PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 60 (74), 1996, 121–136

HOMOGENEOUS k-SYMMETRIC SPACES OF INTERIOR TYPE WITH SIMPLE REAL FUNDAMENTAL GROUPS AND THEIR CONNECTION WITH PARABOLIC SPACES

Boris A. Rosenfeld and **Michael P. Zamakhovsky**

Communicated by Mileva Prvanović

Abstract. It is shown that homogeneous k-symmetric spaces of interior type with real fundamental Lie groups G are homogeneous spaces G/H, where H are biparabolic or dyparabolic subgroups (defined in the paper) of groups G. Geometric interpretations of these spaces are given.

1. Homogeneous k-symmetric spaces. Nomizu [1954] has defined homogeneous symmetric spaces with fundamental group G and isotropy group H as homogeneous spaces satisfying the condition

$$G_0^\Phi \subset H \subset G^\Phi \tag{1.1}$$

where G^Φ is closed subgroup of all fixed elements of an involutive automorphism Φ ($\Phi^2 = \mathrm{id}$) and G_0^Φ is the identity component of G^Φ. Particular cases of homogeneous symmetric spaces are Riemannian symmetric spaces first defined by Shirokov [1925] and Levy [1926], pseudo-Riemannian symmetric spaces, and symmetric spaces with affine connections. Theory of symmetric Riemannian and pseudo-Riemannian spaces and spaces with affine connection was built by Cartan who established that all these spaces are determined by involutive automorphisms of their fundamental groups and called them "symmetric spaces". He found deep connections of all these spaces with Lie groups, and in particular of Riemannian and pseudo-Riemannian spaces with simple Lie groups.

Homogeneous symmetric space G/H, where G is a Lie group, is locally determined by the Cartan decomposition of tangent Lie algebra $\mathfrak{G}$ of Lie group G

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e}, \tag{1.2}$$

AMS Subject Classification (1991): Primary 53C35, 17B20

Sections 1–4 are written by B.A. Rosenfeld, and 5–7 – by M.P. Zamakhovsky.

The paper is partially supported by International Scientific Foundation, grant M3G000.

where $\mathfrak{h}$ is subalgebra of $\mathfrak{g}$ which is tangent Lie algebra of subgroup H, and the subalgebra $\mathfrak{h}$ and subspace $\mathfrak{e}$ satisfy conditions

$$[\mathfrak{h}\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}\mathfrak{e}] \subset \mathfrak{e}, \quad [\mathfrak{e}\mathfrak{e}] \subset \mathfrak{h}. \tag{1.3}$$

First generalization of symmetric spaces with affine connections was done by Rashevsky [1951], who defined symmetric spaces with affine connections with torsion as spaces with affine connection for which covariant derivatives of both curvature and torsion tensors are equal to zero. Nomizu [1954] gave another definition of these spaces and called them *reductive spaces*. Reductive space G/H is locally determined by the Cartan decomposition (1.2) of tangent Lie algebra $\mathfrak{g}$ of the group G where subalgebra $\mathfrak{h}$ and subspace $\mathfrak{e}$ satisfy only first two conditions (1.3).

Another natural generalization of symmetric spaces with affine connections was done by Vedernikov [1966, 1972] who proposed to relinquish from involutivity of automorphism Φ and defined homogeneous Φ-spaces as homogeneous spaces G/H determined by arbitrary analytic endomorphism Φ of group G satisfying condition (1.1). Stepanov [1967, 1972] considered a large class of reductive homogeneous Φ-spaces including homogeneous k-symmetric spaces G/H determined by automorphisms Φ of order k, that is, satisfying condition $\Phi^k = \mathrm{id}$. These spaces were researched by Wolf and Gray [1958], Ledger [1971], Fedenko [1977], and Kowalski [1980]. Wolf and Gray found by means of root systems of semisimple Lie group G the condition of k-symmetricity of spaces G/H and gave classification of 3-symmetric spaces with simple fundamental groups. The term "k-symmetric space" was introduced by Ledger. Fedenko gave a classification of k-symmetric spaces with simple real fundamental groups, complete for classical groups and "rough" for exceptional groups and found geometric interpretations of k-symmetric spaces with classical fundamental groups. Note that "trisymmetric spaces" defined by Sabinin [1961, 1972] are not 3-symmetric homogeneous spaces.

Note also another generalization of symmetric Riemannian spaces, "bisymmetric Riemannian spaces" defined by Kantor, Sirota, and Solodovnikov [1995].

2. Simple real Lie algebras. If $\mathfrak{g}$ is a simple real Lie algebra, all real Lie algebras having common complex form $\mathbf{C}\mathfrak{g}$ with $\mathfrak{g}$ can be obtained by the *Cartan algorithm* defined by Cartan [1929]: each involutive automorphism σ in $\mathfrak{g}$ determines its Cartan decomposition (1.2) where $\mathfrak{h}$ and $\mathfrak{e}$ are eigensubspaces of automorphism σ corresponding to eigenvalues 1 and -1 respectively, and the space

$$\mathfrak{g}(\sigma) = \mathfrak{h} \oplus i\mathfrak{e} \qquad (i^2 = -1) \tag{2.1}$$

is a real Lie algebra having common complex form with $\mathfrak{g}$. In the case when automorphism σ is interior or exterior, Lie algebra (2.1) is called algebra of interior or exterior type, respectively. An example of Lie algebra of exterior type is a *compact* Lie algebra obtained from a splitable Lie algebra $\mathfrak{g}$ by the Cartan algorithm corresponding to its automorphism determined by automorphism $\alpha \to -\alpha$ of its root system.

Let us consider a simple real splitable Lie algebra $\mathfrak{g}$ (see Bourbaki [1975, ch. 8, §2]). Let $\mathfrak{k}$ be its splitting Cartan subalgebra, that is, maximal commutative subalgebra, such that for each element X the operator ad$X$ of adjoint representation of $\mathfrak{g}$ can be reduced to diagonal form. If subalgebra $\mathfrak{k}$ is fixed and a base $\pi(\Delta)$ of the root system Δ of $\mathfrak{g}$ consisting of simple roots is chosen, then we obtain *marked split Lie algebra* represented by direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^{\alpha} \tag{2.2}$$

where α are elements of Δ and $\mathfrak{g}^{\alpha}$ is an 1-dimensional subspace consisting of all elements X satisfying condition $[KX] = \alpha(K)X$, for any $K \in \mathfrak{k}$.

