us first notice, that the sequence $u_{n,m}$, in some special cases represents the functions $n!$, $!n$ and y_n.

PROPOSITION 3.1. *For every $n\in N\cup\{0\}$ following equalities hold:*

$$u_{n,0}=n!, \tag{3.3}$$
$$u_{n,1}=\,!n,\quad (!0=0), \tag{3.4}$$
$$u_{0,-n}=y_n. \tag{3.5}$$

Proof. Referring to (2.1) and (3.1) and bearing in mind (2.7), it follows immediately that

$$u_{n,0}=\left[\frac{1}{1-x}\right]_{x=0}^{(n)}=n!,$$
$$u_{n,1}=\left[e^x\int\limits_0^x f(t)dt\right]_{x=0}^{(n)}=\,!n,$$
$$u_{0,-n}=\left[e^x f^{(n)}(x)\right]_{x=0}=f^{(n)}(0)=y_n,$$

which proves the assertion.

Considering further properties of the sequence $u_{n,m}$, let us show, that the sequence $u_{n,m}$ has some properties similar to those of binomial coefficients.

PROPOSITION 3.2. *For every $n \in N \cup \{0\}$ we have:*

$$u_{n,m} + u_{n,m+1} = u_{n+1,m+1}, \quad m \in Z, \tag{3.6}$$

$$m > n \Rightarrow u_{n,m} = 0, \quad m \in N, \tag{3.7}$$

$$u_{n,n} = 1, \tag{3.8}$$

$$u_{n,n-1} = n. \tag{3.9}$$

Proof. Referring to (3.1) we have

$$\begin{aligned} u_m(x) &= e^x (u_{m+1}(x) e^{-x})' = u'_{m+1}(x) - u_{m+1}(x) \Rightarrow \\ &\qquad u_{n,m} = u_m^{(n)}(0) = [u'_{m+1}(x) - u_{m+1}(x)]_{x=0}^{(n)} = u_{n+1,m+1} - u_{n,m+1}, \end{aligned}$$

and thus, the equality (3.6) holds.

Let $m > n$, then

$$\begin{aligned} u_m^{(n)}(x) &= e^x \sum_{k=0}^{n} \binom{n}{k} \Big[\int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} f(t_m) dt_m \Big]^{(k)} \\ &= e^x \sum_{k=0}^{n} \binom{n}{k} \int_0^x dt_{k+1} \cdots \int_0^{t_{m-1}} f(t_m) dt_m, \end{aligned}$$

and after substituting $x = 0$, we obtain (3.7).

Further, from (3.6) and (3.7) it follows

$$u_{n,n} = u_{n-1,n-1} + u_{n-1,n} = u_{n-1,n-1} = \cdots = u_{0,0} = 1$$

$$u_{n,n-1} = u_{n-1,n-2} + u_{n-1,n-1} = u_{n-1,n-2} + 1 = \cdots = u_{1,0} + (n-1) = n$$

and thus, the assertion is proved.

We can easily calculate members of the sequence $u_{n,m}$. For example, for $|m| < 5$ and $n < 6$, we have:

	$\cdots$ **−4**	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**	**4** $\cdots$
0	9	2	1	0	1	0	0	0	0
1	53	11	3	1	1	1	0	0	0
2	362	64	14	4	2	2	1	0	0
3	2790	426	78	18	6	4	3	1	0
4	24024	3216	504	96	24	10	7	4	1
5	229080	27240	3720	600	120	34	17	11	5

PROPOSITION 3.3. *For every $n \in N$, and every $m \in Z$, the following equalities hold:*

$$\sum_{k=m}^{n}(-1)^{k-m}u_{n,k} = u_{n-1,m-1}, \tag{3.10}$$

$$\sum_{k=0}^{n-1}u_{k,m} = u_{n,m+1} - u_{0,m+1}. \tag{3.11}$$

Proof. Referring to (3.6) and (3.7) we obtain

$$\begin{aligned}\sum_{k=m}^{n}(-1)^{k-m}u_{n,k} &= \sum_{k=m}^{n}(-1)^{k-m}(u_{n-1,k-1} - u_{n-1,k})\\ &= u_{n-1,m-1} + \sum_{k=m+1}^{n}(-1)^{k-m}u_{n-1,k-1} + (-1)^{n-m}u_{n-1,n} + \sum_{k=m}^{n-1}(-1)^{k-m}u_{n-1,k}\\ &= u_{n-1,m-1} + \sum_{k=m}^{n-1}(-1)^{k-m+1}u_{n-1,k} + \sum_{k=m}^{n-1}(-1)^{k-m}u_{n-1,k} = u_{n-1,m-1}.\end{aligned}$$

