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ON SOME GENERALIZED EINSTEIN METRIC CONDITIONS

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Abstract. We present a construction of compact warped product manifolds realizing certain generalized Einstein metric conditions.

1. Introduction

A semi-Riemannian manifold (M,g), $n=\dim M\geq 2$, is said to be an Einstein manifold if the following condition

$$S=(\kappa/n)g \tag{1}$$

holds on M, where S and κ denote the Ricci tensor and the scalar curvature of (M,g), respectively. According to [**1**, p. 432], (1) is called the Einstein metric condition. Einstein manifolds form a natural subclass of various classes of semi-Riemannian manifolds determined by a curvature condition imposed on their Ricci tensor [**1**, Table, pp. 432–433]. For instance, every Einstein manifold belongs to the class of semi-Riemannian manifolds (M,g) realizing the following relation

$$(\nabla(S-(\kappa/(2(n-1)))g))(X,Y;Z)=(\nabla(S-(\kappa/(2(n-1)))g))(X,Z;Y), \tag{2}$$

for all $X,Y,Z\in\Xi(M)$, which means that $S-(\kappa/(2(n-1)))g$ is a Codazzi tensor on M. In the above formula ∇ denotes the Levi-Civita connection of (M,g), $\Xi(M)$ being the Lie algebra of vector fields on M. Manifolds of dimensions ≥ 4 fulfilling (2) are called manifolds with harmonic Weyl tensor [**1**, p. 440]. In the following we will denote by S^n a sphere of radius 1 in the Euclidean space $\mathbf{E}^{n+1}$, $n\geq 1$. It is known that every warped product $S^1\times_F M$ of the sphere S^1, with a positive smooth function F, and an Einstein manifold (M,g), $\dim M\geq 2$, realizes (2) [**1**, p. 433]. Such warped product is a non-Einstein manifold, in general. We say that (2) is a generalized Einstein metric condition [**1**, Chapter XVI]. On the other hand, such warped product realize too another curvature condition, so called

AMS Subject Classification (1991): Primary 53C25, 53C21; Secondary 53C35, 53B20 a condition of pseudosymmetry type (see section 2 of this paper). Namely, the warped product $S^1 \times_F M$ of the sphere S^1, with a positive smooth function F, and an Einstein manifold (M, g), $\dim M \geq 2$, is a Ricci-pseudosymmetric manifold [**15**, Corollary 3.2]. Thus, in particular, the warped product $S^1 \times_F \mathbf{C}P^n$ of S^1, with a positive smooth function F, and the complex projective space $\mathbf{C}P^n$ (considered with its standard Riemannian locally symmetric metric) is a Ricci-pseudosymmetric manifold. A semi-Riemannian manifold (M, g), $\dim M \geq 3$, is said to be Ricci-pseudosymmetric [**6**], [**15**] if at every point of M the following condition is satisfied:

(*) the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The tensors $R \cdot S$ and $Q(g, S)$ are defined by

$$\begin{aligned} (R \cdot S)(X_1, X_2; X, Y) &= (\tilde{\mathcal{R}}(X, Y) \cdot S)(X_1, X_2) \\ &= -S(\tilde{\mathcal{R}}(X, Y)X_1, X_2) - S(X_1, \tilde{\mathcal{R}}(X, Y)X_2), \\ Q(g, S)(X_1, X_2; X, Y) &= ((X \wedge Y) \cdot S)(X_1, X_2) \\ &= -S((X \wedge Y)X_1, X_2) - S(X_1, (X \wedge Y)X_2), \end{aligned}$$

respectively. In the above formulas $\tilde{\mathcal{R}}(X, Y)\cdot$ and $X \wedge Y\cdot$ denote the derivations of the algebra of the tensor fields on M, determined by the curvature operator $\tilde{\mathcal{R}}(X, Y)$ and the wedge product $X \wedge Y$, respectively. The pseudosymmetric manifolds form very important subclass of the class of Ricci-pseudosymmetric manifolds (see section 2). Evidently, any Einstein manifold is Ricci-pseudosymmetric. Thus we see that (*) is a generalized Einstein metric condition. Recently, some examples of compact and non-Einstein Ricci-pseudosymmetric manifolds were found in [**19**]. Namely, in [**19**, Theorem 1] it was shown that the Cartan hypersurfaces M in the spheres S^7, S^{17} or S^{25} are non-pseudosymmetric, Ricci-pseudosymmetric manifolds with non-pseudosymmetric Weyl tensor. The Cartan hypersurfaces M in S^4 are non-semisymmetric, pseudosymmetric manifolds. It is known that Cartan hypersurfaces are manifolds, with non- parallel Ricci tensor, satisfying a generalized Einstein metric condition of the following form [**20**, Theorem 4.1]

$$(\nabla S)(X, Y; Z) + (\nabla S)(Y, Z; X) + (\nabla S)(Z, X; Y) = 0, \tag{3}$$

for all $X, Y, Z \in \Xi(M)$. Evidently, the Cartan hypersurfaces do not satisfy (2).

