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ON THE CONVERGENCE RATE ESTIMATES FOR FINITE DIFFERENCE SCHEMES APPROXIMATING HOMOGENEOUS INITIAL-BOUNDARY VALUE PROBLEM FOR HYPERBOLIC EQUATION

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Abstract. Applying the interpolation theory of the function spaces, we obtain a new convergence rate estimate for the weak solution of hyperbolic initial-boundary value problem.

1. Introduction. In the case of elliptic boundary value problem, the convergence rate estimates for finite difference schemes of the form

$$\|u-v\|_{H_h^k} \le Ch^{s-k}\|u\|_{H^s}, \quad s>k,$$

are said to be compatible with the smoothness of data [**3**]. Here u denotes the solution of the boundary value problem, v denotes the corresponding discrete approximation, h is the discretisation parameter, H^s denotes the standard Sobolev space and H_h^k is the discrete Sobolev space. The compatible estimates may also be derived in parabolic case [**4**]. But in the hyperbolic case, the usual estimates are not compatible with the smoothness of data [**5**]:

$$\|u-v\|_{C_\tau(H_h^k)} \le Ch^{s-k-1}\|u\|_{H^s(Q)}, \quad s>k+1,$$

These estimates are usually obtained using the Brumble-Hilbert lemma [**2**].

A few years ago, Zlotnik [**12**] applied the interpolation theory to obtain for the hyperbolic projection difference scheme a convergence rate estimate of the order $2(s-k)/3$. Using also the interpolation theory, B.S. Jovanović derived in [**6**] the convergence rate estimate of the same order for the finite difference schemes in the case of homogeneous hyperbolic equation with constant coefficients. Here we show

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how the same estimate can be obtained in the case of homogeneous hyperbolic equation with variable coefficients.

2. Statement of the problem Let $L_q = L_q(0,1)$ $(1 \le q \le \infty)$ be Lebesgue spaces of integrable functions, $H^s = H^s(0,1)$ standard Sobolev spaces, $\mathcal{D}$ the space of infinitely differentiable functions with compact support in $(0,1)$ and H_0^s is the closure of $\mathcal{D}$ in H^s. $(,)$ and $\| \, \|$ denote the inner product and the norm in L_2, respectively. Suppose $a \in L_\infty$ such that

$$a \ge a_0 > 0 \quad \text{in} \quad (0,1) \text{ a.e.} \tag{1}$$

For the operator $L\colon H_0^1 \to H^{-1}$ defined by $Lv = (av')'$ there exist $0 < \lambda_1 < \lambda_2 < \ldots$, $\lim_k \lambda_k = \infty$, such that $L\varphi_k = \lambda_k \varphi_k$ $(k \in N)$; the sequence of eigenfunctions $(\varphi_k)_{k\in N} \subset H_0^1$ is an orthonormed topological basis of L_2 (see [**8**]). Introduce the spaces V^α $(\alpha \ge 0)$ by $V^\alpha = \{v \in L_2 | \|v\|_{V^\alpha}^2 = \sum_{k=1}^{\infty} \lambda_k^\alpha \tilde{v}_k^2 < \infty\}$, where $\tilde{v}_k = (v, \varphi_k)$ are the Fourier coefficients of v in the basis $(\varphi_k)_{k\in N}$.

Consider the initial-boundary value problem for the homogeneous second-order hyperbolic equation (IBVP) in the domain $Q = (0,1) \times (0,T]$:

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big), \quad (x,t) \in Q \\
u(0,t) = u(1,t) = 0 \quad t \in [0,T] \\
u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in (0,1)
\end{gathered}$$

There is the unique weak solution of this problem for $u_0 \in V^1$, $u_1 \in V^0$ (see [**10**], [**12**]). It can be represented as the Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} \tilde{u}_k \varphi_k(x), \tag{2}$$

where,

$$\tilde{u}_k(t) = \tilde{u}_k^{(0)} \cos\left(\sqrt{\lambda_k} t\right) + \frac{\tilde{u}_k^{(1)}}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k} t\right) \tag{3}$$

