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m-FAMILIES AND COMMON TRANSVERSALS

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Abstract. A classical separation theorem was generalized in R^n, for m-families of open convex sets by Klee [**5**] and for m-families of compact convex sets by Gallivan and Zaks [**2**]. In our paper we extend these results to topological vector spaces. Finally, two applications are pointed out.

1. Introduction. A collection $\mathcal{A}$ of $m+1$ subsets of a vector space E will be called an m-family if $\bigcap \mathcal{A} = \emptyset$, but each m members of $\mathcal{A}$ have a common point.

From Helly's theorem [**4**], it immediately follows that, if E is finite-dimensional then, for $m > \dim E$, there is no m-family of convex sets in E.

It is well known that if $\{A_1, A_2\}$ is a 1-family of convex sets in a finite-demensional Euclidean space, then A_1 and A_2 can be separated by a hyperplane. This result was generalized to R^n, for m-families of open convex sets by Klee [**5**] and for m-families of compact convex sets by Gallivan and Zaks [**2**]. We present, in the mentioned order, the following two generalizations.

Theorem 1. *If $\{A_1, A_2, \dots, A_{m+1}\}$ is an m-family of open convex sets in R^n, then there is a flat L of deficiency m such that:*

(i) $L \cap \left(\bigcup_{i=1}^{m+1} A_i\right) = \emptyset$;

(ii) *if F is any flat of deficiency $m-1$ which contains L, and S is either of the half-spaces into which L separates F, then S intersects some set A_i.*

Theorem 2. *If $\{A_1, A_2, \dots, A_{m+1}\}$ is an m-family of compact convex sets in R^n, then there is a flat L of deficiency m such that:*

$$\text{(i)} \quad L \cap \left(\bigcup_{i=1}^{m+1} A_i\right) = \emptyset; \qquad \text{(ii)} \quad L \cap \operatorname{conv}\left(\bigcup_{i=1}^{m+1} A_i\right) \neq \emptyset.$$

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In Theorem 3 we shall show that the conclusion of Theorem 2 holds if $\{A_1, A_2, \ldots, A_{m+1}\}$ is an m-family of open convex subsets of a topological vector space or an m-family of closed convex subsets of a locally convex space, at least one being compact.

A well-known consequence of Helly's theorem asserts that for a finite family $\mathcal{A}$ of convex sets and for a convex set L in R^n, the existence of some translate of L which intersects all members of $\mathcal{A}$ is guaranteed by the existence of such a translate for each $n+1$ members of $\mathcal{A}$. When L is a flat we shall prove (Theorem 4) that the integer $n+1$ can be reduced.

Finally, we give two applications of Theorem 3, closely related to Theorem 4.

Throughout this paper we assume that the vector spaces (and implicitly the topological vector spaces) are real.

2. Results. The main result is the following

THEOREM 3. *Let $\{A_1, A_2, \ldots, A_{m+1}\}$ be an m-family of (a) open convex subsets of a topological vector space E, or (b) closed convex subsets of a locally convex space E, at least one being compact.*

Then there is a closed flat L in E, of deficiency m, such that:

$$\text{(i)} \qquad L \cap \left(\bigcup_{i=1}^{m+1} A_i \right) = \emptyset; \qquad\qquad \text{(ii)} \qquad L \cap \operatorname{conv} \left(\bigcup_{i=1}^{m+1} A_i \right) \neq \emptyset.$$

Proof. The proof is by induction on m. In the case $m = 1$ the required flat is a hyperplane. For two disjoint convex sets A_1, A_2 both open in a topological vector space or both closed and at least one compact in a locally convex space, by two standard separation theorems [**1**, pp. 32 and 64] there is a closed hyperplane that strictly separates A_1 from A_2.

Assume inductively that the statement is true for $m-1$ ($m \geq 2$) and let $\{A_1, A_2, \ldots, A_{m+1}\}$ be an m-family as described in the theorem.

In the case (b) we suppose, without loss of generality, that at least one of the sets $A_1, A_2, \ldots, A_m$ is compact.

Since $\bigcap_{i=1}^{m+1} A_i = \emptyset$, A_{m+1} and $\bigcap_{i=1}^{m} A_i$ are two nonempty disjoint convex sets, both open in a topological vector space in the case (a), respectively both closed, at least one compact ($\bigcap_{i=1}^{m} A_i$), in a locally convex space, in the case (b). Therefore there exists a closed hyperplane H that strictly separates them, i.e. $A_{m+1} \subset H^-$ and $\bigcap_{i=1}^{m} A_i \subset H^+$, where H^- and H^+ are the two open half-spaces determined by H.

Obviously, when E is a locally convex space, H is a locally convex space too, with respect to the induced topology.