Let f be a non-constant function satisfying on S^p, $p \geq 2$, the equality [**20**]

$$\bar{\nabla}(df) + f\bar{g} = 0, \tag{4}$$

where $\bar{g}$ is the standard metric on S^p and $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$. We put

$$F = (f + c)^2, \tag{5}$$

where c is a non-zero constant such that $f+c$ is a positive or negative function on S^p. In this paper we shall describe a family of compact Ricci-pseudosymmetric warped product manifolds. Our main result states that the warped product $S^p \times_F M$ of S^p, $p \geq 2$, with the function F defined by (5), and an Einstein manifold M, $\dim M \geq 2$, is a Ricci-pseudosymmetric manifold. From this it follows immediately that the warped product $S^p \times_F M$ of S^p, $p \geq 2$, with the function F defined by (5), and a compact irreducible symmetric space M is a compact Ricci-pseudosymmetric manifold. In particular, the warped product S^2 (resp., S^3), with the function F defined by (5), and the quaternion projective space $\mathbf{H}P^n$ (considered with its standard Riemannian locally symmetric metric), is a non-pseudosymmetric, Ricci-pseudosymmetric manifold with non-pseudosymmetric Weyl tensor. Applying this construction of the warped product manifolds to the generalized Hopf fibrations [**1**, p. 258]: $S^2 \to \mathbf{C}P^{2n+1} \to \mathbf{H}P^n$ and $S^3 \to S^{4n+3} \to \mathbf{H}P^n$, we obtain a local curvature property of the projective complex space $\mathbf{C}P^{2n+1}$ and of the sphere S^{4n+3}, respectively. More precisely, at the end of section 4, we shall state that every point of $\mathbf{C}P^{2n+1}$ (resp., S^{4n+3}) has a neighbourhood on which is defined a Riemannian metric isometric with the Ricci-pseudosymmetric warped product metric of an open submanifold of the manifold $S^2 \times_F \mathbf{H}P^n$, (resp., $S^3 \times_F \mathbf{H}P^n$). We note that by an application of Theorem 4.1 of [**11**] to the generalized Hopf fibration [**1**, p. 258]: $S^7 \to S^{15} \to S^8$, we can obtain the following local curvature property of S^{15}. Namely, every point of S^{15} has a neighbourhood on which is defined a Riemannian metric isometric with the non-conformally flat, pseudosymmetric warped product metric of an open submanifold of the manifold $S^7 \times_F S^8$. Furthermore, applying Corollary 3.2 of [**15**] to the Hopf fibration [**1**, p. 257]: $S^1 \to S^{2n+1} \to \mathbf{C}P^n$ we see that every point of S^{2n+1}, $n \geq 2$, has a neighbourhood on which is defined a Riemannian metric isometric with a non-conformally flat and non-pseudosymmetric, Ricci-pseudosymmetric warped product metric of an open submanifold of the manifold $S^1 \times_F \mathbf{C}P^n$. In section 3 we will consider warped products manifolds. We compute the local components of some tensors defined on the warped product $S^p \times_F M$ of the sphere S^p, with the function F defined by (5), and an Einstein manifold $(M, \tilde{g})$. The main results of this paper are presented in section 4. Throughout this paper all manifolds are assumed to be connected, paracompact manifolds of class C^∞.

2. Curvature conditions of pseudosymmetry type

Let (M,g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞. We define on M the endomorphisms $\tilde{\mathcal{R}}(X,Y)$ and $X \wedge Y$ by

$$\tilde{\mathcal{R}}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \ (X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$

respectively, where $X,Y,Z \in \Xi(M)$. We define the Riemann- Christoffel curvature tensor R and the concircular tensor $Z(R)$ of (M,g) by $R(X_1,\dots,X_4) = g(\tilde{\mathcal{R}}(X_1,X_2)X_3,X_4)$, $Z(R) = R - (\kappa/(n(n-1)))G$, respectively, where G is defined by $G(X_1,\dots,X_4) = g((X_1 \wedge X_2)X_3,X_4)$. For a $(0,k)$-tensor field T, $k \geq 1$, we define the $(0,k+2)$-tensors $R\cdot T$ and $Q(g,T)$ by

$$\begin{aligned}
&(R\cdot T)(X_1,\dots,X_k;X,Y) = (\tilde{\mathcal{R}}(X,Y)\cdot T)(X_1,\dots,X_k)\\
&\quad = -T(\tilde{\mathcal{R}}(X,Y)X_1,X_2,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1},\tilde{\mathcal{R}}(X,Y)X_k),\\
&Q(g,T)(X_1,\dots,X_k;X,Y) = ((X\wedge Y)\cdot T)(X_1,\dots,X_k)\\
&\quad = -T((X\wedge Y)X_1,X_2,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1},(X\wedge Y)X_k).
\end{aligned}$$

The semi-Riemannian manifold (M,g) is said to be pseudosymmetric [**13**] if at every point of M the following condition is satisfied:

(**) the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent.

