PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 60 (74), 1996, 88–100

GENERALIZED CONNECTION ON $T(T^2M)$

Irena Čomić

Communicated by Mileva Prvanović

Abstract. The geometry of some manifolds fibered over a given manifold M is in the first place characterized by the group of allowable coordinate transformations. For the tangent manifold TM these are given by $x^{i'} = x^{i'}(x) y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i$, rank $\left[\frac{\partial x^{i'}}{\partial x^i}\right] = n$, and for the total space of a vector bundle $E \to M$, we have $x^{i'} = x^{i'}(x)$, $y^{a'} = M_a^{a'}(x) y^a$, $\operatorname{rank}(M_a^{a'}) = m =$ dimension of type fiber.

In the last years R. Miron, Gh. Atanasiu and others examined the $\operatorname{Osc}^k M$ spaces, [**10**], [**11**], [**12**]. Here the case $k = 2$ will be investigated. Instead of $\operatorname{Osc}^2 M$ the notation T^2M will be used ($\operatorname{Osc}^1 M$ coincides with TM). Instead of d-connection used in [**10**], [**11**], [**12**], we consider here the generalized connection and determine its torsion tensor. As a special case the known d-connection is obtained.

1. Adapted basis in $T(T^2M)$. Let T^2M be a $3n$ dimensional C^∞ manifold. A point $u \in T^2M$ in the local charts (U, φ) and (U', φ') has coordinates (x^i, y^i, z^i) and $(x^{i'}, y^{i'}, z^{i'})$ respectively. In $U \cap U'$ the allowed coordinate transformations are given by the equations:

$$(1.1) \qquad \text{(a)}\ x^{i'} = x^{i'}(x) \quad \text{(b)}\ y^{i'} = \frac{\partial x^{i'}}{\partial x^j} y^j \quad \text{(c)}\ z^{i'} = \frac{1}{2} \frac{\partial^2 x^{i'}}{\partial x^k \partial x^j} y^k y^j + \frac{\partial x^{i'}}{\partial x^j} z^j.$$

If rank $\left[\frac{\partial x^{i'}}{\partial x^i}\right] = n$, then the inverse transformation of (1.1) exists:

$$(a)\ x^i = x^i(x^{i'}) \quad (b)\ y^i = \frac{\partial x^i}{\partial x^{j'}} y^{j'} \quad (c)\ z^k = \frac{1}{2} \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} y^{i'} y^{j'} + \frac{\partial x^k}{\partial x^{i'}} z^{i'}.$$

(1.2) (a) and (1.2) (b) are obvious. To obtain (1.2) (c) we start from

$$(1.2) \qquad \frac{\partial x^{i'}}{\partial x^h} \frac{\partial x^k}{\partial x^{i'}} = \delta_h^k.$$

AMS Subject Classification (1991): Primary 53B25; Secondary 53B40
Key Words and Phrases: Second order connection, Torsion tensor

From (1.2) it follows:

$$\frac{\partial^2 x^{i'}}{\partial x^h \partial x^j}\frac{\partial x^k}{\partial x^{i'}} + \frac{\partial x^{i'}}{\partial x^h}\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^j} = 0.$$

The multiplication of the above equation with $y^h y^j$ gives

$$\frac{\partial^2 x^{i'}}{\partial x^h \partial x^j} y^h y^j \frac{\partial x^k}{\partial x^{i'}} = -\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}} y^{i'} y^{j'}. \tag{1.3}$$

If we multiply (1.1) (c) by $\partial x^k/\partial x^{i'}$, use (1.3) and (1.4) we obtain (1.2) (c).

The identity transformation is a special case of (1.1), namely if we put $x^{i'} = x^i$ in (1.1) (a), then $y^{i'} = y^i$, $z^{i'} = z^i$ follows.

If in the local chart (U'', φ'') the point u has coordinates $(x^{i''}, y^{i''}, z^{i''})$, then in $(U'' \cap U', \varphi'')$ (1.1) are valid, if the index i' is substituted by i'' and the indices without $'$ obtain $'$. After some calculation it can be obtained, that the connection between $(x^{i''}, y^{i''}, z^{i''})$ and (x^i, y^i, z^i) in $U \cap U' \cap U''$ is given by (1.1) if the index i' is substituted by i''.

From the above follows:

Theorem 1.1. *The transformations of type* (1.1) *form a group.*

In $T(T^2M)$ the natural bases are:

$$\overline{B} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^i}\right\} \text{ and } \overline{B}' = \left\{\frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial y^{i'}}, \frac{\partial}{\partial z^{i'}}\right\}. \tag{1.4}$$

The bases vectors of $\overline{B}$ and $\overline{B}'$ are connected by:

$$\begin{aligned}
\frac{\partial}{\partial x^i} &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial x^{i'}} + \frac{\partial y^{i'}}{\partial x^i}\frac{\partial}{\partial y^{i'}} + \frac{\partial z^{i'}}{\partial x^i}\frac{\partial}{\partial z^{i'}},\\
\frac{\partial}{\partial y^i} &= \frac{\partial y^{i'}}{\partial y^i}\frac{\partial}{\partial y^{i'}} + \frac{\partial z^{i'}}{\partial y^i}\frac{\partial}{\partial z^{i'}},\\
\frac{\partial}{\partial z^i} &= \frac{\partial z^{i'}}{\partial z^i}\frac{\partial}{\partial z^{i'}}.
\end{aligned} \tag{1.5}$$

From (1.1) (c) it follows

$$\frac{\partial}{\partial z^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial z^{i'}}. \tag{1.6}$$

Let us introduce the notation

$$\frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \mathcal{H}_i^j(x,y)\frac{\partial}{\partial z^i}. \tag{1.7}$$

Proposition 1.1. $\frac{\delta}{\delta y^i}$ *defined by* (1.8) *is transformed as tensor, i.e.*

$$\frac{\delta}{\delta y^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\delta}{\delta y^{i'}} \tag{1.8}$$

if $\mathcal{H}^i_j(x,y)$ *is transformed in the following way:*

$$\mathcal{H}^{j'}_{i'} = \mathcal{H}^j_i \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j} - \frac{\partial^2 x^{j'}}{\partial x^i \partial x^j} y^j \frac{\partial x^i}{\partial x^{i'}}. \tag{1.9}$$

The proof is obtained by direct calculation.