Subalgebras

$$\mathfrak{b}_{+} = \mathfrak{k} \oplus \bigoplus_{\alpha\in\Delta_{+}} \mathfrak{g}^{\alpha} \quad \text{and} \quad \mathfrak{b}_{-} = \mathfrak{k} \oplus \bigoplus_{\alpha\in\Delta_{-}} \mathfrak{g}^{\alpha} \tag{2.3}$$

where Δ_{+} and Δ_{-} are sets of positive and negative roots of Δ are *opposite Borel subalgebras* of algebra (2.2). Any subalgebra of $\mathfrak{g}$ containing its Borel subalgebra is called *parabolic subalgebra*. All parabolic subalgebras of simple real split Lie algebra $\mathfrak{g}$ can be obtained as follows: for any set $A = \{\alpha_i, \alpha_j, \dots, \alpha_k\}$ of r simple roots from $\pi(\Delta)$ we denote by $\Delta_q(A)$ the set of all such roots from Δ in whose decompositions as linear combinations of simple roots, the sum of coefficients at roots of A is equal to q. If we denote

$$\mathfrak{g}_0(A) = \mathfrak{k} \oplus \bigoplus_{\alpha\in\Delta_0} \mathfrak{g}^{\alpha} \tag{2.4}$$

and

$$\mathfrak{g}_q = \bigoplus_{\substack{\alpha\in\Delta_q \\ q\neq 0}} \mathfrak{g}^{\alpha}, \tag{2.5}$$

Lie algebra (2.2) can be written as

$$\mathfrak{g} = \mathfrak{g}_{-s}(A) \oplus \dots \oplus \mathfrak{g}_{-1}(A) \oplus \mathfrak{g}_0(A) \oplus \mathfrak{g}_1(A) \oplus \dots \oplus \mathfrak{g}_s(A) \tag{2.6}$$

for which

$$[\mathfrak{g}_q(A)\mathfrak{g}_p(A)] \subset \mathfrak{g}_{q+p}(A). \tag{2.7}$$

Decomposition (2.6) was proposed by Kantor [1966] and Koecher [1967]. Subalgebras

$$\mathfrak{p}_{+}(A) = \mathfrak{g}_0(A) \oplus \mathfrak{g}_1(A) \oplus \dots \oplus \mathfrak{g}_s(A) \tag{2.8}$$

and

$$\mathfrak{p}_-(A) = \mathfrak{g}_{-s}(A) \oplus \ldots \oplus \mathfrak{g}_{-1}(A) \oplus \mathfrak{g}_0(A) \tag{2.9}$$

contain subalgebras (2.3) and therefore are opposite parabolic subalgebras of $\mathfrak{g}$. Automorphism in $\mathfrak{g}$ determined by transformation $\alpha \to -\alpha$ maps (2.8) to (2.9) and vice versa.

Subalgebras $\mathfrak{p}_+(A)$ and $\mathfrak{p}_-(A)$ intersect in $\mathfrak{g}_0(A)$, that is (2.4), which is direct sum of a semisimple Lie algebra $\mathfrak{g}_0'(A)$ and an r-dimensional commutative algebra $\mathfrak{d}^r$. The base of the root system of $\mathfrak{g}_0'(A)$ can be obtained from the base $\pi(\Delta)$ by removing all roots of the set A.

Let us call subalgebras $\mathfrak{g}_0(A)$ and

$$\mathfrak{g}^0(A) = \mathfrak{g}_{-s}(A) \oplus \mathfrak{g}_0(A) \oplus \mathfrak{g}_s(A) \tag{2.10}$$

in $\mathfrak{g}$ its *biparabolic* and *dyparabolic* (that is, *extended biparabolic*; *dyo* is Greek equivalent to Latin word *bis*) subalgebras, respectively.

For $r > 1$ algebra (2.10) is direct sum of the semisimple algebra $\mathfrak{g}^{0\prime}(A)$ and the commutative algebra $\mathfrak{d}^{r-1}$. The base of root system of algebra $\mathfrak{g}^{0\prime}(A)$ can be obtained from the *extended base* $\pi^0(A) = \pi(A) \cup \{-\alpha^0\}$ of the system Δ, where α^0 is the maximal root of this system, by removing all roots of A. For $r = 1$ algebra (2.10) coincides with $\mathfrak{g}^{0\prime}(A)$. Note that removal from $\pi^0(\Delta)$ a simple root entering in the decomposition of the maximal root with coefficient 1 leads anew to the base of Δ. Therefore for $s = 1$ algebra (2.10) coincides with $\mathfrak{g}$ and if the set A does not contain a root entering in the decomposition of the maximal root with coefficient 1 the set A determines two subalgebras of $\mathfrak{g}$, biparabolic and dyparabolic, and if the set A contains such root it determines only a biparabolic subalgebra.

For real nonsplitable algebra $\mathfrak{g}(\sigma)$ obtained from split algebra by Cartan algorithm subalgebras in $\mathfrak{g}(\sigma)$, obtained from subalgebras (2.8), (2.9), (2.4), and (2.10) in $\mathfrak{g}$ by Cartan algorithm corresponding to involutive automorphism σ, are called *opposite parabolic, biparabolic*, and *dyparabolic subalgebras* $\mathfrak{p}_+(A,\sigma)$, $\mathfrak{p}_-(A,\sigma)$, $\mathfrak{g}_0(A,\sigma)$, and $\mathfrak{g}^0(A,\sigma)$, respectively. If $\mathfrak{g}$ is simple splitable Lie algebra the algebra (2.1) coincides with $\mathfrak{g}$ for $\sigma = \mathrm{id}$. Therefore we will later denote any simple real Lie algebra by $\mathfrak{g}(\sigma)$.

Classification of simple Lie algebras can be reduced to classification of bases of their root systems. There are four infinite series of *classical simple Lie algebras* A_n, B_n, C_n $(n \geq 1)$, and $D_n (n \geq 3)$ with isomorphisms

$$A_1 = B_1 = C_1, \quad B_2 = C_2, \quad A_3 = D_3, \tag{2.11}$$

and five classes G_2, F_4, E_6, E_7, and E_8 of *exceptional simple Lie algebras.*

3. Geometric interpretations of simple real Lie groups. Each Lie algebra is a tangent Lie algebra of a Lie group and determines this Lie group up to local automorphism, simple Lie groups are denoted by the notations obtained from notations of their tangent Lie algebras by replacing of gothic letters by corresponding italic letters. All simple Lie groups admit geometric interpretations described by the first author [1955, 1993]. Compact simple Lie groups of classes A_n, B_n, C_n, and D_n are isomorphic on locally isomorphic to groups of motions of elliptic spaces: complex Hermitian space $\mathbf{C}\bar{S}^n$, real space S^{2n}, quaternionic Hermitian space $\mathbf{H}\bar{S}^{n-1}$, and real space S^{2n-1}, respectively. Splitable simple Lie groups of the same classes are isomorphic or local isomorphic to groups of motions of split complex elliptic space $\mathbf{C}'\bar{S}^n$, real pseudoelliptic space S^{2n}_n, split quaternionic elliptic space $\mathbf{H}'\bar{S}^{n-1}$, and real pseudoelliptic space S^{2n-1}_n respectively, these groups for $\mathbf{C}'\bar{S}^n$ and $\mathbf{H}'\bar{S}^{n-1}$ are isomorphic to fundamental groups of real projective space P^n and symplectic space S^{2n-1}_y. Noncompact and nonsplitable simple Lie groups of the same classes are isomorphic or locally isomorphic to groups of motions of real, complex and quaternionic hyperbolic, pseudoelliptic, or pseudohyperbolic spaces or to fundamental groups of quaternionic projective and symplectic spaces having the same complex forms as mentioned elliptic spaces.