(here $\tilde{u}_k^{(0)}$, $\tilde{u}_k^{(}1)$ are the Fourier coefficients of the functions u_0, u_1, respectively). The relation (3) shows that the series (2) also has meaning for $t < 0$. In such a way, the solution of (IBVP) can be extended in t on $[-T,T]$; this extension we shall also denote by u. If $u_0 \in V^\alpha$, $u_1 \in V^{\alpha-1}$, it satisfies the relation

$$\max_{t\in[-T,T]} \|\partial^l u/\partial t^l\|_{V^{\alpha-1}} \le C(\|u_0\|_{V^\alpha} + \|u_1\|_{V^{\alpha-1}}), \tag{4}$$

where $l \in Z$, $0 \le l \le \alpha$ (see the analogous relation in [**9**] and the proofs of Propositions 1.1 and 1.3 in [**12**]). Then, in the perfectly same way as we deduced Theorem 3 in [**9**], we obtain, applying (4) that for $a \in C^3$ satisfying (1), the following assertion holds:

If $u_0 \in V^\alpha$, $u_1 \in V^{\alpha-1}$ then

$$\max_{t\in[-T,T]} \|\partial^l u/\partial t^l\|_{H^{\alpha-1}} \le C(\|u_0\|_{V^\alpha} + \|u_1\|_{V^{\alpha-1}}), \tag{5}$$

where $1 \le \alpha \le 4$, $l \in Z$, $0 \le l \le \alpha$.

3. Discretisation. Lower estimate. Let $\bar\omega_h$ be a uniform mesh on $[0,1]$ with the stepsize $h = 1/n$, $\omega_h = \bar\omega_h \cap (0,1)$ and $\omega_h^- = \omega_h \cup \{0\}$. We set $\overset{0}{H}(\omega)$ to be the space of all functions defined on $\bar\omega_h$ vanishing at 0 and 1. Introduce the finite differences in x:

$$v_x = (v(x+h) - v(x))/h, \qquad v_{\bar x} = (v(x) - v(x-h))/h.$$

We define the following discrete norms

$$\|v\|_h = \Big(h\sum_{x\in\omega_h} v^2(x)\Big)^{1/2}, \quad |[v\|_h = \Big(h\sum_{x\in\omega_{\bar h}} v^2(x)\Big)^{1/2}$$
$$\|v\|_{H_h^1} = (\|v\|^2 + |[v_x\|^2)^{1/2}.$$

The operator $L_h\colon \overset{0}{H}(\omega) \to \overset{0}{H}(\omega)$ defined by

$$L_h v = \begin{cases} -\frac{1}{2}[(av_x)_{\bar x} + (av_{\bar x})_x], & x \in \omega_h \\ 0, & x \in \{0,1\} \end{cases}$$

is positive on $\overset{0}{H}(\omega)$ and satisfies the inequalities

$$c|[v_x\|_h \le \|v\|_{(L_h)} \le C|[v_x\|_h. \tag{6}$$

Let $\bar\omega_\tau$ be a uniform mesh on $[-\tau/2, T]$ with the stepsize $\tau = T/(m-1/2)$, $\omega_\tau = \bar\omega_\tau \cap (0,T)$, and $\omega_\tau^- = \omega_\tau \cup \{-\tau/2\}$ (see [**6**]). Let us introduce the following notations:

$$v = v(t), \quad \hat v = v(t+\tau), \quad \check v = v(t-\tau), \quad v^j = v((j-1/2)\tau),$$
$$\bar v = (v+\hat v)/2, \quad v_t = (\hat v - v)/\tau, \quad v_{\bar t} = (v - \check v)/\tau.$$

For functions defined on $\bar\omega_h \times \bar\omega_\tau$ we define the norms

$$\|v\|_{C_\tau(H_h^1)} = \max_{t\in\omega_t^-} \|v(\cdot,t)\|_{H_h^1} \quad \text{and} \quad \|v\|_{L_{q,\tau}(L_{2,h})} = \Big(\tau\sum_{t\in\omega_\tau} \|v(\cdot,t)\|_h^q\Big)^{1/q}.$$

One can easily deduce

Lemma 1. *For* $v \in \overset{0}{H}(\omega)$ *the inequality* $\|v\|_{(I+0.25\tau^2(\sigma-1/4)L_h)} \le C\|v\|_h$ *holds if one of the following two conditions is satisfied:*