From (1.9) and (1.10) we obtain the first connection coefficient of Berwald type:

$$\mathcal{H}_{i'}{}^{j'}{}_{k'} = \frac{\partial \mathcal{H}^{j'}_{i'}}{\partial y^{k'}} = \frac{\partial \mathcal{H}^j_i}{\partial y^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^k}{\partial x^{k'}} - \frac{\partial^2 x^{j'}}{\partial x^i \partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}. \tag{1.10}$$

Let us introduce the notation:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{M}^j_i(x,y)\frac{\partial}{\partial y^j} - \mathcal{N}^j_i(x,y,z)\frac{\partial}{\partial z^j}. \tag{1.11}$$

Proposition 1.2. $\frac{\delta}{\delta x^i}$ *defined by* (1.12) *is transformed in the form:*

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\delta}{\delta x^{i'}}, \tag{1.12}$$

if $\mathcal{M}^j_i(x,y)$ *and* $\mathcal{N}^j_i(x,y,z)$ *are transformed in the following way:*

$$\mathcal{M}^{j'}_{i'} = \mathcal{M}^j_i \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j} - \frac{\partial^2 x^{j'}}{\partial x^i \partial x^j} y^j \frac{\partial x^i}{\partial x^{i'}}, \tag{1.13}$$

$$\begin{aligned}\mathcal{N}^{j'}_{i'} &= \mathcal{N}^j_i \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j} - \frac{\partial^2 x^{j'}}{\partial x^i \partial x^j} z^j \frac{\partial x^i}{\partial x^{i'}} \\ &\quad - \frac{1}{2}\frac{\partial^3 x^{j'}}{\partial x^i \partial x^h \partial^k} y^h y^k \frac{\partial x^i}{\partial x^{i'}} + \frac{\partial^2 x^{j'}}{\partial x^k \partial x^j} y^k \mathcal{M}^j_h \frac{\partial x^h}{\partial x^{i'}}.\end{aligned} \tag{1.14}$$

Remark. $\mathcal{M}^j_i$ and $\mathcal{N}^j_i$ used here in [**10**], [**11**], [**12**] are denoted by $(1)\mathcal{N}^j_i$ and $(2)\mathcal{N}^j_i$ respectively.

Proof. If we add the following equations (which follow from (1.6), (1.1) and (1.3)):

$$\begin{aligned}\frac{\partial}{\partial x^i} &= \left(\frac{\partial x^{i'}}{\partial x^i}\right)\frac{\partial}{\partial x^{i'}} + \left(\frac{\partial^2 x^{j'}}{\partial x^k \partial x^j} y^j \frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^i}\right)\frac{\partial}{\partial y^{j'}} \\ &\quad + \left(\frac{1}{2}\frac{\partial^3 x^{j'}}{\partial x^h \partial x^k \partial x^j} y^k y^j \frac{\partial x^h}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^i} + \frac{\partial^2 x^{j'}}{\partial x^k \partial x^j} z^j \frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^i}\right)\frac{\partial}{\partial z^{j'}}, \\ -\mathcal{M}^j_i \frac{\partial}{\partial y^j} &= -\mathcal{M}^j_h \frac{\partial x^h}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^i}\left(\frac{\partial x^{j'}}{\partial x^j}\frac{\partial}{\partial y^{j'}} + \frac{\partial^2 x^{j'}}{\partial x^k \partial x^j} y^k \frac{\partial}{\partial z^{j'}}\right), \\ -\mathcal{N}^j_i \frac{\partial}{\partial z^j} &= -\mathcal{N}^j_h \frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^h}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial z^{j'}}\ ,\end{aligned}$$

we obtain $\dfrac{\delta}{\delta x^i} = \dfrac{\partial x^{i'}}{\partial x^i}\left(\dfrac{\partial}{\partial x^{i'}} - \mathcal{M}_{i'}^{j'}\dfrac{\partial}{\partial y^{j'}} - \mathcal{N}_{i'}^{j'}\dfrac{\partial}{\partial z^{j'}}\right) = \dfrac{\partial x^{i'}}{\partial x^i}\dfrac{\delta}{\delta x^{i'}}$, where $\mathcal{M}_{i'}^{j'}$ is given by (1.14) and $\mathcal{N}_{i'}^{j'}$ by (1.15).

The basis

$$B = \left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial z^i}\right\} \tag{1.15}$$

is called *adapted basis* for $T(T^2M)$. Its elements are transformed by (1.7), (1.9) and (1.13).

From (1.14) and (1.15) we obtain two other connection coefficients of Berwald type, namely

$$\mathcal{M}_{i'}{}^{j'}{}_{k'} = \frac{\partial \mathcal{M}_{i'}^{j'}}{\partial x^{k'}} = \mathcal{M}_{i}{}^{j}{}_{k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{j'}}{\partial x^j} - \frac{\partial^2 x^{j'}}{\partial x^i \partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}. \tag{1.16}$$

$$\mathcal{N}_{i'}{}^{j'}{}_{k'} = \frac{\partial \mathcal{N}_{i'}^{j'}}{\partial z^{k'}} = \mathcal{N}_{i}{}^{j}{}_{k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{j'}}{\partial x^j} - \frac{\partial^2 x^{j'}}{\partial x^i \partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}. \tag{1.17}$$

2. The adapted basis in $T^*(T^2M)$. The natural basis of $T^*(T^2M)$ is

$$\overline{B}^* = \{dx^i, dy^i, dz^i\}, \tag{2.1}$$

where the following relations are valid with respect to (1.1):

$$(a)\ dx^{i'} = \frac{\partial x^{i'}}{\partial x^i}dx^i \quad (b)\ dy^{i'} = \frac{\partial^2 x^{i'}}{\partial x^j \partial x^i}y^i dx^j + \frac{\partial x^{i'}}{\partial x^i}dy^i \tag{2.2}$$

$$(c)\ dz^{i'} = \left(\frac{1}{2}\frac{\partial^3 x^{i'}}{\partial x^k \partial x^h \partial x^j}y^k y^h + \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}z^i\right)dx^j + \left(\frac{\partial^2 x^{i'}}{\partial x^k \partial x^j}y^k\right)dy^j + \left(\frac{\partial x^{i'}}{\partial x^j}\right)dz^j.$$

From (2.2) (b) and (2.2) (c) it is obvious, that the transformations of $dy^{i'}$ and $dz^{i'}$ are not tensorial. If we put

$$\delta y^i = dy^i + \mathcal{H}_j^i dx^j \tag{2.3}$$

then from (1.10) and (2.1) we get

$$\delta y^{i'} = \frac{\partial x^{i'}}{\partial x^i}\delta y^i. \tag{2.4}$$

Let us introduce the notation

$$\delta z^i = dz^i + \mathcal{M}_j^i(x,y)dy^j + \mathcal{N}_j^i(x,y,z)dx^j + \mathcal{G}_j^i(x,y)dx^j, \tag{2.5}$$

where the transformation laws of $\mathcal{M}_j^i$ and $\mathcal{N}_j^i$ are prescribed by (1.14) and (1.15).