Compact and splitable simple Lie groups of class G_2 are isomorphic or locally isomorphic to fundamental groups of G-elliptic and G-pseudoelliptic spaces Sg^6 and Sg^6_3 respectively. Compact simple Lie groups of classes F_4, E_6, E_7, and E_8 are isomorphic or locally isomorphic to groups of motions of octonionic Hermitian elliptic plane $\mathbf{O}\bar{S}^2$ and analogous planes over tensor products $\mathbf{C}\otimes\mathbf{O}$, $\mathbf{H}\otimes\mathbf{O}$, and $\mathbf{O}\otimes\mathbf{O}$, respectively. Splitable simple Lie groups of the same classes are isomorphic or locally isomorphic to groups of motions of split octonionic Hermitian elliptic plane $\mathbf{O}'\bar{S}^2$ and analogous planes over tensor products $\mathbf{C}'\otimes\mathbf{O}'$, $\mathbf{H}'\otimes\mathbf{O}'$, and $\mathbf{O}'\otimes\mathbf{O}'$, respectively. Noncompact and nonsplitable simple Lie groups of the same classes are isomorphic or locally isomorphic to groups of motions of octonionic Hermitian hyperbolic plane and analogous planes over tensor products $\mathbf{C}\otimes\mathbf{O}$, $\mathbf{H}\otimes\mathbf{O}$, and $\mathbf{O}\otimes\mathbf{O}$, and to groups of motions of Hermitian elliptic planes over tensor products $\mathbf{C}'\otimes\mathbf{O}$, $\mathbf{C}\otimes\mathbf{O}'$, $\mathbf{H}'\otimes\mathbf{O}$, $\mathbf{H}\otimes\mathbf{O}'$, and $\mathbf{O}'\otimes\mathbf{O}$. Certain of these groups are isomorphic to fundamental groups of octonionic projective plane $\mathbf{O}P^2$ and symplectic 5-space $\mathbf{O}\bar{S}y^5$ or of complex, quaternionic, and octonionic metasymplectic geometries $\mathbf{C}\bar{M}s$, $\mathbf{H}\bar{M}s$, and $\mathbf{O}\bar{M}s$ defined by H. Freudenthal [1954-1964]. Splitable simple Lie algebras of classes F_4, E_6, E_7, and E_8 are also isomorphic to fundamental groups of real, split complex, split quaternionic, and split octonionic metasymplectic geometries Ms, $\mathbf{C}'\bar{M}s$, $\mathbf{H}'\bar{M}s$, and $\mathbf{O}'\bar{M}s$, respectively, and, for F_4, E_6 and E_7 also of elliptic plane $\mathbf{O}'\bar{S}^2$, projective plane $\mathbf{O}'P^2$ and symplectic 5-space $\mathbf{O}'\bar{S}y^5$, respectively.

4. Parabolic spaces with simple real fundamental groups. Cartan [1913] proved that all linear representations of complex simple Lie group G can be obtained by means of *fundamental representations* which correspond to simple roots of G which Cartan also called fundamental roots. Cartan has shown also a geometric figure connected with each fundamental representation and therefore with each simple (fundamental) root of Lie groups Tits [1956] called these geometric figures *fundamental elements* of Lie groups G, therefore we call these figures *fundamental figures* of G. Stabilizers of fundamental figures are subgroups of G whose tangent Lie algebras are parabolic subalgebras of tangent Lie algebra $\mathfrak{g}$ of G determined by one simple root of $\mathfrak{g}$ corresponding to these figures.

Closed subgroups of G whose tangential Lie algebras are parabolic subalgebras of $\mathfrak{g}$ are called *parabolic subgroups* P of G, these subgroups can be also defined as closed subgroups in G containing the *Borel subgroup* B defined by Borel [1956] as maximal connected resoluble subgroup in G. Tangent Lie algebras of Borel subgroups B in G are Borel subalgebras $\mathfrak{b}$ in $\mathfrak{g}$ and tangent Lie algebras of parabolic subgroups P in G are parabolic subalgebras $\mathfrak{p}$ in $\mathfrak{g}$. Tits [1957] considered also geometric figures whose stabilizers are arbitrary parabolic subgroups in P determined by sets of simple roots of G. Tits called manifolds of these figures *R-spaces*, Wolf [1959] called them *flag manifolds*, in the paper of the authors with Timoshenko [1990] these manifolds are called *parabolic spaces*. Thus parabolic space with fundamental group G is a homogeneous space G/P where P is a parabolic subgroup in G.

These definitions are valable also for real Lie groups G. Simple Lie algebra $\mathfrak{g}(\sigma)$, for $\sigma = \mathrm{id}$ splitable algebra determines up to local isomorphism simple real group $G(\sigma)$, for $\sigma = \mathrm{id}$ splitable group, and subalgebras $\mathfrak{p}_+(A,\sigma)$ and $\mathfrak{p}_-(A,\sigma)$ – opposite parabolic subgroups $P_+(A,\sigma)$ and $P_-(A,\sigma)$ of $G(\sigma)$. The spaces $\Pi_+(A,\sigma) = G(\sigma)/P_+(A,\sigma)$ and $\Pi_-(A,\sigma) = G(\sigma)/P_-(A,\sigma)$ are *opposite parabolic spaces*. The local automorphism of $G(\sigma)$ determined by automorphism $\alpha \to -\alpha$ of its root system maps opposite parabolic spaces to each other. Pairs of points in these spaces corresponding under this automorphism are called *pairs of opposite points*.

If $\mathfrak{r}$ is a homogeneous space with fundamental group $G(\sigma)$, let us call minimal geometric figures in $\mathfrak{r}$ whose stabilizers are subgroups $P_+(A,\sigma)$ and $P_-(A,\sigma)$ in $G(\sigma)$ *opposite parabolic A-figures* in $\mathfrak{r}$.

Two parabolic figures in $\mathfrak{r}$ are called *incident* if intersection of their stabilizers is a parabolic subgroup, that is, they contain the same Borel subgroup. Since parabolic $(\alpha_i, \alpha_j, \dots, \alpha_k)$-figure in $\mathfrak{r}$ is a set of incident $\alpha_{i^-}, \alpha_{j^-}, \dots, \alpha_k$-figures in $\mathfrak{r}$, for description of all parabolic figures in $\mathfrak{r}$ it is sufficient to describe only parabolic α_i-figures in $\mathfrak{r}$.

In the paper of authors with Timoshenko [1990] parabolic figures of homogeneous spaces with simple real fundamental groups are described and it is proved that real simple roots of these groups determine real parabolic figures, imaginary roots of these groups determine imaginary parabolic figures and pairs of conjugate parabolic simple roots determine pairs of imaginary parabolic figures determining real parabolic figures.

5. Interior automorphisms of finite order of simple real Lie algebras. Kac [1969] has described automorphisms Φ of finite order ($\Phi^k = \mathrm{id}$) of semisimple Lie algebras over algebraically closed field. Let us find all interior automorphisms of finite order of simple real Lie algebra $\mathfrak{g}(\sigma)$. First let us prove

LEMMA 5.1. *The Dynkin diagram of a closed and symmetric subset Δ' of irreducible reduced root system Δ can be obtained from the Dynkin diagram or extended Dynkin diagram of Δ by removing certain set of dots representing simple roots.*

Proof. Let Δ' be a closed and symmetric subset of root system Δ, that is, the intersection of $\Delta' + \Delta'$ and Δ enters into Δ' and $\Delta' = -\Delta'$. Then it is a root system and for each base B' of Δ' there is a base $B \subset B'$ of Δ (Bourbaki [1968, ch. 6, §1, propositions 23 and 24]). Hence, since removal from extended base $\pi^0(\Delta)$ of a simple root entering into the decomposition of the maximal root α^0 with coefficient 1 leads to the base of Δ, we obtain the assertion of the lemma.