(i) *If* $\sigma > 1/4$*, then* $\tau/h < C$*, where* C *is an arbitrary constant;*

(ii) *If* $\sigma < 1/4$*, then* $\tau/h \le 4\sqrt{\frac{1-s_0}{(1-4\sigma)c_1^2}}$ *for an* $s_0 \in (0,1)$*, where* c_1 *is a constant depending only on the function* a*.* $\square$

Let S_x and S_t denote the Steklov smoothing operators in x and t:

$$S_x f(x,t) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(s,t)\,ds, \quad S_t f(x,t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f(x,\eta)\,d\eta.$$

For the approximation of (IBVP) we shall use a weighted finite difference scheme (FDS) (see [**7**]):

$$\begin{gathered} v_{t\bar t} = -L_h(\sigma\hat v + (1-2\sigma)v + \sigma\check v), \\ v(0,t) = v(1,t) = 0, \qquad t \in \bar\omega_\tau \\ v^0 = u_0 - \frac{\tau}{2}S_x^2 u_1, \;\; v^1 = u_0 + \frac{\tau}{2}S_x^2 u_1 \end{gathered}$$

Let $z = u - v$ denote the error of the approximation.

Suppose $u_0 \in V^1$, $u_1 \in V^0$, $a \in C^3$ satisfying (1) and that one of conditions in Lemma 1 holds. Then applying the a priori estimate for FDS (see [**7**]) one obtains

$$N(v) \le CN(v^0), \tag{7}$$

where $N^2(w) = \|w_t\|^2_{(I+0.25\tau^2(\sigma-1/4)L_h)} + \|\bar w\|^2_{(L_h)}$. Using Lemma 1, we have

$$\|v_t^0\|_{(I+0.25\tau(\sigma-1/4)L_h)} \le C\|v_t^0\|_h = C\|S_x^2 u_1\|_h \le C\|u_1\| \tag{8}$$

(the last inequality in (8) follows from the Cauchy-Schwartz inequality). The inequality (6) yields

$$\|\bar v^0\|_{(L_h)} = \|u_0\|_{(L_h)} \le C|[(u_0)_x\|_h \le C\|u_0\|_{H^1}$$

(for the last inequality see [**6**]). From this, using equivalency of the norms in H^1 and V^1 (see [**9**]), we obtain

$$\|\bar v^0\|_{(L_h)} \le C\|u_0\|_{V^1}.$$

The estimates (7), (8) and the last inequality yield

$$\max_{t\in\omega_t^-} \|\bar v\|_{(L_h)} \le C(\|u_0\|_{V^1} + \|u_1\|_{V^0}). \tag{9}$$

Further, thanks to (5) (for $l=0$, $\alpha=1$), we obtain

$$\|\bar{u}\|_{(L_h)} \le C\|[(\bar{u})_x\|_h \le C \max_{t\in[-T,T]} \|\partial u/\partial x\| \le C(\|u_0\|_{V^1} + \|u_1\|_{V^0}),$$

whence,

$$\max_{t\in\omega_\tau^-} \|\bar{u}\|_{(L_h)} \le C(\|u_0\|_{V^1} + \|u_1\|_{V^0}). \tag{10}$$

Finally, from (7), (9), (10) follows the lower estimate

$$\|\bar{z}\|_{C_\tau(H_h^1)} \le C(\|u_0\|_{V^1} + \|u_1\|_{V^0}). \tag{11}$$

4. Upper estimate. In this section we suppose that $u_0 \in V^4$, $u_1 \in V^3$, $a \in C^3$ satisfying (1) and that one of the conditions in Lemma 1 holds. Then the inequality (5) implies $\partial^2 u/\partial t^2, \partial^2/\partial x^2 \in H^2(Q)$. Thus, applying the embedding theorem $H^2(Q) \subset C(\bar{Q})$ (see [**11**]) we conclude that $\partial^2 u/\partial t^2, \partial^2 u/\partial x^2$ are continuous. The error z is the solution of the following finite difference scheme:

$$\begin{gathered} z_{\overline{tt}} = -L_h(\sigma\hat{z} + (1-2\sigma)z + \sigma\check{z}) + \psi, \\ z(0,t) = z(1,t) = 0, \qquad t \in \bar{\omega}_\tau \\ z^0 = u\left(x, -\frac{\tau}{2}\right) - u_0(x) + \frac{\tau}{2}S_x^2 u_1, \;\; z^1 = u\left(x, \frac{\tau}{2}\right) - u_0(x) - \frac{\tau}{2}S_x^2 u_1, \end{gathered}$$

where $\psi = u_{t\bar{t}} + L_h(\sigma\hat{u} + (1-2\sigma)u + \sigma\check{u})$. The application of the a priori estimate to z yields

$$N(z) \le C\big(N(z^0) + \frac{1}{\sqrt{c}}\|\psi\|_{L_{1,\tau}(L_{2,h})}\big), \tag{12}$$

where $c=1$ if (i) in Lemma 1 is satisfied or $c=s_0$ if (ii) is satisfied. Let us first estimate $N(z^0)$. Decompose the first term in $N(z^0)$:

$$\begin{aligned} z_t^0 &= \frac{u\left(x,\frac{\tau}{2}\right) - u\left(x,-\frac{\tau}{2}\right)}{\tau} - S_x^2 u_1 \\ &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \frac{\partial u}{\partial t}(x,\eta)d\eta - u_1(x) + u_1(x) - S_x^2 u_1 = g_1(x) + g_2(x). \end{aligned}$$

From

$$g_1(x) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \frac{\partial u}{\partial t}(x,\eta)d\eta - \frac{\partial u}{\partial t}(x,0) = -\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\int_0^{\eta} (\zeta-\eta)\frac{\partial^3 u}{\partial t^3}(x,\zeta)d\zeta d\eta,$$

$$(g_1)_x = -\frac{1}{h\tau}\int_{-\tau/2}^{\tau/2}\int_0^{\eta}\int_x^{x+h} (\zeta-\eta)\frac{\partial^4 u}{\partial t^3\partial x}(\xi,\zeta)d\xi d\zeta d\eta,$$

we have

$$
\begin{aligned}
|[g_1\|_h &\le C|[(g_1)_x\|_h \\
&\le \left[h\sum_{x\in\omega_h} h^{-2}\tau^{-2}\tau^2\Big(\int_x^{x+h}\int_{-\tau/2}^{\tau/2}\int_0^t\left|\frac{\partial^4 u}{\partial t^3\partial x}(\xi,\zeta)\right|d\zeta dt d\xi\Big)^2\right]^{1/2} \qquad (13)\\
&\le C\tau^2\max_{t\in[-T,T]}\left\|\frac{\partial^4 u}{\partial t^3\partial x}\right\|.
\end{aligned}
$$

Further, the equality

$$g_2(x)=u_1(x)-S_x^2u_1=\frac{1}{2h}\int_{x-h}^{x+h}\int_x^s(\xi-s)u_1''(\xi)d\xi ds,$$

and the equivalence of the norms in H^2 and V^2 imply

$$\|g_2\|_h\le Ch^2\|u_1''\|\le Ch^2\|u_1\|_{V^2}. \qquad (14)$$

Then Lemma 1, together with (13), (5), (14) yields

$$\|z_t^0\|_{\left(I+\frac{\tau^2}{4}\left(\sigma-\frac14\right)L_h\right)}\le C(h^2+\tau^2)(\|u_0\|_{V^4}+\|u_1\|_{V^3}). \qquad (15)$$

For

$$\bar z^0=\frac12\left(u\left(x,\frac{\tau}{2}\right)+u\left(x,-\frac{\tau}{2}\right)\right)-u_0(x)=\hat u\left(x,\frac{\tau}{2}\right)-\hat u(x,0),$$

where $\hat u(x,t)=(u(x,t)+u(x,-t))/2$, the identity

$$\bar z^0=\int_0^{\tau/2}\int_0^t\frac{\partial^2\hat u}{\partial t^2}(x,\zeta)d\zeta dt,$$

holds. Hence,

$$
\begin{aligned}
\|\bar z^0\|_{(L_h)}\le C|[(\bar z^0)_x\|_h &\le C\tau^2\max_{t\in[0,T]}\left\|\frac{\partial^3\hat u}{\partial t^2\partial x}\right\|\\
&\le C\tau^2\max_{t\in[-T,T]}\left\|\frac{\partial^3 u}{\partial t^2\partial x}\right\|\le C\tau^2(\|u_0\|_{V^4}+\|u_1\|_{V^3}).
\end{aligned}
$$