Proposition 2.1. *If* $\mathcal{G}_j^i(x,y)$ *has the following law of transformation* (*with respect to* (1.1)):

$$\mathcal{G}_j^i = \mathcal{G}_{j'}^{i'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} + \mathcal{M}_{j'}^{i'} \frac{\partial^2 x^{j'}}{\partial x^k \partial x^j} y^k \frac{\partial x^i}{\partial x^{i'}} + \mathcal{M}_j^k \frac{\partial^2 x^{j'}}{\partial x^k \partial x^h} y^h \frac{\partial x^i}{\partial x^{j'}}, \tag{2.6}$$

then δz^i *defined by* (2.5) *satisfies the equation*

$$\delta z^{j'} = \frac{\partial x^{j'}}{\partial x^j} \delta z^j \tag{2.7}$$

Proof. The substitution of (2.2), (1.14) and (1.15) into $\delta z^{j'} = dz^{j'} + \mathcal{M}_{i'}^{j'} dy^{i'} + \mathcal{N}_{i'}^{j'} dx^{i'} + \mathcal{G}_{i'}^{j'} dx^{i'}$ gives (2.7) if (2.6) is true.

The adapted basis of $T^*(T^2F)$ is

$$B^* = \{dx^i, \delta y^i, \delta z^i\}, \tag{2.8}$$

where the elements of B^* defined by (2.3) and (2.5) satisfy the transformation law prescribed by (2.2) (a), (2.4) and (2.7).

Theorem 2.1. *The bases* B (1.16) *and* B^* (2.8) *of* $T(T^2M)$ *and* $T^*(T^2M)$ *respectively are dual to each other if*

$$\mathcal{H}_j^i(x,y) = \mathcal{M}_j^i(x,y), \tag{2.9}$$

$$\mathcal{G}_j^i(x,y) = \mathcal{M}_k^i(x,y)\mathcal{M}_j^k(x,y) \tag{2.10}$$

and if $\overline{B}$ (1.6) *and* $\overline{B}^*$ (2.1) *are dual to each other.*

Proof. By direct calculation using (1.8), (1.12), (2.3) and (2.5) we obtain:

$$\begin{gathered}
\langle dx^j, \frac{\delta}{\delta x^i}\rangle = \delta_i^j, \quad \langle dx^j, \frac{\delta}{\delta y^i}\rangle = 0, \quad \langle dx^j \frac{\partial}{\partial z^i}\rangle = 0, \\
\langle \delta y^j, \frac{\delta}{\delta x^i}\rangle = \mathcal{H}_i^j - \mathcal{M}_i^j, \quad \langle \delta y^j, \frac{\delta}{\delta y^i}\rangle = \delta_i^j, \quad \langle \delta y^j, \frac{\partial}{\partial z^i}\rangle = 0, \\
\langle \delta z^j, \frac{\delta}{\delta x^i}\rangle = \mathcal{G}_i^j - \mathcal{M}_k^j \mathcal{M}_i^k, \quad \langle \delta z^j, \frac{\delta}{\delta y^i}\rangle = -\mathcal{H}_i^j + \mathcal{M}_i^j, \quad \langle \delta z^j, \frac{\partial}{\partial z^i}\rangle = \delta_i^j.
\end{gathered} \tag{2.11}$$

The duality follows from (2.11) and (2.9).

Proposition 2.2. $\mathcal{G}_i^j(x,y)$ *defined by* (2.10) *satisfies* (2.6).

Proof. Using (1.4) and (1.14), it can be proved that

$$\mathcal{M}_r^s = \mathcal{M}_{i'}^{j'} \frac{\partial x^{i'}}{\partial x^r} \frac{\partial x^s}{\partial x^{j'}} - \frac{\partial^2 x^s}{\partial x^{j'} \partial x^{t'}} y^{t'} \frac{\partial x^{j'}}{\partial x^r}. \tag{2.12}$$

From (2.12) and (2.10) after some calculation we obtain (2.6).

Remark. It is important that the bases B and B^* be dual to each other, because if they are not, then the contraction of tensors doesn't result tensors. If $\overline{B}$ and $\overline{B}^*$ ((1.6) and (2.1)) are dual to each other, it doesn't follow that $\overline{B}'$ and $\overline{B}^{*'}$ are dual.

Now we have:

THEOREM 2.2. *The bases* $B(\mathcal{M},\mathcal{N}) = \left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial z^i}\right\}$ *and* $B^*(\mathcal{M},\mathcal{N}) = \{dx^i, \delta y^i, \delta z^i\}$, *where their elements are given by*

$$\begin{gathered}\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{M}_i^j\frac{\partial}{\partial y^j} - \mathcal{N}_i^j\frac{\partial}{\partial z^j}, \quad \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \mathcal{M}_i^j\frac{\partial}{\partial z^j},\\ \delta y^i = dy^i + \mathcal{M}_j^i dx^j, \quad \delta z^i = dz^i + \mathcal{M}_j^i\delta y^j + \mathcal{N}_j^i dx^j\end{gathered} \tag{2.13}$$

are adapted basis for $T(T^2M)$ *and* $T^*(T^2M)$ *respectively, dual to each other, and they satisfy the law of transformation:*

$$\begin{gathered}\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\delta}{\delta x^{i'}}, \frac{\delta}{\delta y^i} = \frac{\partial y^{i'}}{\partial y^i}\frac{\delta}{\delta y^{i'}}, \frac{\partial}{\partial z^i} = \frac{\partial z^{i'}}{\partial z^i}\frac{\partial}{\partial z^{i'}},\\ dx^i = \frac{\partial x^i}{\partial x^{i'}}dx^i, \;\; \delta y^i = \frac{\partial y^i}{\partial y^{i'}}\delta y^{i'}, \;\; \delta z^i = \frac{\partial z^i}{\partial z^{i'}}\delta z^{i'}.\end{gathered} \tag{2.14}$$

It must be noted that there exist as many adapted bases as many functions $\mathcal{M}_i^j(x,y)$ and $\mathcal{N}_i^j(x,y,z)$ can be found, satisfying (1.14) and (1.15) respectively.