The manifold (M,g) is pseudosymmetric if and only if

$$R\cdot R = LQ(g,R) \tag{6}$$

holds on the set $U_R = \{x\in M \mid Z(R)\neq 0 \text{ at } x\}$, where L is some function on U_R. It is clear that any semisymmetric manifold ($R\cdot R=0$) is pseudosymmetric. The condition (**) arose on the study on totally umbilical submanifolds of semisymmetric manifolds as well as when considering of geodesic mappings of semisymmetric manifolds [**10**], [**23**]. There exist many examples of pseudosymmetric manifolds which are not semisymmetric (see, e.g. [**11**], [**12**], [**13**]). Among these examples we can distinguish also compact pseudosymmetric manifolds. For instance, in [**11**] (see Example 3.1 and Theorem 4.1) it was proved that the warped product $S^p\times_F S^{n-p}$, $p\geq 2$, $n-p\geq 1$, with the function F defined by (5), is a pseudosymmetric manifold. Another example of a compact pseudosymmetric manifold is the warped product $S^1\times_F S^{n-1}$, with a positive smooth function F, as well as n-dimensional tori T^n with a certain metric (see [**11**, Examples 4.1 and 4.2]).

The manifold (M,g) is Ricci-pseudosymmetric if and only if

$$R\cdot S = L_S Q(g,S) \tag{7}$$

holds on the set $U_S = \{x\in M \mid S-(\kappa/n)g\neq 0 \text{ at } x\}$, where L_S is some function on U_S. It is clear that if at a point x of a manifold (M,g) (**) is satisfied then also (*) holds at x. The converse statement is not true. E.g. every warped product $M_1\times_F M_2$, $\dim M_1 = 1, \dim M_2 = n-1\geq 3$, of a manifold $(M_1,\bar{g})$ and a non-pseudosymmetric, Einstein manifold $(M_2,\tilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold (cf. [**15**, Remark 3.4] and [**13**, Theorem 4.1]). Warped products realizing (*) were considered in [**6**] and [**15**]. Recently, Ricci-pseudosymmetric hypersurfaces immersed isometrically in a semi-Riemannian manifolds of constant curvature were investigated in [**4**].

For any $X,Y\in\Xi(M)$ we define the endomorphism $\tilde{\mathcal{C}}(X,Y)$ by

$$\tilde{\mathcal{C}}(X,Y) = \tilde{\mathcal{R}}(X,Y) - (1/(n-2))(X\wedge\tilde{\mathcal{S}}Y + \tilde{\mathcal{S}}X\wedge Y - (\kappa/(n-1))X\wedge Y),$$

where the Ricci operator $\tilde{\mathcal{S}}$ of (M,g) is defined by $g(\tilde{\mathcal{S}}X,Y) = S(X,Y)$. The Weyl curvature tensor C of (M,g) is defined by $C(X_1,\dots,X_4) = g(\tilde{\mathcal{C}}(X_1,X_2)X_3,X_4)$. Now we define on M the $(0,6)$-tensor $C \cdot C$ by

$$\begin{aligned}(C\cdot C)(X_1,\dots,X_4;X,Y) &= (\tilde{\mathcal{C}}(X,Y)\cdot C)(X_1,\dots,X_4)\\ &= -C(\tilde{\mathcal{C}}(X,Y)X_1,\dots,X_4) - \cdots - C(X_1,X_2,X_3,\tilde{\mathcal{C}}(X,Y)X_4).\end{aligned}$$

A semi-Riemannian manifold (M,g), $\dim M \geq 4$, is said to be a manifold with pseudosymmetric Weyl tensor [**18**] if at every point of M the following condition is satisfied:

(***) the tensors $C \cdot C$ and $Q(g,C)$ are linearly dependent.

The manifold (M,g) is a manifold with pseudosymmetric Weyl tensor if and only if

$$C \cdot C = L_C Q(g,C) \tag{8}$$

holds on the set $U_C = \{x \in M | C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. The condition (***) arose in the study of 4-dimensional warped products [**9**]. Namely, in [**9**, Theorem 2] it was proved that at every point of a warped product $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$, (***) is fulfilled. Many examples of manifolds satisfying (***) are present in [**3**]. For instance, the Cartesian product of two manifolds of constant curvature is a manifold realizing (***). Warped products satisfying (***) were considered in [**18**]. In [**3**] it was shown that the classes of manifolds realizing (**) and (***) do not coincide. However, there exist pseudosymmetric manifolds fulfilling (8), e.g. Einsteinian pseudosymmetric manifolds [**3**, Theorem 3.1].

Remark 2.1. The above mentioned warped product $S^p \times_F S^{n-p}$, $p \geq 2$, $n-p \geq 2$, is a pseudosymmetric manifold with pseudosymmetric Weyl tensor which cannot be realized as a hypersurface isometrically immersed in a space of constant curvature $M^{n+1}(c)$, $n \geq 4$ [**11**, Theorem 4.1].