Define $A_i^* = A_i \cap H$, for $1 \leq i \leq m$. Obviously the sets A_i^* are open in the case (a), respectively closed and at least one compact in the case (b). We shall prove that $\{A_1^*, A_2^*, \ldots, A_m^*\}$ is an $(m-1)$-family in H.

Let T^* be an arbitrary intersection of $m-1$ sets A_i^* $(1 \leq i \leq m)$ and let T be the intersection of the corresponding sets A_i. Clearly $T^* = T \cap H$. Then $T \cap A_{m+1}$ is the intersection of m members of the m-family $\{A_1, A_2, \ldots, A_{m+1}\}$, hence $T \cap A_{m+1} \neq \emptyset$. This relation and the inclusion $A_{m+1} \subset H^-$ imply $T \cap H^- \neq \emptyset$.

From $\bigcap_{i=1}^m A_i \subset H^+$ and $\cap_{i=1}^m A_i \subset T$ we obtain $T \cap H^+ \neq \emptyset$. Since T is a convex set, from $T \cap H^+ \neq \emptyset$ and $T \cap H^- \neq \emptyset$, it follows that $T \cap H \neq \emptyset$. Therefore $T^* = T \cap H \neq \emptyset$.

The inductive assumption is now applicable to the $(m-1)$-family $\{A_1^*, A_2^*, \ldots, A_m^*\}$ in H. Thus there is a closed flat L of deficiency $m-1$ in H, hence of deficiency m in E, such that:

$$L \cap \left(\bigcup_{i=1}^m A_i^* \right) = \emptyset, \qquad L \cap \operatorname{conv} \left(\bigcup_{i=1}^m A_i^* \right) \neq \emptyset.$$

Since $A_{m+1} \subset H^-$, $L \subset H$ and $A_i^* = A_i \cap H$ it follows that $L \cap (\bigcup_{i=1}^{m+1} A_i) = \emptyset$. Also

$$L \cap \operatorname{conv} \left(\bigcup_{i=1}^{m+1} A_i \right) \supset L \cap \operatorname{conv} \left(\bigcup_{i=1}^{m} A_i \right) \supset L \cap \operatorname{conv} \left(\bigcup_{i=1}^{m} A_i^* \right) \neq \emptyset$$

and the proof is complete.

Theorem 4. *Let $\mathcal{A}$ be a finite family of at least $m+1$ convex subsets of a vector space E and L a vector subspace of deficiency m. Suppose that, given any $m+1$ members of $\mathcal{A}$, there exists a translate of L intersecting all of them. Then there exists a translate of L which intersects all members of $\mathcal{A}$.*

Moreover, if E is a Hausdorff topological vector space, the subspace L is closed and the members of the family $\mathcal{A}$ are compact sets, then the conclusion holds for infinite families $\mathcal{A}$.

Proof. Let K be a complementary subspace of L in E. Then K is m-dimensional.

Each point $x \in E$ has unique representation of the form $x = u + v$, with $u \in L$, $v \in K$. Let $p: E \to K$ be the projection of E on K parallel to L, i.e. $p(x) = v$. Denote by $\mathcal{B}$ the family $\{p(A): A \in \mathcal{A}\}$. Since p is a linear mapping, each $B \in \mathcal{B}$ is a convex set.

Also, since each $m+1$ members of $\mathcal{A}$ are intersected by a translate of L it follows that each $m+1$ members of $\mathcal{B}$ have a common point. By Helly's theorem, there exists a point $x \in \bigcap \mathcal{B}$. Then $x + L$ is a translate of L which intersects all members of $\mathcal{A}$.

For the topological variant we remark that in the given conditions, the mapping p is continuous (see [**7**, p. 17]. Therefore the members of $\mathcal{B}$ are compact sets in K, with respect to the induced topology. Since K is finite dimensional and the induced topology is separated, this coincides with the Euclidean topology (see [**8**, p. 52]).

Thus, the second part of the theorem has an analogous proof using the variant of Helly's theorem concerning families of compact convex sets of arbitrary cardinal.

Simple examples show that the existence of some translate of L which intersects all members of $\mathcal{A}$ is not assured by the assumption of the same property for all subfamilies with m members of $\mathcal{A}$. Some information in this case we shall get in the next theorems.

THEOREM 5. *Let E be a Hilbert space and L a closed subspace of E having the deficiency at least m. Let $\{A_1, A_2, \dots, A_{m+1}\}$ be a family of convex subsets of E, all open or all compact. Suppose that for each $i \in \{1, 2, \dots, m+1\}$ there is a translate of L, denoted by L_i, such that:*

(a) $L_i \cap A_i = \emptyset$; (b) $L_i \cap A_j \neq \emptyset$, *for each* $j \in \{1, 2, \dots, m+1\} \backslash \{i\}$.