If we denote by T_H, T_{V_1}, T_{V_2} the subspaces of $T(T^2M)$ spanned by $\{\frac{\delta}{\delta x^i}\}$, $\{\frac{\delta}{\delta y^i}\}$, $\{\frac{\partial}{\partial z^i}\}$, and by T_H^*, $T_{V_1}^*$, $T_{V_2}^*$ the subspaces of $T^*(T^2M)$ spanned by $\{dx^i\}$, $\{\delta y^i\}$, $\{\delta z^i\}$ respectively, then

$$T(T^2M) = T_H \oplus T_{V_1} \oplus T_{V_2}, \quad T^*(T^2M) = T_H^* \oplus T_{V_1}^* \oplus T_{V_2}^*.$$

For the further examinations it is useful to introduce different kinds of indices. Indices $i,j,h,k,l = \overline{1,n}$ will be used in T_H and T_H^*, $a,b,c,d,e,f = \overline{n+1,2n}$ in T_{V_1} and $T_{V_1}^*$, $p,q,r,s,t = \overline{2n+1,3n}$ in T_{V_2} and $T_{V_2}^*$. The Greek letters as indices will take values from 1 to $3n$. Using this notation the adapted bases have the form:

$$B = \left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^a}, \frac{\partial}{\partial z^p}\right\} = \{\delta_\alpha\}, \quad B^* = \{dx^j, \delta y^b, \delta z^q\} = \{\delta^\beta\}, \tag{2.15}$$

where

$$\begin{gathered}\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{M}_i^b\frac{\partial}{\partial y^b} - \mathcal{N}_i^q\frac{\partial}{\partial z^q}, \quad \frac{\delta}{\delta y^a} = \frac{\partial}{\partial y^a} - \mathcal{M}_a^q\frac{\partial}{\partial z^q},\\ \delta y^a = dy^a + \mathcal{M}_i^a dx^i, \quad \delta z^q = dz^q + \mathcal{M}_a^q\delta y^a + \mathcal{N}_j^q dx^j.\end{gathered} \tag{2.16}$$

If $i=a \pmod n$ and $j=b=q \pmod n$, then $\mathcal{M}_i^j=\mathcal{M}_i^b=\mathcal{M}_a^q$, $\mathcal{N}_i^q=\mathcal{N}_i^j$, in (2.16).

Some tensor field T expressed in the bases B and B^* ((2.15)) has the form:

$$T=T^{\dots i\,\dots a\,\dots r\,\dots}_{\dots. j\dots\, b\dots\, s\dots}\frac{\delta}{\delta x^i}\otimes dx^j\cdots\frac{\delta}{\delta y^a}\otimes\delta y^b\cdots\frac{\partial}{\partial z^r}\otimes\delta z^s\cdots$$

The components of the tensor T, with respect to the coordinate transformations (1.1) are transformed in the following way:

$$T^{\dots i'\,\dots a'\,\dots r'\,\dots}_{\dots. j'\dots\, b'\dots\, s'\dots}=T^{\dots i\,\dots a\,\dots r\,\dots}_{\dots j\dots\, b\dots\, s\dots}\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial x^j}{\partial x^{j'}}\cdots\frac{\partial y^{a'}}{\partial y^a}\,\frac{\partial y^b}{\partial y^{b'}}\cdots\frac{\partial z^{r'}}{\partial z^r}\,\frac{\partial z^s}{\partial z^{s'}}\cdots$$

For some vector field $X\in T(T^2M)$ and some 1-form $\omega\in T^*(T^2M)$ we have:

$$\begin{aligned} X&=X^i\frac{\delta}{\delta x^i}+X^a\frac{\delta}{\delta y^a}+X^p\frac{\partial}{\partial z^p}=X^\alpha\delta_\alpha,\\ \omega&=\omega_j dx^j+\omega_b\delta y^b+\omega_q\delta z^q=\omega_\beta\delta^\beta. \end{aligned}\tag{2.17}$$

With respect to (1.1) the coordinates of X and ω transform in the following way:

$$\begin{aligned} X^{i'}&=X^i\frac{\partial x^{i'}}{\partial x^i},\quad X^{a'}=X^a\frac{\partial y^{a'}}{\partial y^a},\quad X^{p'}=X^p\frac{\partial z^{p'}}{\partial z^p},\\ \omega_{j'}&=\omega_j\frac{\partial x^j}{\partial x^{j'}},\quad \omega_{b'}=\omega_b\frac{\partial y^b}{\partial y^{b'}},\quad \omega_{q'}=\omega_q\frac{\partial z^q}{\partial z^{q'}}, \end{aligned}$$

because for $i=a=p \pmod n$ we have:

$$\frac{\partial x^{i'}}{\partial x^i}=\frac{\partial y^{a'}}{\partial y^a}=\frac{\partial z^{p'}}{\partial z^p}.\tag{2.18}$$

3. Generalized covariant derivatives. The generalized connection in Lagrange and Hamilton spaces was studied among others in [**2**]–[**6**]. In $T(T^2M)$ it is introduced in the following way. Let $\nabla: T(T^2M)\times T(T^2M)\to T(T^2M)$ ($\times$ is the Descartes product) be a linear connection, such that $\nabla:(X,Y)\to\nabla_XY\in T(T^2M)$, $\forall X,Y\in T(T^2M)$. The operator ∇ is called generalized connection. It is called d-connection if ∇_XY is in T_H, T_{V_1} or T_{V_2} if Y is in T_H, T_{V_1} or T_{V_2}, respectively, $\forall X\in T(T^2M)$. It has been studied by many authors, mostly romanian geometers.

We shall not make that restriction on ∇ here. In the following we shall use the abbreviations: $\delta_k=\frac{\delta}{\delta x^k}$, $\delta_a=\frac{\delta}{\delta y^a}$, $\partial_k=\frac{\partial}{\partial x^k}$, $\partial_a=\frac{\partial}{\partial y^a}$, $\partial_p=\frac{\partial}{\partial z^p}$.