Further, we define on M the $(0,6)$-tensor $Q(S,R)$ by

$$\begin{aligned}Q(S,R)(X_1,\dots,X_4;X,Y) &= ((X\wedge_S Y)\cdot R)(X_1,\dots,X_4)\\ &= -R((X\wedge_S Y)X_1,X_2,X_3,X_4) - \cdots - R(X_1,X_2,X_3,(X\wedge_S Y)X_4),\end{aligned}$$

where $X \wedge_S Y$ is the endomorphism defined by $(X\wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y$. A semi-Riemannian manifold (M,g) is said to be Ricci-generalized pseudosymmetric [**10**] if at every point of M the following condition is satisfied:

(****) the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent.

A very important subclass of Ricci-generalized pseudosymmetric manifolds form manifolds fulfilling (see, e.g. [**2**]):

$$R \cdot R = Q(S,R). \tag{9}$$

Any 3-manifold (M, g) fulfils (9) [**8**, Theorem 3.1]. Moreover, any hypersurface M immersed isometrically in an $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$, $n \geq 4$, satisfies (9) [**16**, Corollary 3.1].

Remark 2.3. It is easy to see that if (**) holds on a semi-Riemannian manifold (M, g), $n \geq 4$, then at every point of M the following condition is satisfied:

(*****) the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

Manifolds fulfilling (*****) has been studied in [**7**], [**9**] and [**14**].

As it was proved in [**16**], at every point of a hypersurface M immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, the following condition is satisfied:

(******) the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.

Remark 2.3. In [**16**, Proposition 3.1] it was proved that every hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, satisfies the following equality $R \cdot R - Q(S, R) = -(((n-2)\tilde{\kappa})/(n(n+1)))Q(g, C)$, where $\tilde{\kappa}$ is the scalar curvature of $M^{n+1}(c)$ and R, S and C is the Riemann-Christoffel curvature tensor, the Ricci tensor and the Weyl tensor of M, respectively.

Using Theorem 3.1 of [8, which was mentioned above, and the fact that the Weyl tensor of any 3-dimensional semi- Riemannian manifold vanishes identically, we conclude that (******) is trivially satisfied on any 3-dimensional semi-Riemannian manifold. Recently, warped products realizing (******) were considered in [**5**]. For instance, in [**5**] it was stated that every warped product $M_1 \times_F M_2$, with $\dim M_1 = 1$, $\dim M_2 = 3$, with an arbitrary positive smooth function F, satisfies (******). The relations (*)–(******) are called conditions of pseudosymmetry type. We refer to [**10**] and [**23**] as the review papers on semi-Riemannian manifolds satisfying such conditions.

Remark 2.4. Every Einsteinian, as well as every conformally flat hypersurface M in $M^{n+1}(c)$, $n \geq 4$, is a pseudosymmetric manifold [**16**, Proposition 3.2]. Thus every quasi-umbilical hypersurface M in $M^{n+1}(c)$, $n \geq 4$, is pseudosymmetric. We recall that an n-dimensional hypersurface M, in a Riemannian manifold N, $\dim N \geq 4$, is called quasi-umbilical if M has at every point a principal curvature of multiplicity $\geq n-1$. In [**17**, Theorem 1] it was stated that every hypersurface M in $M^{n+1}(c)$, $n \geq 3$, having at every point at most two distinct principal curvatures is also pseudosymmetric. Necessary and sufficient conditions for hypersurfaces in $M^4(c)$, to be pseudosymmetric were found in [**17**]. In [**17**] it was shown that the Cartan hypersurface M in $S^4(c)$ is a non-semisymmetric pseudosymmetric manifold. The Cartan hypersurfaces in the sphere $S^{n+1}(c)$ are compact, minimal hypersurfaces with constant principal curvatures $-\sqrt{3c}$, 0, $\sqrt{3c}$ having the same multiplicity. The Cartan hypersurfaces exist only for $n = 3, 6, 12, 24$. More precisely, the Cartan hypersurfaces are tubes of constant radius over the standard

Veronese embeddings $i : \mathbf{F}P^2 \to S^{3d+1}(c) \to \mathbf{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbf{F}P^2$ in the sphere $S^{3d+1}(c)$ in a Euclidean space $\mathbf{E}^{3d+2}$, where $\mathbf{F} = \mathbf{R}$ (real numbers), $\mathbf{C}$ (complex numbers), $\mathbf{Q}$ (quaternions) or $\mathbf{O}$ (octonions), respectively. These hypersurfaces were discovered by E. Cartan in his work about isoparametric hypersurfaces. In [**19**, Theorem 1] it was proved that every Cartan hypersurface M in $S^{n+1}(c)$, $n = 6, 12, 14$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold with non-pseudosymmetric Weyl tensor.