From (15) and the last estimate one obtains

$$N(z^0)\le C(h^2+\tau^2)(\|u_0\|_{V^4}+\|u_1\|_{V^3}). \qquad (16)$$

To estimate ψ, we shall rewrite it in the following manner:

$$\psi=u_{\overline{t}t}+L_hu+\sigma\tau^2L_hu_{\overline{t}t}=\alpha+\beta+\gamma,$$

where

$$\alpha = u_{t\bar{t}} - \frac{\partial^2 u}{\partial t^2}, \quad \beta = -Lu + L_h u, \quad \gamma = \sigma\tau^2 L_h u_{t\bar{t}}.$$

Obviously, $\alpha = \alpha_1 + \alpha_2$, where

$$\alpha_1(x,t) = u_{t\bar{t}} - S_x^2 S_t^2 \frac{\partial^2 u}{\partial t^2}, \;\; \alpha_2(x,t) = S_x^2 S_t^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t^2}.$$

But,

$$\begin{aligned} \alpha_1(x,t) &= S_t^2 \frac{\partial^2 u}{\partial t^2} - S_x^2 S_t^2 \frac{\partial^2 u}{\partial t^2} \\ &= \frac{1}{h\tau} \int\limits_{x-h}^{x+h} \int\limits_{x}^{s} \int\limits_{t-\tau}^{t+\tau} (s-\xi) \left(1 - \frac{|s-x|}{h}\right) \left(1 - \frac{|\zeta - t|}{\tau}\right) \frac{\partial^4 u(\xi,\zeta)}{\partial t^2 \partial x^2} d\zeta d\xi ds \end{aligned}$$

wherefrom,

$$\|\alpha_1(\cdot,t)\|_h \leq Ch^2 \max_{t\in[-T,T]} \left\| \frac{\partial^4 u}{\partial t^2 \partial x^2} \right\|,$$

and consequently, referring to (5), we obtain

$$\|\alpha_1\|_{L_{1,\tau}(L_{2,h})} \leq Ch^2 (\|u_0\|_{V^4} + \|u_1\|_{V^3}). \tag{17}$$

The term α_2 can be represented in the form

$$\begin{aligned} \alpha_2(x,t) = &- \frac{1}{h\tau} \int\limits_{x-h}^{x+h} \int\limits_{x}^{s} \int\limits_{t-\tau}^{t+\tau} (\xi - s) \left(1 - \frac{|s-x|}{h}\right) \left(1 - \frac{|\zeta - t|}{\tau}\right) \frac{\partial^4 u(\xi,t)}{\partial t^2 \partial x^2} d\zeta d\xi ds \\ &- \frac{1}{h\tau} \int\limits_{x-h}^{x+h} \int\limits_{t-\tau}^{t+\tau} \int\limits_{t}^{\eta} (\zeta - \eta) \left(1 - \frac{|s-x|}{h}\right) \left(1 - \frac{|\eta - t|}{\tau}\right) \frac{\partial^4 u(s,\zeta)}{\partial t^4} d\zeta d\eta ds, \end{aligned}$$

whence,

$$\|\alpha_2\|_{L_{1,\tau}(L_{2,h})} \leq Ch^2 \max_{t\in[-T,T]} \left\| \frac{\partial^4 u}{\partial t^2 \partial x^2} \right\| + C\tau^2 \max_{t\in[-T,T]} \left\| \frac{\partial^4 u}{\partial t^4} \right\|,$$

and therefore

$$\|\alpha_2\|_{L_{1,\tau}(L_{2,h})} \leq C(h^2 + \tau^2)(\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