Then there is a closed flat F of deficiency m, parallel to L, satisfying:

$$\text{(i)} \qquad F \cap \left(\bigcup_{i=1}^{m+1} A_i \right) = \emptyset; \qquad\qquad \text{(ii)} \qquad F \cap \operatorname{conv} \left(\bigcup_{i=1}^{m+1} A_i \right) \neq \emptyset.$$

Proof. Let $L^\perp$ be the orthogonal complement of L and $p\colon E \to L^\perp$ the orthogonal projection on $L^\perp$. Using the closed graph theorem it immediately follows that p is continuous (see [**9**, p. 180]). In these conditions p is an open mapping (see [**3**, p. 127]). The same conclusion can be obtained by using the open mapping theorem (see [**10**, p. 75]).

Denote by $B_i = p(A_i)$, $1 \le i \le m+1$. Since p is a linear mapping, each B_i is convex. From (a) and (b) it follows that $\{B_1, B_2, \dots, B_{m+1}\}$ is a m-family in $L^\perp$. Moreover, the sets B_i are all open or all compact.

Thus, by Theorem 3, there exists a closed flat G of deficiency m in $L^\perp$, such that

$$G \cap \left(\bigcup_{i=1}^{m+1} B_i \right) = \emptyset \quad \text{and} \quad G \cap \operatorname{conv} \left(\bigcup_{i=1}^{m+1} B_i \right) \neq \emptyset.$$

Let F be the flat $L + G$. It is easy to check that if K is a complement of G in $L^\perp$, then K is also a complement of F in E, hence F is a flat of deficiency m. Since L is closed and G is closed in $L^\perp$, it is also easy to see that $F = L + G$ is closed.

From $p(A_i \cap F) \subset p(A_i) \cap p(F) = B_i \cap G = \emptyset$ we obtain $A_i \cap F = \emptyset$, for each $i \in \{1, 2, \dots, m+1\}$, hence $F \cap (\bigcup_{i=1}^{m+1} A_i) = \emptyset$.

Let $y \in G \cap \operatorname{conv} (\bigcup_{i=1}^{m+1} B_i)$. Then $y = \sum_{i=1}^{m+1} \lambda_i y_i$, where $y_i \in B_i$, $\lambda_i \ge 0$, $\sum_{i=1}^{m+1} \lambda_i = 1$. For each $i \in \{1, 2, \dots, m+1\}$ we choose an element $x_i \in A_i$ such that $y_i = p(x_i)$ and denote by x the convex combination $\sum_{i=1}^{m+1} \lambda_i x_i$. Obviously $x \in \operatorname{conv}(\bigcup_{i=1}^{m+1} A_i)$ and since $p(x) = y \in G$, it follows that $x = (x - p(x)) + p(x) \in L + G = F$.

THEOREM 6. *Let E be a Hausdorff locally convex space and L a vector subspace of E of deficiency at least m. Suppose that L is either finite-dimensional or closed and of finite deficiency. Let $\{A_1, A_2, \ldots, A_{m+1}\}$ be a family of convex subsets of E, all open or all compact. If for each $i \in \{1, 2, \ldots, m+1\}$ there is a translate of L, denoted by L_i, such that*

(a) $L_i \cap A_i = \emptyset$,

(b) $L_i \cap A_j \neq \emptyset$, *for each $j \in \{1, 2, \ldots, m+1\} \setminus \{i\}$, then there is a closed flat F of deficiency m, parallel to L, satisfying:*

$$\text{(i)} \quad F \cap \left(\bigcup_{i=1}^{m+1} A_i \right) = \emptyset; \qquad \text{(ii)} \quad F \cap \operatorname{conv} \left(\bigcup_{i=1}^{m+1} A_i \right) \neq \emptyset.$$

Proof. In the conditions of the theorem, L is topologically complementable, i.e., there exists a closed complement K of L(see [**6**, p. 191]). Then the projection of E on K parallel to L, denoted by p, is continuous and an open mapping (see [**3**, p. 127]). Denoting $B_i = p(A_i)$, $1 \le i \le m+1$, further the same reasoning, as in the previous proof, works.

Note that if G is the flat of K guaranteed by Theorem 3, under the assumptions of Theorem 6, at least one of the closed flats L and G is finite-dimensional, hence $F = L + G$ is closed (see [**7**, p. 13]).

Remark. The assertion of the previous theorem holds if E is a Hausdorff topological vector space, L is a closed subspace of finite deficiency and the sets A_i, $1 \le i \le m+1$, are open. The proof in this case is analogous.

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