Definition 3.1. The generalized connection ∇ is defined by

$$\begin{aligned}
&\nabla_{\delta_i}\delta_j = \underline{F}_{j\ i}^{\ k}\delta_k + F_{j\ i}^{\ c}\delta_c + F_{j\ i}^{\ r}\partial_r, \quad &&\nabla_{\delta_i}\delta_b = F_{b\ i}^{\ k}\delta_k + \underline{F}_{b\ i}^{\ c}\delta_c + F_{b\ i}^{\ r}\partial_r\\
&\nabla_{\delta_i}\partial_q = F_{q\ i}^{\ k}\delta_k + F_{q\ i}^{\ c}\delta_c + \underline{F}_{q\ i}^{\ r}\partial_r, \quad &&\nabla_{\delta_a}\delta_j = \underline{C}_{j\ a}^{\ k}\delta_k + C_{j\ a}^{\ c}\delta_c + C_{j\ a}^{\ r}\partial_r,\\
&\nabla_{\delta_a}\delta_b = C_{b\ a}^{\ k}\delta_k + \underline{C}_{b\ a}^{\ c}\delta_c + C_{b\ a}^{\ r}\partial_r, \quad &&\nabla_{\delta_a}\partial_q = C_{q\ a}^{\ k}\delta_k + C_{q\ a}^{\ c}\delta_c + \underline{C}_{q\ a}^{\ r}\partial_r,\\
&\nabla_{\partial_p}\delta_j = \underline{L}_{j\ p}^{\ k}\delta_k + L_{j\ p}^{\ c}\delta_c + L_{j\ p}^{\ r}\partial_r, \quad &&\nabla_{\partial_p}\delta_b = L_{b\ p}^{\ k}\delta_k + \underline{L}_{b\ p}^{\ c}\delta_c + L_{b\ p}^{\ r}\partial_r,\\
&\nabla_{\partial_p}\partial_q = L_{q\ p}^{\ k}\delta_k + L_{q\ p}^{\ c}\delta_c + \underline{L}_{q\ p}^{\ r}\partial_r.
\end{aligned} \tag{3.1}$$

The d-connection is defined if in (3.1) all terms on the right-hand side vanish, except the underlined ones.

For the vector field X defined by (2.17) we have

$$\begin{aligned}
&\nabla_{\delta_i}(X^j\delta_j + X^b\delta_b + X^q\partial_q)\\
&\quad = (\delta_i X^j)\delta_j + X^j\nabla_{\delta_i}\delta_j + (\delta_i X^b)\delta_b + X^b\nabla_{\delta_i}\delta_b + (\delta_i X^q)\partial_q + X^q\nabla_{\delta_i}\partial_q\\
&= (\delta_i X^k + F_{j\ i}^{\ k}X^j + F_{b\ i}^{\ k}X^b + F_{q\ i}^{\ k}X^q)\delta_k + (\delta_i X^c + F_{j\ i}^{\ c}X^j + F_{b\ i}^{\ c}X^b + F_{q\ i}^{\ c}X^q)\delta_c +\\
&\qquad (\delta_i X^r + F_{j\ i}^{\ r}X^j + F_{b\ i}^{\ r}X^b + F_{q\ i}^{\ r}X^q)\partial_r
\end{aligned}$$

From the above equation it follows

$$\nabla_{\delta_i}X = X^k_{|i}\delta_k + X^c_{|i}\delta_c + X^r_{|i}\partial_r, \tag{3.2}$$

where

$$X^x_{|i} = \delta_i X^x + F_{j\ i}^{\ x}X^j + F_{b\ i}^{\ x}X^b + F_{q\ i}^{\ x}X^q, \quad x \in \{k, c, r\} \tag{3.3}$$

or shorter

$$X^x_{|i} = \delta_i X^x + F_{\alpha\ i}^{\ x}X^\alpha, \quad x \in \{k, c, r\}. \tag{3.4}$$

The summation over α is the sum of summations over j, b and q as is written in (3.3). The sign $|i$ is the covariant derivative in direction of the basis vector δ_i.

The covariant derivative of X in the direction of δ_a has the form:

$$\nabla_{\delta_a}X = X^k|_a\delta_k + X^c|_a\delta_c + X^r|_a\partial_r, \tag{3.5}$$

where

$$X^x|_a = \delta_a X^x + C_{j\ a}^{\ x}X^j + C_{b\ a}^{\ x}X^b + C_{q\ a}^{\ x}X^q, \quad x \in \{k, c, r\}, \tag{3.6}$$

or shorter

$$X^x|_a = \delta_a X^x + C_{\alpha\ a}^{\ x}X^\alpha. \tag{3.7}$$

The covariant derivative of the vector field X in the direction of ∂_p is given by

$$\nabla_{\partial_p} X = X^k\|_p \delta_k + X^c\|_p \delta_c + X^r\|_p \partial_r, \tag{3.8}$$

where

$$X^x\|_p = \partial_p X^x + L_{j\ p}^{\ x} X^j + L_{b\ p}^{\ x} X^b + L_{q\ p}^{\ x} X^q, \quad x \in \{k, c, r\}, \tag{3.9}$$

or abbreviated

$$X^x\|_p = \partial_p X^x + L_{\alpha\ p}^{\ x} X^\alpha. \tag{3.10}$$

In (3.7) and (3.10) the summation over α is the sum of summations over j, b and q (as in (3.4)).

Theorem 3.1. *If X and Y are vector fields in $T(T^2M)$, ∇ the generalized connection defined by (3.1), then the following equation is valid:*

$$\begin{aligned}\nabla_Y X = {} & (X^k_{|i} Y^i + X^k|_a Y^a + X^k\|_p Y^p)\delta_k \\ & + (X^c_{|i} Y^i + X^c|_a Y^a + X^c\|_p Y^p)\delta_c + (X^r_{|i} Y^i + X^r|_a Y^a + X^r\|_p Y^p)\partial_r.\end{aligned} \tag{3.11}$$

Proof. The proof follows from (3.2)–(3.10) and the bilinearity of ∇.

The equation (3.11) can be written in the abbreviated form as follows

$$\nabla_Y X = X^\alpha_{|\beta} Y^\beta \delta_\alpha, \tag{3.12}$$

$$X^\alpha_{|\beta} = \delta_\beta X^\alpha + \Gamma_{\gamma\ \beta}^{\ \alpha} X^\gamma. \tag{3.13}$$

If $\beta = i$, then $\Gamma = F$; if $\beta = a$, then $\Gamma = C$; if $\beta = p$ then $\Gamma = L$.