3. Warped products

Let $(M_1, \bar{g})$ and $(M_2, \tilde{g})$, $\dim M_1 = p$, $\dim M_2 = n - p$, $1 \leq p < n$, be Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let F be a positive C^∞ function on M_1. The warped product $M_1 \times_F M_2$ of $(M_1, \bar{g})$ and $(M_2, \tilde{g})$ is the product manifold $M_1 \times M_2$ with the metric $g = \bar{g} \times_F \tilde{g} = \Pi_1^* \bar{g} + (F \circ \Pi_1)\Pi_2^* \tilde{g}$, where $\Pi_i : M_1 \times M_2 \longrightarrow M_i$ being the natural projections, $i = 1, 2$. Let $\{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $M_1 \times M_2$. The local components of the metric $g = \bar{g} \times_F \tilde{g}$ with respect to this chart are the following $g_{rs} = \bar{g}_{ab}$ if $r = a$ and $s = b$, $g_{rs} = F\tilde{g}_{\alpha\beta}$ if $r = \alpha$ and $s = \beta$, $g_{rs} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p + 1, \ldots, n\}$ and $r, s, t, \ldots \in \{1, 2, \ldots, n\}$. We shall denoted by bars (resp., tildes) tensors formed from $\bar{g}$ (resp., $\tilde{g}$). The local components of the tensors R and S of $M_1 \times_F M_2$ which may not vanish identically are the following []:

$$R_{abcd} = \bar{R}_{abcd}, \tag{10}$$

$$R_{\alpha ab\beta} = -(1/(2F))T_{ab}g_{\alpha\beta}, \tag{11}$$

$$R_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} - ((\Delta_1 F)/(4F^2))G_{\alpha\beta\gamma\delta}, \tag{12}$$

$$S_{ab} = \bar{S}_{ab} - ((n - p)/(2F))T_{ab}, \tag{13}$$

$$S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - (1/(2F))(tr(T) + ((n - p - 1)/(2F))\Delta_1 F)g_{\alpha\beta}, \tag{14}$$

$$T_{ab} = \bar{\nabla}_b F_a - (1/(2F))F_a F_b,\ tr(T) = \bar{g}^{ab}T_{ab}\Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{ab}F_a F_b, \tag{15}$$

where T is the $(0, 2)$-tensor with the local components T_{ab}.

Example 3.1. (cf. [**11**, Example 2.1]) Let $(M_1, \bar{g}) = S^p$ be the sphere in a Euclidean $\mathbf{E}^{p+1}$, $p \geq 2$, defined by $(x^0)^2 + (x^1)^2 + \cdots + (x^p)^2 = 1$, with the induced metric $\bar{g}$. Let f be a non-constant function on S^p satisfying (4). We put

$$L = 1 - c\tau,\ \tau = 1/\sqrt{F}, \tag{16}$$

where c is a non-zero constant such that $f + c$ is a positive or a negative function on S^p and F is defined by (5). Now, using (15), (4), (5) and (16), we can easily verify that the tensor $(1/2)T + FL\bar{g}$ vanishes on S^p. Furthermore, from (4) we get

$$\Delta_1 f = -f^2 + c_2,\ c_2 \in \mathbf{R}. \tag{17}$$

Combining (17) with (5) we can state that

$$(1/(4F^2))\Delta_1 F = c_1\tau^2 + 2c\tau - 1, \quad c_1 \in \mathbf{R}, \tag{18}$$

holds on S^p. Now we prove that τ cannot be constant on a non-empty open subset of S^p. We suppose that $\tau = const.$ on a non- empty open subset $U \subset S^p$. Evidently, we have also $F = textconst.$ on U. From (5) it follows that $f = \text{const.}$ on U. Thus, by (4), we have

$$f = 0 \text{ on } U. \tag{19}$$

On the other hand, it is known [**21**], [**22**] that every solution of (4) is a function Φ, defined by $\Phi = \lambda_0 x^0 + \lambda_1 x^1 + \cdots + \lambda_p x^p$, $\lambda_i = const.$, $i = 0, 1, \ldots, p$, restricted to the sphere S^p. Now (19) implies that $\lambda_0 = \lambda_1 = \ldots = \lambda_p = 0$, i.e. f vanishes identically on S^p, a contradiction with the fact that f is non-constant on S^p.

Example 3.2. Let $(M, \tilde{g})$, $\dim M = n - p \geq 2$, $p \geq 2$, be a semi-Riemannian Einstein manifold. We consider the warped product $S^p \times_F M$, where F is defined by (5). By (16), (18) and the fact that the tensor $(1/2)T + FL\bar{g}$, defined in Example 3.1, is a zero tensor, (10)–(14) turns into

$$R_{abcd} = G_{abcd}, \tag{20}$$

$$R_{a\alpha\beta b} = (1 - c\tau)G_{a\alpha\beta b}, \tag{21}$$

$$R_{\alpha\beta\gamma\delta} = FZ(\tilde{R})_{\alpha\beta\gamma\delta} + ((l - c_1)\tau^2 - 2c\tau + 1)G_{\alpha\beta\gamma\delta}, \tag{22}$$

$$S_{ab} = (n - 1 - (n - p)c\tau)g_{ab}, \tag{23}$$

$$S_{\alpha\beta} = ((n - p - 1)(l - c_1)\tau^2 - (2n - p - 2)c\tau + n - 1)g_{\alpha\beta}, \tag{24}$$