THEOREM 5.1. *Any interior automorphism Φ of finite order of simple real Lie algebra $\mathfrak{g}(\sigma)$ leaves invariant a biparabolic or dyparabolic subalgebra $\mathfrak{g}_0(A,\sigma)$ or $\mathfrak{g}^0(A,\sigma)$ and its order is equal, respectively, to $s+1$ or s, where s is the sum of coefficients with which simple roots of A enter into the decomposition of the maximal root α^0 of the root system of $\mathfrak{g}(\sigma)$.*

Proof. It is sufficient to prove this theorem for a simple real splitable Lie algebra $\mathfrak{g}$. Let Φ be an arbitrary interior automorphism of finite order of Lie algebra (2.2) and $\mathfrak{g}^\Phi$ is its subalgebra of fixed elements. Since the automorphism Φ is interior its restriction on the Cartan subalgebra $\mathfrak{k}$ is identity and therefore Φ multiplies each root subspace $\mathfrak{g}^\alpha$ by a nonzero real number t_α. Therefore the subalgebra $\mathfrak{g}^\Phi$ has the form

$$\mathfrak{g}^\Phi = \mathfrak{k} \bigoplus_{\alpha\in\Delta^\Phi} \mathfrak{g}^\alpha \tag{5.1}$$

where Δ^Φ is a closed set of roots from Δ (Bourbaki [1975, ch. VIII, §3, lemma 2]). Since $[\mathfrak{g}^\alpha\mathfrak{g}^{-\alpha}] \subset \mathfrak{k}$, Δ^Φ is a symmetric set of roots, therefore by Lemma 5.1 there is such set A of simple roots of $\mathfrak{g}$, for which algebra (5.1) coincides with $\mathfrak{g}_0(A)$ or $\mathfrak{g}^0(A)$.

A set A of simple roots of Lie algebra $\mathfrak{g}$ corresponds to two graduations

$$\begin{aligned}\mathfrak{g}(A) &= \mathfrak{g}_0(A) \oplus \mathfrak{a}_1(A) \oplus \ldots \oplus \mathfrak{a}_s(A)\\ \mathfrak{g}(A) &= \mathfrak{g}^0(A) \oplus \mathfrak{a}_1(A) \oplus \ldots \oplus \mathfrak{a}_{s-1}(A),\end{aligned} \tag{5.2}$$

where $\mathfrak{a}_k(A) = \mathfrak{g}_k(A) \oplus \mathfrak{g}_{-k}(A)$, $k \neq 0$ and s is the sum of coefficients with which simple roots of A enter in the decomposition of maximal root α^0 (it is well known (Bourbaki [1975, p. 105]) that they determine automorphisms of orders $s+1$ and s in Lie algebra $\mathfrak{g}$ leaving invariant subalgebras $\mathfrak{g}_0(A)$ and $\mathfrak{g}^0(A)$, respectively. Since they leave invariant the subalgebra $\mathfrak{k}$, they are interior. The theorem is proved.

Let us denote an interior automorphism of finite order of $\mathfrak{g}(\sigma)$ leaving invariant its subalgebra $\mathfrak{g}_0(A,\sigma)$ or $\mathfrak{g}^0(A,\sigma)$ by $\Phi_0(A,\sigma)$ or $\Phi^0(A,\sigma)$, respectively. Therefore if a simple root α_m enters into the decomposition of the maximal root α^0 with coefficient 1, automorphisms $\Phi_0(\alpha_m,\sigma)$ or $\Phi^0(\alpha_m,\sigma)$ are, respectively, involutive and identical, and if root α_m enters into the decomposition of the maximal root α^0 with coefficient 2, automorphism $\Phi^0(\alpha_m, \sigma)$ is involutive.

Note that exterior automorphisms there are only in simple Lie groups of classes A_n, D_n, and E_6, that is in the groups whose Dynkin diagrams have symmetries; two-fold for groups A_n, D_n $(n \neq 4)$ and E_6, and three-fold for group D_4, that is in fundamental groups of spaces with duality or triality principle, and these automorphisms are connected with symmetries of dual or trial figures.

6. Homogeneous k-symmetric spaces of interior type with real fundamental Lie groups and their connection with parabolic spaces. It is well known that certain usual symmetric spaces, that is, 2-symmetric spaces, are connected with parabolic spaces: for instance the symmetric space with fundamental group A_n admitting a model in the manifold of m-pairs in real projective space P^n, that is, manifold of pairs m-plane $+(n-m-1)$-plane in P^n is connected with two parabolic spaces with the same fundamental group admitting models in manifolds of m-planes and $(n-m-1)$-planes in P^n, as well as symmetric spaces with fundamental groups B_n and D_n admitting models in manifolds of hyperbolic lines in hyperbolic spaces H^{2n} and H^{2n-1} are connected with two parabolic spaces with the same fundamental group admitting models on absolute quadrics of spaces H^{2n} and H^{2n-1}, respectively. Let us show that these facts can be generalized onto all k-symmetric spaces.

We call the subgroups of simple real group $G(\sigma)$ whose tangent Lie algebras are $\mathfrak{g}_0(A, \sigma)$ and $\mathfrak{g}^0(A, \sigma)$ *biparabolic* and *dyparabolic subgroups* $G_0(A, \sigma)$ and $G^0(A, \sigma)$ of $G(\sigma)$, respectively. Note that biparabolic group $G_0(A, \sigma)$ is an intersection of two opposite parabolic subgroups $P_+(A, \sigma)$ and $P_-(A, \sigma)$ of the group $G(\sigma)$.

Theorem 5.1 implies that the isotropy group H of homogeneous k-symmetric space of interior type with real fundamental Lie group $G(\sigma)$ defined by an interior automorphism of order k is a biparabolic or dyparabolic subgroup of $G(\sigma)$. In the first case we call the homogeneous space $G(\sigma)/H$ *space of interior type 1* and denote it $\Sigma_0(A; G(\sigma))$, in the second case we call the space $G(\sigma)/H$ *space of interior type 2* and denote it $\Sigma^0(A; G(\sigma))$.

THEOREM 6.1. *If s is sum of coefficients with which simple roots of the set A enter into the decomposition of the maximal root α^0 of the root system of $\mathfrak{g}(\sigma)$, order of symmetry of spaces $\Sigma_0(A; G(\sigma))$ and $\Sigma^0(A; G(\sigma))$ is equal to $s+1$ and s, respectively.*

Proof. Graduations (5.2) of simple Lie algebra $\mathfrak{g}$ determine automorphisms of orders $s+1$ and s, respectively.