Also

$$\sum_{k=0}^{n-1}u_{k,m} = \sum_{k=0}^{n-1}(u_{k+1,m+1} - u_{k,m+1}) = \sum_{k=1}^{n}u_{k,m+1} - \sum_{k=0}^{n-1}u_{k,m+1} = u_{n,m+1} - u_{0,m+1}.$$

and that proves the proposition.

PROPOSITION 3.4. *For every $k, n \in N \cup \{0\}$ and every $m \in Z$, the following equalities hold:*

$$\sum_{i=0}^{n}(-1)^{n-1}\binom{n}{i}u_{k+i,m} = u_{k,m-n}, \tag{3.12}$$

$$\sum_{i=0}^{n}\binom{n}{i}u_{k,m-i} = u_{n+k,m}. \tag{3.13}$$

Proof. According to (3.1), we have

$$u_{m-n}(x) = e^{x}(u_m(x)e^{-x})^{(n)} = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}u_m^{(i)}(x).$$

Differentiating the last equality k times and substituting $x = 0$ we obtain (3.12). Further we have

$$u_m^{(n)}(x) = \sum_{i=0}^{n}\binom{n}{i}u_{m-i}(x) \Rightarrow u_m^{(n+k)}(x) = \sum_{i=0}^{n}\binom{n}{i}u_{m-i}^{(k)}(x),$$

and by substituting $x = 0$ we obtain (3.13).

Let us notice, that by substituting $k = 0$ in (3.13) and considering (3.5) and (3.7) we obtain

$$u_{n,m} = \sum_{i=0}^{n} \binom{n}{i} u_{0,m-i} = \sum_{i=0}^{n} \binom{n}{i} y_{i-m} \quad (m \leq 0), \text{ i.e.}$$
$$u_{n,m} = \sum_{i=0}^{m-1} \binom{n}{i} u_{0,m-i} + \sum_{i=m}^{n} \binom{n}{i} u_{0,m-i} = \sum_{i=m}^{n} \binom{n}{i} y_{i-m}, \quad (0 < m \leq n).$$

We can conclude that, for every $n \in N \cup \{0\}$ and every $m \in Z$, $m \leq n$ holds

$$u_{n,m} = \sum_{i=s}^{n} \binom{n}{i} y_{i-m}, \qquad s = \max(0, m). \tag{3.14}$$

Referring to (2.8), substitution $m = -k$, $k \in N \cup \{0\}$ in (3.14), gives

$$u_{n,-k} = \sum_{i=0}^{n} \binom{n}{i} y_{i+k} = n! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{n+i}{i} i!. \tag{3.15}$$

Especially, for $k = 1$ we obtain

$$u_{n,-1} = n \cdot n!. \tag{3.16}$$

It is obvious that for $m \leq 0$

$$u_{n,m} \equiv 0 \pmod{n!}. \tag{3.17}$$

Substitution $n = p$, $p \in P$ in (3.13), gives us

$$u_{k,m} + u_{k,m-p} \equiv u_{p+k,m} \pmod{p}. \tag{3.18}$$

According to (3.17) and (3.18), for $m \leq p \leq k$, we obtain

$$u_{k,m} \equiv u_{p+k,m} \pmod{p}. \tag{3.19}$$

Let us notice, that for $n = p + 1$, $p \in P$, $p > 2$, in (3.14), referring to (2.10) we obtain:

$$u_{p+1,2} \equiv 1 \pmod{p}. \tag{3.20}$$

At last, let us prove two recurrent formulas for the sequence $u_{n,m}$.