$$l = (\tilde{\kappa}/((n - p)(n - p - 1))). \tag{25}$$

Next, by using (19)–(23) and the relations

$$\begin{aligned} C_{rstu} = R_{rstu} - (1/(n - 2))((&g_{ru}S_{ts} + g_{ts}S_{ru} - g_{rt}S_{us} - g_{us}S_{rt}) \\ &- (\kappa/(n - 1))G_{rstu}), \end{aligned}$$

$$\kappa = (n - p)(n - p - 1)(l - c_1)\tau^2 - 2(n - 1)(n - p)c\tau + n(n - 1), \tag{26}$$

we find the non-zero components of C

$$C_{abcd} = (\rho/(p(p - 1)))G_{abcd}, \tag{27}$$

$$C_{a\alpha\beta b} = -(\rho/(p(n - p)))G_{a\alpha\beta b}, \tag{28}$$

$$C_{\alpha\beta\gamma\delta} = FZ(\tilde{R})_{\alpha\beta\gamma\delta} + (\rho/((n - p)(n - p - 1)))G_{\alpha\beta\gamma\delta}, \tag{29}$$

$$\rho = ((p(p - 1)(n - p)(n - p - 1))/((n - 1)(n - 2)))(l - c_1)\tau^2. \tag{30}$$

Furthermore, applying (18), (20)–(24), (27)–(29), we can easily verify that only components of $R \cdot R$, $Q(g,R)$, $R \cdot S$, $Q(g,S)$, $C \cdot C$, $Q(g,C)$ and $Q(S,R)$, which are not identically equal to zero are those related to

(31) $$(R \cdot R)_{\alpha abcd\beta} = -c\tau(1-c\tau)G_{dabc}g_{\alpha\beta},$$

(32) $$(R \cdot R)_{a\alpha\beta\gamma d\delta} = F(1-c\tau)g_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} - \tau(c+(c_1-c^2-l)\tau + (l-c_1)c\tau^2)g_{ad}G_{\delta\alpha\beta\gamma},$$

(33) $$(R \cdot R)_{\alpha\beta\gamma\delta\lambda\mu} = F(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} - F(c_1\tau^2+2c\tau-1)Q(g,\tilde{R})_{\alpha\beta\gamma\delta\lambda\mu},$$

(34) $$Q(g,R)_{\alpha abcd\beta} = -c\tau G_{dabc}g_{\alpha\beta},$$

(35) $$Q(g,R)_{a\alpha\beta\gamma d\delta} = Fg_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} - (c+(c_1-l)\tau)\tau g_{ad}G_{\delta\alpha\beta\gamma},$$

(36) $$Q(g,R)_{\alpha\beta\gamma\delta\lambda\mu} = FQ(g,\tilde{R})_{\alpha\beta\gamma\delta\lambda\mu},$$

(37) $$(R \cdot S)_{a\alpha\beta b} = \tau(1-c\tau)((n-p-1)(c_1-l)\tau + (n-2)c)g_{ab}g_{\alpha\beta},$$

(38) $$Q(g,S)_{a\alpha\beta b} = \tau((n-p-1)(c_1-l)\tau + (n-2)c)g_{ab}g_{\alpha\beta},$$

(39) $$(C \cdot C)_{\alpha abcd\beta} = (((n-1)\rho^2)/(p^2(n-p)^2(p-1)))G_{dabc}g_{\alpha\beta},$$

(40) $$(C \cdot C)_{a\alpha\beta\gamma d\delta} = -(\rho/(p(n-p)))Fg_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} - (((n-1)\rho^2)/(p^2(n-p)^2(n-p-1)))g_{ad}G_{\delta\alpha\beta\gamma},$$

(41) $$(C \cdot C)_{\alpha\beta\gamma\delta\lambda\mu} = F^2(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} + F(\rho/((n-p)(n-p-1))) - l)Q(g,\tilde{R})_{\alpha\beta\gamma\delta\lambda\mu},$$

(42) $$Q(g,C)_{\alpha abcd\beta} = -(((n-1)\rho)/(p(p-1)(n-p)))G_{dabc}g_{\alpha\beta},$$

(43) $$Q(g,C)_{a\alpha\beta\gamma d\delta} = Fg_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} + (((n-1)\rho)/(p(n-p)(n-p-1)))g_{ad}G_{\delta\alpha\beta\gamma},$$

(44) $$Q(g,C)_{\alpha\beta\gamma\delta\lambda\mu} = FQ(g,\tilde{R})_{\alpha\beta\gamma\delta\lambda\mu},$$

(45) $$Q(S,R)_{\alpha abcd\beta} = -(-(n-p-1) + (2n-2p-1)c\tau + (n-p-1)(((l-c)-(n-p)c^2)\tau^2 - c_1\tau^3))G_{dabc}g_{\alpha\beta},$$