In particular if m_i is coefficient with which the simple root α_i enters into the decomposition of the maximal root α^0 order of symmetry of the spaces $\Sigma_0(\alpha_i; G(\sigma))$ and $\Sigma^0(\alpha_i; G(\sigma))$ is equal to $m_i + 1$ and m_i, respectively.

Therefore if the simple root α_i enters into the decomposition of maximal root α^0 with coefficient 1 or 2, the spaces $\Sigma_0(\alpha_i; G(\sigma))$ or $\Sigma^0(\alpha_i; G(\sigma))$ are 2-symmetric (that is, usual symmetric) spaces, and in the first case the root α_i determines only symmetric space of interior type 1.

Since decomposition (2.10) of Lie algebra $\mathfrak{g}^0(A)$ is the Cartan decomposition, the space $\Sigma(A;G(\sigma)) = G^0(A;G(\sigma))/G_0(A;G(\sigma))$ is a 2-symmetric space.

Decomposition (2.6) of Lie algebra (2.2) implies that tangent space to $\Sigma_0(A;G)$ where G is a splitable group is the direct sum of tangent spaces to opposite parabolic spaces $\Pi_+(A;G)$ and $\Pi_-(A;G)$ or the direct sum of tangent spaces to $\Sigma^0(A;G)$ and $\Sigma(A;G)$. This fact implies

THEOREM 6.2. *Space $\Sigma_0(A;G(\sigma))$ is a space of local trivial fibration with the base $\Pi_+(A;G(\sigma))$ and the fiber $\Pi_-(A;G(\sigma))$ or with the base $\Sigma^0(A;G(\sigma))$ and the fiber $\Sigma(A;G(\sigma))$.*

Therefore

$$\begin{aligned}\dim \Sigma_0(A;G(\sigma)) &= 2\dim \Pi_+(A;G(\sigma)) = \dim G(\sigma) - \dim G_0(A,\sigma),\\ \dim \Sigma^0(A;G(\sigma)) &= \dim \Sigma_0(A;G(\sigma)) - \dim \Sigma(A;G(\sigma))\\ &= \dim G(\sigma) - \dim G^0(A,\sigma).\end{aligned}$$

Therefore also points in $\Sigma_0(A;G(\sigma))$ can be represented by pairs of opposite points in $\Pi_+(A;G(\sigma))$ and $\Pi_-(A;G(\sigma))$ or by pairs of points, one of which is a point in $\Sigma^0(A;G(\sigma))$ and the second is a point in $\Sigma(A;(G(\sigma))$.

If we look the coefficients with which simple roots of simple splitable Lie algebras enter into the decompositions of the maximal roots α^0 of these algebras we find that the spaces

$$\begin{gathered}\Sigma_0(\alpha_m,A_n) \text{ for all } M,\ \Sigma_0(\alpha_1,B_n),\ \Sigma_0(\alpha_n,C_n),\\ \Sigma_0(\alpha_m,D_n) \text{ for } m=1,n-1,n, \Sigma_0(\alpha_m,E_6) \text{ for } m=1,6, \text{ and}\\ \Sigma_0(\alpha_7,E_7)\end{gathered}$$

are symmetric spaces of interior type 1, and

$$\begin{gathered}\Sigma^0(\alpha_m,B_n) \text{ for } m>1,\ \Sigma^0(\alpha_m,C_n) \text{ for } m<n,\ \Sigma^0(\alpha_m,D_n) \text{ for } m\neq 1,n-1,n,\\ \Sigma^0(\alpha_2,G_2), \Sigma^0(\alpha_m,F_4) \text{ for } m=1,4,\ \Sigma^0(\alpha_m,E_6) \text{ for } m=2,3,5,\\ \Sigma^0(\alpha_m,E_7) \text{ for } m=1,2,6, \text{ and } \Sigma^0(\alpha_m,E_8) \text{ for } m=1,8\end{gathered}$$

are symmetric spaces of interior type 2.

Classification of spaces $\Sigma_0(A;G(\sigma))$ and $\Sigma^0(A;G(\sigma))$ can be reduced to classification of subgroups $G_0(A)$ and $G^0(A)$ of simple splitable group G.

7. Figures of k-symmetry in homogeneous spaces with simple real fundamental groups. Like usual symmetric spaces admit models by manifolds of symmetry figures (m-pairs in P^n and lines in H^{2n} and H^{2n-1}, mentioned at the beginning of §6, as well as points, lines, and m-planes in all elliptic, hyperbolic, pseudoelliptic and pseudohyperbolic spaces, are examples of symmetry figures), k-symmetric spaces also admit models by manifolds of figures of k-symmetry in spaces $\mathfrak{r}$ listed in §3.

We call *k-symmetry A-figure of interior type 1 or 2* in $\mathfrak{r}$ minimal geometric figure in $\mathfrak{r}$ whose stabilizer is, respectively, subgroup $G_0(A,\sigma)$ or $G^0(A,\sigma)$ in fundamental group of $\mathfrak{r}$. Theorem 6.1 implies

THEOREM 7.1. *$(s+1)$-symmetry A-figure of interior type 1 in $\mathfrak{r}$ is pair of opposite parabolic A-figures in this space.*

First let us consider s-symmetry A-figures of interior type 2 in spaces $\mathfrak{r}$.

LEMMA 7.1. *Intersection of opposite parabolic A-figures in $\mathfrak{r}$ is empty.*

Proof. Let us denote intersection of opposite parabolic A-figures $p_+(A)$ and $p_-(A)$ in $\mathfrak{r}$ by X_0 and by X_+ and X_- complements of X_0 to $p_+(A)$ and $p_-(A)$, respectively. Then opposite parabolic subgroups $P_+(A,\sigma)$ and $P_-(A,\sigma)$ in the group $G(\sigma)$ admit linear representations in the space $[X-X_0X_+]$ by linear operators with matrices

$$\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix}, \tag{7.1}$$

where 0 are zero matrices. Operators (7.1) transform subspaces X_+ and X_- into themselves, therefore these subspaces are opposite parabolic A-figures, and X_0 is empty set. The lemma is proved.

The opposite parabolic A-figures $p_+(A)$ and $p_-(A)$ in $\mathfrak{r}$ generate a subspace in $\mathfrak{r}$. Let us call this subspace *sum* of $p_+(A)$ and $p_-(A)$ and denote that sum $p_+(A)+p_-(A)$.