Theorem 3.2. *All covariant derivatives $X^\alpha_{|i}$, $X^\alpha|_a$, $X^\alpha\|_p$ ($\alpha = k$, or $\alpha = c$ or $\alpha = r$) from* (3.11) *are transformed as tensors with respect to* (1.1) *if all connection coefficients from* (3.1) *are transformed as tensors, except the following, which have the form*

$$\begin{aligned}F_{j\ i}^{\ k} &= F_{j'\ i'}^{\ k'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{j'}}{\partial x^j} + \frac{\partial^2 x^{k'}}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial x^{k'}} \\ F_{b\ i}^{\ c} &= F_{b'\ i'}^{\ c'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial y^{b'}}{\partial y^b} \frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial x^i \partial y^b} \frac{\partial y^c}{\partial y^{c'}} \\ F_{q\ i}^{\ r} &= F_{q'\ i'}^{\ r'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial z^{q'}}{\partial z^q} \frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial x^i \partial z^q} \frac{\partial z^r}{\partial z^{r'}}.\end{aligned} \tag{3.14}$$

The proof is obtained by direct calculation.

Remark. From (2.17) it follows that $F_{j\ i}^{\ k} = F_{b\ i}^{\ c} = F_{q\ i}^{\ r}$ if $k = c = r \pmod n$, $j = b = q \pmod n$. The connection coefficients $\mathcal{H}_{j\ k}^{\ i}$, $\mathcal{M}_{i\ k}^{\ j}$ and $\mathcal{N}_{i\ k}^{\ j}$ defined by (1.11), (1.17) and (1.18) respectively, satisfy the transformation laws prescribed by (3.14).

4. The torsion tensor of the generalized connection. The torsion tensor $T(X,Y)$ is defined in the usual way by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

THEOREM 4.1. *The torsion tensor for the generalized connection has the form* $T(X,Y) = T^k\delta_k + T^c\delta_c + T^r\partial_r$, *where*

$$\begin{aligned}
T^k = {} & T_{j\ i}^{\ k} Y^j X^i + T_{j\ b}^{\ k} Y^j X^b + T_{j\ q}^{\ k} Y^j X^q \\
& + T_{b\ i}^{\ k} Y^b X^i + T_{b\ a}^{\ k} Y^b X^a + T_{b\ q}^{\ k} Y^b X^q \\
& + T_{p\ i}^{\ k} Y^p X^i + T_{p\ b}^{\ k} Y^p X^b + T_{p\ q}^{\ k} Y^p X^q,
\end{aligned}$$

$$\begin{aligned}
&T_{j\ i}^{\ k} = F_{j\ i}^{\ k} - F_{i\ j}^{\ k}, \quad T_{j\ b}^{\ k} = C_{j\ b}^{\ k} - F_{b\ j}^{\ k}, \quad T_{j\ q}^{\ k} = L_{j\ q}^{\ k} - F_{q\ j}^{\ k}, \\
&T_{b\ i}^{\ k} = F_{b\ i}^{\ k} - C_{i\ b}^{\ k}, \quad T_{b\ a}^{\ k} = C_{b\ a}^{\ k} - C_{a\ b}^{\ k}, \quad T_{b\ q}^{\ k} = L_{b\ q}^{\ k} - C_{q\ b}^{\ k}, \\
&T_{p\ i}^{\ k} = F_{p\ i}^{\ k} - L_{i\ p}^{\ k}, \quad T_{p\ b}^{\ k} = C_{p\ b}^{\ k} - L_{b\ p}^{\ k}, \quad T_{p\ q}^{\ k} = L_{p\ q}^{\ k} - L_{q\ p}^{\ k},
\end{aligned}$$

$$\begin{aligned}
T^c = {} & T_{j\ i}^{\ c} Y^j X^i + T_{j\ b}^{\ c} Y^j X^b + T_{j\ r}^{\ c} Y^j X^r + T_{b\ i}^{\ c} Y^b X^i + T_{a\ b}^{\ c} Y^a X^b + \\
& T_{b\ r}^{\ c} Y^b X^r + T_{q\ i}^{\ c} Y^q X^i + T_{q\ b}^{\ c} Y^q X^b + T_{q\ r}^{\ c} Y^q X^r,
\end{aligned}$$

$$\begin{aligned}
&T_{j\ i}^{\ c} = F_{j\ i}^{\ c} - F_{i\ j}^{\ c} - K_{i\ j}^{\ c}, \quad T_{a\ b}^{\ c} = C_{a\ b}^{\ c} - C_{b\ a}^{\ c}, \quad T_{q\ i}^{\ c} = F_{q\ i}^{\ c} - L_{i\ q}^{\ c}, \\
&T_{j\ b}^{\ c} = C_{j\ b}^{\ c} - F_{b\ j}^{\ c} + K_{j\ b}^{\ c}, \quad T_{b\ r}^{\ c} = L_{b\ r}^{\ c} - C_{r\ b}^{\ c}, \quad T_{q\ b}^{\ c} = C_{q\ b}^{\ c} - L_{b\ q}^{\ c}, \\
&T_{b\ i}^{\ c} = F_{b\ i}^{\ c} - C_{i\ b}^{\ c} - K_{i\ b}^{\ c}, \quad T_{j\ r}^{\ c} = L_{j\ r}^{\ c} - F_{r\ j}^{\ c}, \quad T_{q\ r}^{\ c} = L_{q\ r}^{\ c} - L_{r\ q}^{\ c},
\end{aligned}$$

$$\begin{aligned}
T^r = {} & T_{j\ i}^{\ r} Y^j X^i + T_{j\ b}^{\ r} Y^j X^b + T_{j\ q}^{\ r} Y^j X^q + T_{b\ i}^{\ r} Y^b X^i + T_{b\ a}^{\ r} Y^b X^a + \\
& T_{b\ q}^{\ r} Y^b X^q + T_{p\ i}^{\ r} Y^p X^i + T_{p\ b}^{\ r} Y^p X^b + T_{p\ q}^{\ r} Y^p X^q
\end{aligned}$$

$$\begin{aligned}
&T_{j\ i}^{\ r} = F_{j\ i}^{\ r} - F_{i\ j}^{\ r} - K_{i\ j}^{\ r}, \quad T_{j\ b}^{\ r} = C_{j\ b}^{\ r} - F_{b\ j}^{\ r} + K_{j\ b}^{\ r}, \quad T_{b\ q}^{\ r} = L_{b\ q}^{\ r} - C_{q\ b}^{\ r}, \\
&T_{j\ q}^{\ r} = L_{j\ q}^{\ r} - F_{q\ j}^{\ r} + K_{j\ q}^{\ r}, \quad T_{b\ i}^{\ r} = F_{b\ i}^{\ r} - C_{i\ b}^{\ r} - K_{i\ b}^{\ r}, \quad T_{p\ b}^{\ r} = C_{p\ b}^{\ r} - L_{b\ p}^{\ r}, \\
&T_{b\ a}^{\ r} = C_{b\ a}^{\ r} - C_{a\ b}^{\ r} - K_{a\ b}^{\ r}, \quad T_{p\ i}^{\ r} = F_{p\ i}^{\ r} - L_{i\ p}^{\ r} - K_{i\ p}^{\ r}, \quad T_{p\ q}^{\ r} = L_{p\ q}^{\ r} - L_{q\ p}^{\ r}.
\end{aligned}$$