(46) $$Q(S,R)_{a\alpha\beta\gamma d\delta} = -F^2(n-1-(n-p)c\tau)g_{ad}Z(\tilde{R})_{\delta\alpha\beta\gamma} - \tau(c+((p-2)c^2-p(l-c_1))\tau + (l-c_1)c\tau^2)g_{ad}G_{\delta\alpha\beta\gamma},$$

(47) $$Q(S,R)_{\alpha\beta\gamma\delta\lambda\mu} = F((n-p-1)(l-c_1)\tau^2 - (2n-p-2)c\tau + n - 1)Q(g,\tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}.$$

4. Main results

THEOREM 4.1. *Let $(N,g) = S^p \times_F M$ be the warped product of the sphere S^p, with the function F defined by (5), and an Einstein manifold $(M,\tilde{g})$, $p \geq 2$,* $\dim M = n - p \geq 3$. *Then we have:*
(i) (N,g) is a non-Einstein Ricci-pseudosymmetric manifold.

(ii) (N,g) *is a pseudosymmetric manifold if and only if* $(M,\tilde{g})$ *is a space of constant curvature.*
(iii) If $l \neq c_1$ *then* (N,g) *is a non-conformally flat manifold.*
(iv) (N,g) *is a manifold with pseudosymmetric Weyl tensor if and only if* $(M,\tilde{g})$ *is a space of constant curvature.*
(v) The tensor $R \cdot R - Q(S,R)$ *is a non-zero tensor on* N.
(vi) (N,g) *is a manifold do not satisfying* (3).
(vii) If $l \neq c_1$ *then the tensors* $R \cdot R - Q(S,R)$ *and* $Q(g,C)$ *are not linearly dependent on* N.
(viii) If $l \neq c_1$ *then* (N,g) *is a manifold with non-harmonic Weyl tensor.*
The constants c_1 *and* l *are defined by* (18) *and* (25), *respectively.*

Proof. (i) From (23)–(25) it follows that (1) is not satisfied on N. Further, (37) and (38) lead to (7).

(ii) We assume that (N,g) is pseudosymmetric. Let x be a point of N. If the tensor $Z(R) = R - (\kappa/(n(n-1)))G$ vanishes at x, i.e. $x \in N - U_R$, then (12) implies that the tensor $Z(\tilde{R}) = \tilde{R} - (\tilde{\kappa}/((n-p)(n-p-1)))\tilde{G}$ vanishes at x. Let now $x \in U_R$. We prove that the tensor $Z(\tilde{R})$ must also vanishes at x. Since (6) is fulfilled at x, (31)–(33) together with (34)–(36) imply that (16) and

$$\tilde{R} \cdot \tilde{R} = (c_1\tau^2 + c\tau)Q(\tilde{g},\tilde{R}) \tag{48}$$

hold at x. We suppose that $Z(\tilde{R})$ is non-zero at x. Now, from (48) we can deduce that the function $c_1\tau^2 + c\tau$ is constant on a non- empty subset $U' \subset S^p$. From this fact it follows that τ is constant on U'. But, as we have stated in Example 3.1, the function τ cannot be constant on an open non-empty subset of S^p. So, $Z(\tilde{R})$ must vanishes at x. Thus, $(M,\tilde{g})$ is a space of constant curvature. Conversely, if $(M,\tilde{g})$ is a space of constant curvature, then pseudosymmetry of (N,g) was proved in [**11**, Theorem 4.1(i)].

(iii) This assertion follows immediately from (27)–(30).

(iv) We assume that (N,g) is a manifold with pseudosymmetric Weyl tensor. Let x be a point of N. Using (39)–(44) we can deduce, in the same way as in the proof of the assertion (ii), that $(M,\tilde{g})$ is a space of constant curvature. The converse statement was proved in [**11**, Theorem 4.1(ii)].

(v) (31) and (45) yield

$$\begin{aligned}&((R \cdot R) - Q(S,R))_{\alpha abcd\beta}\\ &\quad = -((n-p-1)(1 - 2c\tau + (c - l + (n-p)c^2)\tau^2 + c_1\tau^3) - c^2\tau^2)g_{\alpha\beta}G_{dabc}.\end{aligned}$$

Thus $R \cdot R - Q(S,R)$ is a non-zero tensor on N.

(vi) From (26) it follows that the scalar curvature κ of (N,g) is not constant. Since any semi-Riemannian manifold satisfying (3) must have constant scalar curvature, (3) can not be fulfilled on N.

(vii) If at a point x of N the tensor $Z(\tilde{R})$ vanishes then our assertion was proved in [**11**, Theorem 4.1(v)]. If at a point x of N the tensor $Z(\tilde{R})$ is non-zero then our assertion follows from (32), (46) and (43).

(viii) Using (23), (24) and (26) we get

$$\begin{aligned}\nabla_c S_{ab} - \nabla_b S_{ac} - (1/(2(n-1)))((\nabla_c \kappa) g_{ab} - (\nabla_b \kappa) g_{ac})\\ = -(((n-p)(n-p-1))/(n-1))(l-c_1)\tau((\nabla_c \tau) g_{ab} - (\nabla_b \tau) g_{ac}).\end{aligned}$$

Thus (2) can not be satisfied on (N, g). Our theorem is thus proved.