LEMMA 7.2. *Points of 2-symmetric space $\Sigma(A;G(\sigma))$ are represented by opposite parabolic A-figures $p_+(A)$ and $p_-(A)$ in $p_+(A)+p_-(A)$.*

Proof. It is sufficient to prove this lemma for space $\mathfrak{r}$ with simple real splitable fundamental group G. Theorem 7.1 implies that group $G_0(A)$ maps each of opposite parabolic A-figures $p_+(A)$ and $p_-(A)$ into itself, therefore group $G_0(A)$ maps into itself the pair $p_+(A)+p_-(A)$. Therefore by Lemma 7.1 group $\mathfrak{g}_0(A)$ admits linear representation in the space of pairs $p_+(A)+p_-(A)$ by operators with the matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{7.2}$$

where submatrices A_{12} and A_{21} are zero matrices. The same form is the form of matrices of corresponding linear representation of Lie algebra $\mathfrak{g}_0(A)$. It is easy to check that the correspondence mapping elements of subalgebra $\mathfrak{g}_0(A)$ of algebra

$\mathfrak{g}^0(A)$ onto operators with matrices (7.2), where submatrices A_{12} and A_{21} are zero matrices, and elements of subspaces $\mathfrak{g}_s(A)$ and $\mathfrak{g}_{-s}(A)$ in this Lie algebra onto operators with matrices (7.2), where, respectively, submatrices A_{11}, A_{21}, A_{22} and A_{11}, A_{12}, A_{22} are zero matrices, is an isomorphism of Lie algebras. Therefore the group $G^0(A)$ is represented in the space of pairs $p_+(A) + p_-(A)$ by operators with the matrices (7.2) and it is fundamental group of this space, and its subgroup $G_0(A)$ is the stabilizer of each of $p_+(A)$ and $p_-(A)$, which form a symmetry figure in the space of pairs $p_+(A) + p_-(A)$. Lemma is proved.

This lemma implies that if a pair of two opposite parabolic A-figures in $\mathfrak{r}$ generates whole space $\mathfrak{r}$, the subgroup $G^0(A)$ coincides with group G and $\mathfrak{r}$, is 2-symmetric space, and corresponding symmetry figures are pairs of opposite parabolic figures.

The manifold of pairs of opposite parabolic A-figures in $\mathfrak{r}$ is a space of local trivial fibration whose base is manifold of sums of opposite parabolic A-figures in $\mathfrak{r}$ and fibers above $p_+(A) + p_-(A)$ are $p_+(A)$ and $p_-(A)$. Therefore Theorem 6.1 and Lemma 7.2 imply also

THEOREM 7.2. *k-symmetry A-figure of interior type 2 in space $\mathfrak{r}$ is the sum of pairs of its opposite parabolic A-figures.*

Thus manifold of k-symmetry A-figures of interior type 1 or 2 in homogeneous space $\mathfrak{r}$ with simple real fundamental group $G(\sigma)$ is a model of homogeneous k-symmetric space of interior type 1 or 2, respectively, determined by a set A of simple roots of simple splitable Lie group G.

Let us find k-symmetry α_m-figures of interior type 1 and 2 in homogeneous spaces with all simple real splitable fundamental groups. We will use the description of parabolic figures in these spaces given in Rosenfeld, Zamakhovsky and Timoshenko [1990] and the numeration of simple roots in the book Bourbaki [1968].

1. Each simple root α_m of group A_n enters into decomposition of maximal root α^0 with coefficient 1. Therefore in projective space P^n there are only 2-symmetry α_m-figures of interior type 1, which are pairs of opposite parabolic figures in P^n, that is $(n-m)$-pairs consisting of nonintersecting $(n-m)$-plane and an $(m-1)$-plane. Stabilizers of these figures are groups $A_{n-m} \times A_{m-1} \times D^1$. Space P^n admits an interpretation in the space $\mathbf{C}'\bar{S}^n$, $(n-m)$-pairs in P^n are interpreted as $(n-m)$-planes in $\mathbf{C}'\bar{S}^n$.

2. Simple root α_m of group B_n enters into decomposition of maximal root α^0 with coefficient 1 for $m = 1$ and with coefficient 2 for $m > 1$. Therefore in space S_n^{2n} there are 2-symmetry α_1-figures of interior type 1 and, for $m > 1$, 3-symmetry α_m-figures of interior type 1 and 2-symmetry α_m-figures of interior type 2. A pair of opposite parabolic α_m-figures in S_n^{2n} is a pair of $(m-1)$-planar generators of absolute quadric of S_n^{2n} which are cut out from it by $(2n-m)$-planes intersecting in a $2(n-m)$-plane. Therefore 2-symmetry α_1-figure of interior type 1 in S_n^{2n} is a pair of points of absolute quadric which are cut out from this quadric by two hyperplanes tangent at these points which intersect in a $2(n-1)$-plane, this figure

determines a line in S_n^{2n-1} and its polar $2(n-1)$-plane. Stabilizers of these figures are groups $B_{n-1} \times D^1$.

For $m > 1$ 3-symmetry α_m-figure of interior type 1 in S_n^{2n} which is a pair of opposite parabolic α_m-figures consists of a $2(n-m)$-plane which is $S_{n-m}^{2(n-m)}$, and of two $(m-1)$-planar generators of absolute quadric of S_n^{2n} which lie in a $(2m-1)$-plane which is S_m^{2m-1} and determine a paratactic congruence of lines in this S_m^{2m-1}, a 2-symmetry figure in S_m^{2m-1}. Thus this 3-symmetry figure consists of $2(n-m)$-plane and $(2m-1)$-plane and of paratactic congruence in the last $(2m-1)$-plane. Stabilizers of these figures are groups $B_{n-m} \times A_{m-1} \times D^1$.

For $m > 1$ 2-symmetry α_m-figure of interior type 2 in S_n^{2n} which is a pair of opposite parabolic α_m-figures consists of a $(2m-1)$-plane and its polar $2(n-m)$-plane. Stabilizers of these figures are groups $B_{n-m} \times D_m$.

3. Simple root α_m of group C_n enters into decomposition of maximal root α^0 with coefficient 1 for $m = n$ and with coefficient 2 for $m < n$. Therefore in space Sy_n^{2n-1} there are 2-symmetry α_n-figures of interior type 1 and, for $m < n$, 3-symmetry α_m-figures of interior type 1 and 2-symmetry α_m-figures of interior type 2.

A pair of opposite parabolic α_m-figures in Sy^{2n-1} for $m < n$ is a pair of null $(m-1)$-planes which lie in $(2n-m-1)$-planes corresponding to them in absolute null-system of Sy^{2n-1} and intersecting in a $(2(n-m)-1)$-plane, and for $m = n$ it consists of two null $(n-1)$-planes determining a symplectic congruence of lines in Sy^{2n-1}. Therefore 2-symmetry α_n-figure of interior type 1 in Sy^{2n-1} determines symplectic congruence of lines in Sy^{2n-1}. Stabilizers of these figures are groups $A_{n-1} \times D^1$.

For $m < n$ 3-symmetry α_m-figure of interior type 1 in Sy^{2n-1} which is pair of opposite parabolic α_m-figures consists of a $(2(n-m)-1)$-plane which is $Sy^{2(n-m)-1}$, and of two null $(m-1)$-planes which lie in a $(2m-1)$-plane being Sy^{2m-1} and determine a symplectic congruence of lines in this Sy^{2m-1}. Thus this figure consists of $(2(n-m)-1)$-plane and $(2m-1)$-plane and of symplectic congruence in the last $(2m-1)$-plane. Stabilizers of these figures are groups $C_{n-m} \times A_{m-1} \times D^1$.

For $m < n$ 2-symmetry α_m-figure of interior type 2 in Sy^{2n-1} which is pair of opposite parabolic α_m-figures consists of a $(2m-1)$-plane and $(2(n-m)-1)$-plane corresponding to it in the absolute null-system of Sy^{2n-1}. Stabilizers of these figures are groups $C_{n-m} \times C_m$.