Proof. The proof is obtained by direct calculation using (3.12), (3.13) and the relations:

$$\begin{aligned}
[X,Y] = {} & [X(Y^j) - Y(X^j)]\delta_j + [X(Y^b) - Y(X^b)]\delta_b + \\
& [X(Y^q) - Y(X^q)]\partial_q + A + B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
&A = X^i Y^j (\delta_i\delta_j - \delta_j\delta_i), \quad B = (X^i Y^b - Y^i X^b)(\delta_i\delta_b - \delta_b\delta_i) \\
&C = (X^i Y^q - Y^i X^q)(\delta_i\partial_q - \partial_q\delta_i), \quad D = X^a Y^b (\delta_a\delta_b - \delta_b\delta_a).
\end{aligned}$$

Using (2.16) we obtain:

$$A = X^iY^j(K_{i\ j}^{\ c}\partial_c + \overline{K}_{i\ j}^{\ r}\partial_r) = X^iY^j(K_{i\ j}^{\ c}\delta_c + K_{i\ j}^{\ r}\partial_r),$$

where

$$K_{i\ j}^{\ c} = \delta_j\mathcal{M}_i^c - \delta_i\mathcal{M}_j^c, \quad K_{i\ j}^{\ r} = \overline{K}_{i\ j}^{\ r} + \mathcal{M}_c^r K_{i\ j}^{\ c}, \overline{K}_{i\ j}^{\ r} = \delta_j\mathcal{N}_i^r - \delta_i\mathcal{N}_j^r.$$
$$B = (X^iY^b - Y^iX^b)(K_{i\ b}^{\ c}\delta_c + K_{i\ b}^{\ r}\partial_r),$$
$$K_{i\ b}^{\ c} = \delta_b\mathcal{M}_i^c, \ \ K_{i\ b}^{\ r} = \overline{K}_{i\ b}^{\ r} + \mathcal{M}_c^r K_{i\ b}^{\ c}, \ \ \overline{K}_{i\ b}^{\ r} = \delta_b\mathcal{N}_i^r - \delta_i\mathcal{M}_b^r.$$
$$C = (X^iY^q - Y^iX^q)K_{i\ q}^{\ r}\partial_r, \ \ K_{i\ q}^{\ r} = \frac{\partial\mathcal{N}_i^r}{\partial z^q}.$$
$$D = X^aY^bK_{a\ b}^{\ r}\partial_r, \ \ K_{a\ b}^{\ r} = \delta_b\mathcal{M}_a^r - \delta_a\mathcal{M}_b^r.$$

Remark. As $\mathcal{M} = \mathcal{M}(x,y)$, in all above formulae $\partial_r\mathcal{M} = 0$.

5. Special cases. As mentioned in Definition 3.1 the special connection $\bar{\nabla}$ (the so called d-connection) is obtained, if in (3.1) only the underlined terms are left. More precisely:

Definition 5.1 The d-connection $\bar{\nabla}$ is defined by:

$$\begin{aligned}
&\bar{\nabla}_{\delta_i}\delta_j = \bar{F}_{j\ i}^{\ k}\delta_k, \quad \bar{\nabla}_{\delta_i}\delta_b = \bar{F}_{b\ i}^{\ c}\delta_c, \quad \bar{\nabla}_{\delta_i}\partial_q = \bar{F}_{q\ i}^{\ r}\partial_r\\
&\bar{\nabla}_{\delta_a}\delta_j = \bar{C}_{j\ a}^{\ k}\delta_k, \quad \bar{\nabla}_{\delta_a}\delta_b = \bar{C}_{b\ a}^{\ c}\delta_c, \quad \bar{\nabla}_{\delta_a}\partial_q = \bar{C}_{q\ a}^{\ r}\partial_r\\
&\bar{\nabla}_{\partial_p}\delta_j = \bar{L}_{j\ p}^{\ k}\delta_k, \quad \bar{\nabla}_{\partial_p}\delta_b = \bar{L}_{b\ p}^{\ c}\delta_c, \quad \bar{\nabla}_{\partial_p}\partial_q = \bar{L}_{q\ p}^{\ r}\partial_r.
\end{aligned} \tag{5.1}$$

From (5.1) the following property of d-connection is obvious:

$$\bar{\nabla}_X : T_H \to T_H, \ \bar{\nabla}_X : T_{V_1} \to T_{V_1}, \ \bar{\nabla}_X : T_{V_2} \to T_{V_2}$$

for any vector field X from $T(T^2M)$.

Theorem 5.1. *If X and Y are vector fields in $T(T^2M)$ expressed in the basis B (2.15), then*

$$\begin{aligned}
\bar{\nabla}_XY = Y^k_{\bar{|}i}X^i + Y^k\bar{|}_aX^a + Y^k\bar{\|}_pX^p)\delta_k + (Y^c_{\bar{|}i}X^i + Y^c\bar{|}_aX^a + Y^c\bar{\|}_pX^p)\delta_c\\
+ (Y^r_{\bar{|}i}X^i + Y^r\bar{|}_aX^q + Y^r\bar{\|}_pX^p)\partial_r,\\
Y^x_{\bar{|}i} = \delta_iY^x + \bar{F}_{y\ i}^{\ x}Y^y, \quad Y^x\bar{|}_a = \partial_aY^x + \bar{C}_{y\ a}^{\ x}Y^y, \quad Y^x\bar{\|}_p = \partial_pY^x + \bar{L}_{y\ p}^{\ x}Y^y,
\end{aligned}$$

where either $x = k$, $y = j$, or $x = c$, $y = b$, or $x = r$, $y = q$.