Using the above theorem and Remark 2.4 we obtain the following statement.

THEOREM 4.3. *Let* $(N, g) = S^p \times_F M$*, be the warped product of the sphere* S^p*, with the function* F *defined in Example* 3.1*, and an Einstein manifold* $(M, \tilde{g})$*,* $p \geq 2$*,* $\dim M = n - p \geq 3$*. If* $(M, \tilde{g})$ *is not of constant curvature then* (N, g) *is a non-conformally flat, non-pseudosymmetric and non-Einstein, Ricci-pseudosymmetric manifold. If the constants* c_1 *and* l *defined by (18) and (25), respectively, are not equal then the manifold* (N, g) *cannot be realized as a hypersurface isometrically immersed in a space of constant curvature.*

Remark 4.1. In [**6**, Corollary 2] an example of a Ricci-pseudosymmetric warped product of the sphere S^p and a compact Einstein manifold (M, g), $\dim M \geq 2$, is presented. To construct this example we have needed Lemma 3.1(ii) of [**12**]. Let now $(\tilde{M}, \tilde{g})$ be a Riemannian manifold isometric with the Cartan hypersurface M in the unit sphere S^{n-p+1}, $n - p \in \{3, 6, 12, 24\}$. We can apply the mentioned above construction of warped product manifolds to obtain a Ricci-pseudosymmetric warped product of the sphere S^p and the manifold $(\tilde{M}, \tilde{g})$.

It is well known that every irreducible locally symmetric space $(M, \tilde{g})$, $\dim M \geq 3$, is an Einstein manifold. Thus we have the following corollary.

COROLLARY 4.1. *Let* $(N, g) = S^p \times_F M$ *be the warped product of the sphere* S^p*, with the function* F *defined in Example* 3.1*, and irreducible locally symmetric space* $(M, \tilde{g})$*,* $p \geq 2$*,* $\dim M = n - p \geq 3$*. If* $(M, \tilde{g})$ *is not of constant curvature then* (N, g) *is a non- conformally flat, non-pseudosymmetric and non-Einstein, Ricci-pseudosymmetric manifold.*

Using the above Corollary we can construct a family of Ricci-pseudosymmetric manifolds. This family of manifolds contains also compact manifolds.

Example 4.1. Let $(M, \tilde{g})$, $\dim M = n - p \geq 3$, be an irreducible symmetric space of nonconstant curvature (see [**1**, Tables 1–4, pp. 201–202]. Then the warped product $(N, g) = S^p \times_F M$, of the sphere S^p with the function F defined in Example 3.1, and the manifold $(M, \tilde{g})$, $p \geq 2$, is a non-conformally flat, non-pseudosymmetric, and non- Einstein, Ricci-pseudosymmetric manifold.

Using Theorem 4.2 and the fact that the projective space $\mathbf{H}P^n$, $n \geq 1$ (with its standard Riemannian locally symmetric metric), is an Einstein manifold which is not of constant curvature, we get the following corollary.

Corollary 4.2. *The warped product $S^{k+1} \times_F \mathbf{H}P^n$, of the sphere S^{k+1}, $k = 1, 2$, with the function F defined in Example 3.1, and the space $\mathbf{H}P^n$ (with its standard Riemannian locally symmetric metric), is a non-conformally flat, non-pseudosymmetric and non- Einstein, Ricci-pseudosymmetric manifold.*

Example 4.2. We consider the generalized Hopf fibrations: $S^2 \to \mathbf{C}P^{2n+1} \to \mathbf{H}P^n$ and $S^3 \to S^{4n+3} \to \mathbf{H}P^n$, with the projections $\pi_1 : \mathbf{C}P^{2n+1} \to \mathbf{H}P^n$ and $\pi_2 : S^{4n+3} \to \mathbf{H}P^n$, respectively. Let $\{U_\alpha\}_{\alpha \in A}$, be an open covering of the manifold $\mathbf{H}P^n$. Thus we have two families of diffeomorphisms $\Phi_{\alpha,k} : \pi_k^{-1} U_\alpha \to U_\alpha \times S^{k+1}$, $k = 1, 2$. Further, we denote by $i_{\alpha,k}$, $k = 1, 2$, the natural diffeomorphisms $i_{k,\alpha} : U_\alpha \times S^{k+1} \to S^{k+1} \times U_\alpha$. Thus on the open submanifolds: $\pi_1^{-1} U_\alpha$ in $\mathbf{C}P^{2n+1}$ and $\pi_2^{-1} U_\alpha$ in S^{4n+3} are given metric tensors $h_{\alpha,k}$, defined by $h_{\alpha,k} = (i_{k,\alpha} \circ \Phi_{k,\alpha})^* g_k$, where g_k is the warped product metric on $S^{k+1} \times_F \mathbf{H}P^n$, $k = 1, 2$, defined in Corollary 4.2.

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