Space Sy^{2n-1} admits an interpretation in the space $\mathbf{H}'\bar{S}^{n-1}$, null m-planes in Sy^{2n-1} are interpreted by $(n-m)/2$-planar generators of absolute Hermitian quadric in $\mathbf{H}'\bar{S}^{n-1}$.

4. Simple root α_m of group D_n enters into decomposition of maximal root α^0 with coefficient 1 for $m = 1$, $n-1$, and n and with coefficient 2 for $1 < m < n-1$. Therefore in space S_n^{2n-1} there are 2-symmetry α_1-figures, α_{n-1}-figures, and α_n-figures of interior type 1 and, for $1 < m < n-1$, 3-symmetry α_m-figures of interior type 1 and 2-symmetry α_m-figures of interior type 2.

A pair of opposite parabolic α_m-figures in S_n^{2n-1} for $m < n-1$ is a pair of

$(m-1)$-planar generators of absolute quadric cut out from it by $(2n-m-1)$-planes intersecting in a $(2(n-m)-1)$-plane and for $m=n-1$ or n it is a pair of $(n-1)$-planar generators of absolute quadric of S_n^{2n-1} which belong to one family of $(n-1)$-planar generators, these two $(n-1)$-planar generators determine a paratactic congruence of lines in S_n^{2n-1}. Therefore 2-symmetry α_1-figure of interior type 1 in S_n^{2n-1} for $m=n-1$ or n determines a paratactic congruence of lines in S_n^{2n-1} and for $m=1$ consists of two points of absolute quadric which are cut out from this quadric by two hyperplanes tangent at these points which intersect in a $2(n-3)$-plane, this figure determines a line in S_n^{2n-1} and its polar $(2n-3)$-plane. Stabilizers of first two of these figures are groups $A_{n-1}\times D^1$, stabilizers of last of these figures are groups $D_{n-1}\times D^1$.

For $1<m<n-1$ 3-symmetry α_m-figure of interior type 1 in S_n^{2n-1} which is pair of opposite parabolic α_m-figures consists of a $(2(n-m)-1)$-plane being $S_{n-m}^{2(n-m)-1}$, and of two $(m-1)$-planar generators of absolute quadric of S_n^{2n-1} which lie in a $(2m-1)$-plane being S_m^{2m-1} and determine a paratactic congruence of lines in this S_m^{2m-1}. Thus this figure consists of $(2(n-m)-1)$-plane and $(2m-1)$-plane and of paratactic congruence in the last $(2m-1)$-plane. Stabilizers of these figures are groups $D_{n-m}\times A_{m-1}\times D^1$.

For $1<m<n-1$ 2-symmetry α_m-figure of interior type 2 in S_n^{2n-1} which is pair of opposite parabolic α_m-figures consists of a $(2m-1)$-plane and its polar $(2(n-m)-1)$-plane. Stabilizers of these figures are groups $D_{n-m}\times D_m$.

Space S_n^{2n-1} admits an interpretation in the space $\mathbf{H}'\bar{S}y^{n-1}$, m-planar generators of absolute quadric in S_n^{2n-1} are interpreted by null $((n-m)/2)$-planes in $\mathbf{H}'\bar{S}y^{n-1}$.

5. Simple roots α_1 and α_2 of group G_2 enter into decomposition of maximal root α^0 with coefficients 3 and 2, respectively. The split group G_2 is locally isomorphic to group of automorphisms in alternative algebra $\mathbf{O}'$ of split octonions and space S_3^6 can be defined as intersection of hypersphere $|\alpha|=1$ in $\mathbf{O}'$ with the metric of pseudo-Euclidean space R_4^8 with hyperplane $\alpha=-\bar{\alpha}$ with identified antipodal points. In space Sg_3^6 there are 4-symmetry α_1-figures and 3-symmetry α_2-figures of interior type 1 and 3-symmetry α_1-figures and 2-symmetry α_2-figures of interior type 2. The 2-symmetry figure is only the last figure, Dynkin diagram of its stabilizer can be obtained from extended Dynkin diagram of the group G_2 by removal of the root α_2, this stabilizer is direct product $A_1\times A_1$.

Simple roots α_1, α_2, α_3, α_4 of group F_4 enter into decomposition of maximal root α^0 with coefficients 2, 3, 4 and 2, respectively. The split group F_4 is the group of motions of Hermitian elliptic plane $\mathbf{O}'\bar{S}^2$ over the algebra $\mathbf{O}'$ of split octonions. In this plane there are 3-symmetry α_1-figures, 4-symmetry α_2-figures, 5-symmetry α_3-figures, and 3-symmetry α_4-figures of interior type 1 and 2-symmetry α_1-figures, 3-symmetry α_2-figures, 4-symmetry α_3-figures, and 2-symmetry α_4-figures of interior type 2. The 2-symmetry figures are only the first and last figures, Dynkin diagrams of their stabilizers can be obtained form extended Dynkin diagram of the group F_4 by removal of the roots α_1 and α_4, theirs stabilizers are direct product $A_1\times C_3$ and group B_4.

Simple roots α_1, α_2, α_3, α_4, α_5, α_6 group E_6 enter into decomposition of maximal root α^0 with coefficients 1, 2, 3, 2 and 1, respectively. The split group E_6 is the group of collineations of projective plane $\mathbf{O}'P^2$ over algebra $\mathbf{O}'$ or group of motions of Hermitian elliptic plane over direct product $\mathbf{O}' \times \mathbf{C}'$. In $\mathbf{O}'P^2$ there are two 2-symmetry figures of exterior type, including "0-pair" point+line, and 2-symmetry α_1-figures and α_6-figures and 3-symmetry α_2-figures, α_3-figures, and α_5-figures of interior type 1 and 2-symmetry α_2-figures, α_3-figures, and α_5-figures of interior type 2. Therefore this plane has only 2-symmetry figures of interior type with stabilizers $D_5 \times D$ and $A_5 \times A_1$ (stabilizers of symmetry figures of exterior type in this plane are C_4 and F^4).

Simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ of group E_7 enter into decomposition of maximal root α^0 with coefficients 2, 2, 2, 4, 3, 2 and 1, respectively. The split group E_7 is the group of motions of Hermitian elliptic plane over direct product $\mathbf{O}' \times \mathbf{H}'$. In this plane there are 3-symmetry α_1-figures, α_2-figures, α_3-figures, and α_6-figures, 5-symmetry α_4-figures, 4-symmetry α_5-figures and 2-symmetry α_7-figures of interior type 1 and 2-symmetry α_1-figures, α_2-figures, α_3-figures, and α_5-figures of interior type 2. The 2-symmetry figures are only α_1-figures, α_2-figures, α_3-figures, and α_5-figures of interior type 2 and α_7-figures of interior type 1, stabilizers of first figures are $D_6 \times A_1$ and A_7, stabilizer of the last figure is $E_6 \times D$.

Simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ of group E_8 enter into decomposition of maximal root α^0 with coefficients 2, 3, 4, 6, 5, 4, 3, and 2, respectively. The split group E_8 is the group of motions of Hermitian elliptic plane over direct product $\mathbf{O}' \times \mathbf{O}'$. In this plane there are 3-symmetry α_1-figures, 4-symmetry α_2-figures, 5-symmetry α_3-figures, 7-symmetry α_4-figures, 6-symmetry α_5-figures, 5-symmetry α_6-figures, 4-symmetry α_7-figures and 3-symmetry α_8-figures of interior type 1 and 2-symmetry α_1-figures, 3-symmetry α_2-figures, 4-symmetry α_3-figures, 6-symmetry α_4-figures, 5-symmetry α_5-figures, 4-symmetry α_6-figures, 3-symmetry α_7-figures, and 2-symmetry α_8-figures of interior type 2. The 2-symmetry figures are only α_1-figures and α_8-figures of interior type 2, stabilizers of these figures are $E_7 \times A_1$ and D_8.

Other k-symmetry figures in spaces with simple real splitable fundamental groups can be obtained analogously for arbitrary set A of simple roots. If the set A consists of roots $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$ the stabilizer $H = G_0(A)$ of correspondent k-figure of interior type 1 is direct product of simple Lie groups whose Dynkin diagrams are obtained from the usual Dynkin diagram of the group G after removal of dots corresponding to simple roots of A and the commutative group D^k and the stabilizer $H = G^0(A)$ of correspondent k-figure of interior type 2 is the direct product of simple Lie groups whose Dynkin diagrams are obtained from the **extended** Dynkin diagram of the group G after removal of dots corresponding to simple roots of A. The dimensions of these k-symmetric spaces are equal to differences $\dim G - \dim H$.

Let us call k_1-symmetry A_1-figure and k_2-symmetry A_2-figure of interior type 1 or 2 *incident* if parabolic A_1-figure and A_2-figure are incident (see n^{0}4). Since parabolic $\{\alpha_i, \alpha_j, \dots, \alpha_h\}$-figure in $\mathfrak{r}$ consists of incident parabolic α_{i}-, α_{j}-, $\dots$, α_h-figures in $\mathfrak{r}$ k-symmetry $\{\alpha_i, \alpha_j, \dots, \alpha_k\}$-figure of interior type 1 or 2 consists of incident $k_{1^-}, k_{2^-}, \dots, k_h$-symmetry $\alpha_{i^-}, \alpha_{j^-}, \dots, \alpha_k$-figures of interior type 1 or 2, respectively.

k-symmetry A-figures of interior type 1 and 2 in homogeneous spaces with simple real nonsplitable fundamental groups can be obtained from k-symmetry A-figures of interior type 1 and 2 in homogeneous spaces with simple real splitable fundamental groups by Cartan algorithm.

References

A. Borel [1956], *Linear Algebraic Groups*, Benjamin, New York – Amsterdam.

N. Bourbaki [1968], *Groupes algebres de Lie, Ch. 4–6*, Hermann, Paris.

N. Bourbaki [1975], *Groupes et algebres de Lie, Ch. 7–8*, Hermann, Paris.

É. Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicite plane, Oeuvres complètes, partie I*, CRNS, Paris, 1984, pp. 493–530.

É. Cartan [1926], *Sur une classe remarquable d'espaces de Riemann, Oeuvres complètes, partie I*, CRNS, Paris, 1984, pp. 587–650.

É. Cartan [1929], *Groupes simples clos et ouverts et géométrie riemannienne, Oeuvres complètes*, partie I, CRNS, Paris, 1984, 1011–1043.

A.S. Fedenko [1977], *Spaces with Symmetries*, Univ. Press, Minsk (in Russian).

H. Freudenthal [1954–1963], *Beziehungen der E_7 und E_8 zur Oktavenebene*, Proc. Kon. Ned. Akad. Wet. **A57**, 218–230, 363–368, **A58**, 151–157, 277–285, **A62**, 165–201, 447–474, **A66**, 457–487.

V.G. Kac, *Automorphisms of finite order in semisimple Lie algebras*, Funkc. Anal. Prilož. **3** (3) 94–96 (in Russian).

I.L. Kantor [1966], *Transitive differential groups and invariant connections on homogeneous spaces*, Trudy Sem. Vekt. Tenz. Anal. **13**, 310–398.

I.L. Kantor, A.I. Sirota, A.S. Solodovnikov, *Bisymmetric Riemannian Spaces*, Izv. RAN, Ser. Mat. **59**, 963–970.

M. Koecher [1967], *Imbedding of Jordan algebras into Lie algebras*, Amer. J. Math. **39**, 787–816, **40**, 476–510.

O. Kowalski [1980], *Generalized Symmetric Spaces*, Springer-Verlag, Berlin – Heidelberg – New York.

A.J. Ledger [1964], *Espaces de Riemann symétriques généralisés*, C. R. Acad. Sci. Paris **A264**, 947–948.

H. Levy [1925], *Forma canonica dei ds^2 per i quali si annulano i simboli di Riemann a cinque indici*, Rend. Accad. Lincei **3**, 65–69.

K. Nomizu [1954], *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76**, 33–65.

P.K. Rashevsky [1951], *On Geometries of Homogeneous Spaces*, Doklady AN SSSR, **80** 169–171 (in Russian).

B.A. Rosenfeld [1955], *Non-Euclidean Geometries*, GITTL Moscow, (in Russian).

B.A. Rosenfeld [1993], *Spaces with exceptioanl fundamental groups*, Publ. Inst. Math. (Beograd) **54** (**68**), 97–119.

B.A. Rosenfeld, M.P. Zamakhovsky and T.A. Timoshenko [1990], *Parabolic spaces*, J. Sov. Math. **47**, 1014–1034.

L.V. Sabinin [1961], *On Geometry of trisymmetric Riemannian spaces*, Sib. Mat. J. **2**, 266–278.

L.V. Sabinin [1972], *Trisymmetric spaces with simple compact groups of motions*, Trudy Sem. Vekt. Tenz. Anal. **16**, 202–226.

P.A. Shirokov [1966], *Constant Fields of Vectors and Tensors of Second Order in Riemannian Spaces, Selected Works in Geometry*, Univ. Press, Kazan, 1966, 256–280 (in Russian).

N.A. Stepanov, *Fundamental facts of the theory of Φ-Spaces*, Izv. VUZov, Matem. **3**, 70–79 (in Russian).

N.A. Stepanov [1972], *Φ-Spaces for General Linear Group*, Izv. VUZov, Mat. **3**, 88–95 (in Russian).

J. Tits [1957], *Sur la géométrie des R-espaces*, J. Math. Pures Appl. **36**, 17–38.

V.I. Vedernikov [1965], *Symmetric spaces and conjugate connections*, Uch. Zap. Kazan. Univ. **1**, 7–59 (in Russian).

V.I. Vedernikov [1972], *On a special class of homogeneous spaces*, Izv. VUZov Mat. **12**, 117–122 (in Russian).

J.A. Wolf [1969], *The action of a real semisimple group on a complex flag manifold*, Bull. Amer. Math. Soc. **75**, 1121–1237.

J.A. Wolf, A. Gray [1968], *Homogeneous spaces defined by Lie group automorphisms*, J. Differential Geom. **1–2**, 77–159.

Department of Mathematics
Pennsylvania State University
University Park, PA 16802
USA

(Received 01 06 1996)