Proposition 3.5. *For every $n \in N$ and every $m \in Z$ following equality holds*

$$(m-1)u_{n,m} = (n - m + 1)u_{n,m-1} - u_{n,m-2} + a(n, m), \tag{3.21}$$

where

$$a(n,m) = \begin{cases} \binom{n}{m-2}, & m \geq 2 \\ 0, & m < 2 \end{cases}.$$

Proof. For $m = 1$, the equality holds, since (3.21) reduces to (3.16). For $m < 1$, according to (2.3), (3.6) and (3.14), we obtain

$$\begin{aligned} u_{n,m-1} + u_{n,m-2} = u_{n+1,m-1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} y_{i-m+1} \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} (i-m+1) y_{i-m} + (-1)^{1-m} \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i \\ &= (n+1) \sum_{i=1}^{n+1} \binom{n}{i-1} y_{i-m} - (m-1) \sum_{i=0}^{n+1} \binom{n+1}{i} y_{i-m} \\ &= (n+1) u_{n,m-1} - (m-1)(u_{n,m} + u_{n,m-1}). \end{aligned}$$

and, after putting this in order, we obtain (3.21).

For $m > 1$, we have

$$\begin{aligned} u_{n,m-1} + u_{n,m-2} = u_{n+1,m-1} &= \sum_{i=m-1}^{n+1} \binom{n+1}{i} y_{i-m+1} \\ &= \binom{n+1}{m-1} + \sum_{i=m}^{n+1} \binom{n+1}{i} y_{i-m}(i-m+1) + (-1)^{1-m} \sum_{i=m}^{n+1} \binom{n+1}{i} (-1)^i \\ &= \binom{n+1}{m-1} + (n+1) \sum_{i=m}^{n+1} \binom{n}{i-1} y_{i-m} - (m-1) \sum_{i=m}^{n+1} \binom{n+1}{i} y_{i-m} \\ &\qquad + (-1)^m \sum_{i=0}^{m-1} \binom{n+1}{i} (-1)^i \\ &= \binom{n+1}{m-1} + (n+1) \sum_{i=m-1}^{n} \binom{n}{i} y_{i-m+1} - (m-1) u_{n+1,m} - \binom{n}{m-1} \\ &= \binom{n}{m-2} + (n+1) u_{n,m-1} - (m-1)(u_{n,m} + u_{n,m-1}), \end{aligned}$$

and, after putting this in order, we obtain (3.21).

Proposition 3.6. *For every $n \in N \cup \{0\}$ and every $m \in Z$, the following equality holds:*

$$u_{n+2,m} = (n-m+3) u_{n+1,m} - (n+1) u_{n,m} + a(n,m). \tag{3.22}$$

Proof. According to (3.6), (3.12) and (3.21), we obtain

$$\begin{aligned} u_{n+2,m} - 2u_{n+1,m} + u_{n,m} &= u_{n,m-2} \\ &= (n-m+1)u_{n,m-1} - (m-1)u_{n,m} + a(n,m) \\ &= (n-m+1)(u_{n+1,m} - u_{n,m}) - (m-1)u_{n,m} + a(n,m), \end{aligned}$$

and, after putting this in order, we get (3.22).

Notice that, according to (3.4) and (3.6), after substituting $m = 2$ in (3.22), we obtain

$$u_{n+2,2} = (n+1)(!n) + 1. \tag{3.23}$$

From properties of the sequence $u_{n,m}$, it is clear that one can state number of assertions equivalent to KH. For example KH is equivalent to all following assertions:

$$(\exists k)(k \geq p \wedge u_{k,2} \not\equiv 1 \pmod{p}), \quad \text{for all primes } p > 2, \tag{3.24}$$

$$u_{p-1,2} \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2, \tag{3.25}$$

$$u_{p-2,2} \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2, \tag{3.26}$$

$$u_{p+1,2} \not\equiv p + 1 \pmod{p^2}, \quad \text{for all primes } p > 2. \tag{3.27}$$

Really, (3.24) and (3.27) are direct corollaries of (1.2) and (3.23), while (3.6) implies (3.25) and (3.26).

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