Theorem 5.2. *The connection coefficients $\bar{C}_{j\ a}^{\ k}$, $\bar{C}_{b\ a}^{\ c}$, $\bar{C}_{q\ a}^{\ r}$, $\bar{L}_{j\ p}^{\ k}$, $\bar{L}_{b\ p}^{\ c}$ and $\bar{L}_{q\ p}^{\ r}$ with respect to* (1.1) *are transformed as tensors. The transformation laws for*

$\bar{F}_{j\,i}^{\,k}$, $\bar{F}_{b\,i}^{\,c}$ and $\bar{F}_{q\,i}^{\,r}$ are given by (3.14) if in these formulae the connection coefficients are overlined.

THEOREM 5.3. *The torsion tensor $T(X,Y)$ of d-connection $\bar{\nabla}$ has the form $T(X,Y)=\bar{T}^k\delta_k+\bar{T}^c\delta_c+\bar{T}^r\partial_r$, where*

$$\begin{aligned}
\bar{T}^k &= \bar{T}_{j\,i}^{\,k}Y^jX^i+\bar{T}_{j\,b}^{\,k}Y^jX^b+\bar{T}_{b\,j}^{\,k}Y^bX^j+\bar{T}_{j\,q}^{\,k}Y^jX^q+\bar{T}_{q\,j}^{\,k}Y^qX^j,\\
\bar{T}^c &= \bar{T}_{j\,i}^{\,c}Y^jX^i+\bar{T}_{i\,b}^{\,c}Y^jX^b+\bar{T}_{b\,j}^{\,c}Y^bX^j+\bar{T}_{a\,b}^{\,c}Y^aX^b+\bar{T}_{b\,q}^{\,c}Y^bX^q+\bar{T}_{q\,b}^{\,c}Y^qX^b,\\
\bar{T}^r &= \bar{T}_{j\,i}^{\,r}Y^jX^i+\bar{T}_{b\,a}^{\,r}Y^bX^a+\bar{T}_{j\,q}^{\,r}Y^jX^q+\bar{T}_{q\,j}^{\,r}Y^qX^j\\
&\quad+\bar{T}_{b\,q}^{\,r}Y^bX^q+\bar{T}_{q\,b}^{\,r}Y^qX^b+\bar{T}_{p\,q}^{\,r}Y^pX^q
\end{aligned}$$

and

$$\begin{aligned}
&\bar{T}_{j\,i}^{\,k}=\bar{F}_{j\,i}^{\,k}-\bar{F}_{i\,j}^{\,k}, && \bar{T}_{j\,b}^{\,k}=-\bar{T}_{b\,j}^{\,k}=\bar{C}_{j\,b}^{\,k},\\
&\bar{T}_{j\,q}^{\,k}=-\bar{T}_{q\,j}^{\,k}=\bar{L}_{j\,q}^{\,k}, && \bar{T}_{j\,i}^{\,c}=-K_{i\,j}^{\,c},\\
&\bar{T}_{b\,j}^{\,c}=-\bar{T}_{j\,b}^{\,c}=\bar{F}_{b\,j}^{\,c}-K_{j\,b}^{\,c}, && \bar{T}_{a\,b}^{\,c}=\bar{C}_{a\,b}^{\,c}-\bar{C}_{b\,a}^{\,c},\\
&\bar{T}_{b\,r}^{\,c}=-\bar{T}_{r\,b}^{\,c}=\bar{L}_{b\,r}^{\,c}, && \bar{T}_{q\,j}^{\,r}=-\bar{T}_{j\,q}^{\,r}=\bar{F}_{q\,j}^{\,r}-K_{j\,q}^{\,r},\\
&\bar{T}_{q\,b}^{\,r}=-\bar{T}_{b\,q}^{\,r}=\bar{C}_{q\,b}^{\,r}, && \bar{T}_{p\,q}^{\,r}=\bar{L}_{p\,q}^{\,r}-\bar{L}_{q\,p}^{\,r},\\
&\bar{T}_{j\,i}^{\,r}=-K_{i\,j}^{\,r}, && \bar{T}_{b\,a}^{\,r}=-K_{a\,b}^{\,r}.
\end{aligned}\tag{5.2}$$

It is easy to see, that all components of the torsion tensor T which appear in (5.2) with respect to (1.1) transform as tensors.

REFERENCES

1. M. Anastasiei, *Geometry of higher order sprays,* (preprint).
2. I.Čomić, *Generalized Miron's d-connection in the recurrent K-Hamilton spaces,* Publ. Inst. Math. Beograd **52(66)** (1992), 136–152.
3. I.Čomić, *Induced generalized connection in vector subbundles,* Publ. Inst. Math. Beograd **53(67)** (1993), 103–114.
4. I.Čomić, *Recurrent Hamilton spaces with generalized Miron's d-connection,* Ann. Univ. Craiova **18** (1990), 90–104.
5. I.Čomić, *Generalized connections on vector bundles and subbundles* Coll. Sci. Papers Fac. Sci. Kragujevac **16** (1994), 13–32.
6. I.Čomić, T. Kawaguchi, H. Kawaguchi, *A theory of dual vector bundles and their subbundles for general dynamical systems of the information geometry,* Tensor **52** (1993), 286–300.
7. A. Kawaguchi, *On the vectors of higher order and the extended affine connections,* Ann. Pura Appl. (IV), **55** (1961), 105–118.
8. M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces,* Kaiseisha Press, Otsu, Japan, 1986.
9. R. Miron, M. Anastasiei, *The geometry of Lagrange space, theory and applications,* Kluwer, Amsterdam, 1993.
10. R. Miron, Gh. Atanasiu, *Compendium sur les espaces Lagrange d'ordre supérieur,* Seminarul de Mecanica 40, Universitatea din Timisoara, 1994.

11. R. Miron, Gh. Atanasiu, *Differential geometry of the k-osculator bundle,* Rev. Roumaine Math. Pures Appl. (to appear).
12. R. Miron, Gh. Atanasiu, *Higher order Lagrange spaces,* Rev. Roumaine Math. Pures Appl. (to appear).
13. Gh. Munteanu, Gh. Atanasiu, *On Miron-connections in Lagrange spaces of second order,* Tensor (N.S.) **50** (1991), 241–247.

Fakultet tehničkih nauka
Trg Dositeja Obradovića 6
21000 Novi Sad
Yugoslavia

(Received 07 02 1996)