From (17) and the last estimate we obtain

$$\|\alpha\|_{L_{1,\tau}(L_{2,h})} \leq C(h^2 + \tau^2)(\|u_0\|_{V^4} + \|u_1\|_{V^3}). \tag{18}$$

Decompose β in the following way: $\beta = \beta_1 + \beta_2 + \beta_3 + \beta_4$, where,

$$\beta_1 = a\left(\frac{\partial^2 u}{\partial x^2} - u_{x\bar{x}}\right), \qquad \beta_2 = a'\left(\frac{\partial u}{\partial x} - \frac{1}{2}(u_x + u_{\bar{x}})\right),$$
$$\beta_3 = \frac{1}{2}(a' - a_x)(u_x - u_{\bar{x}}), \qquad \beta_4 = \frac{1}{4}\left(a' - \frac{1}{2}(a_x + a_{\bar{x}})\right)u_{\bar{x}}.$$

Combining the estimate $\max\limits_{x\in[0,1]} |a(x)| \le C$ and the fact that

$$\frac{\partial^2 u}{\partial x^2} - u_{x\bar{x}} = \frac{1}{h}\int\limits_{x-h}^{x+h}\int\limits_{x}^{s}(\xi - s)\frac{\partial^4 u(\xi,t)}{\partial x^4}d\xi ds,$$

one obtains

$$\|\beta_1\|_{L_{1,\tau}(L_{2,h})} \le Ch^2 \max_{t\in[-T,T]} \|\partial^4 u/\partial x^4\|. \tag{19}$$

Using the relation

$$\frac{\partial u}{\partial x} - \frac{1}{2}(u_x + u_{\bar{x}}) = \frac{1}{2h}\int\limits_{x-h}^{x+h}\int\limits_{x}^{s}(\xi - s)\frac{\partial^3 u(\xi,t)}{\partial x^3}d\xi ds,$$

and the estimate $\max\limits_{x\in[0,1]} |a'(x)| \le C$ we have

$$\|\beta_2\|_{L_{1,\tau}(L_{2,h})} \le Ch^2 \max_{t\in[-T,T]} \|\partial^3 u/\partial x^3\|. \tag{20}$$

Applying Taylor's formula, one has

$$|a'(x) - a_x| \le h \max_{x\in[0,1]} |a''(x)| \le Ch.$$

This estimate and the obvious relation $u_x - u_{\bar{x}} = hu_{x\bar{x}} = hS_x^2(\partial^2 u/\partial x^2)$, imply

$$\|\beta_3\|_{L_{1,\tau}(L_{2,h})} \le Ch^2 \max_{t\in[-T,T]} \|\partial^2 u/\partial x^2\|. \tag{21}$$

From Taylor's formula it follows that

$$\left|a'(x) - \frac{1}{2}(a_x + a_{\bar{x}})\right| \le Ch^2 \max_{x\in[0,1]} |a'''(x)| \le Ch^2.$$

From this, taking into account that

$$u_{\bar{x}} = \frac{1}{h}\int\limits_{x-h}^{x}\frac{\partial u(s,t)}{\partial x}ds,$$

we obtain

$$\|\beta_4\|_{L_{1,\tau}(L_{2,h})} \le Ch^2 \max_{t\in[-T,T]} \|\partial u/\partial x\|. \tag{22}$$

Then (19)–(22), thanks to (5), yield

$$\|\beta\|_{L_{1,\tau}(L_{2,h})} \le Ch^2(\|u_0\|_{V^4} + \|u_1\|_{V^3}). \tag{23}$$

Representing the term γ in the form

$$\gamma = -\frac{\sigma\tau^2}{2}(a_x u_{xt\bar{t}} + a_{\bar{x}} u_{\bar{x}t\bar{t}} + 2au_{x\bar{x}t\bar{t}}),$$

we easily obtain, using preceding techniques, that

$$\|\gamma\|_{L_{1,\tau}(L_{2,h})} \le C\tau^2(\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

The last estimate together with (18), (23) yields

$$\|\psi\|_{L_{1,\tau}(L_{2,h})} \le C(h^2+\tau^2)(\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

From this estimate, (16) and (12) it follows that $N(z) \le C(h^2+\tau^2)(\|u_0\|_{V^4} + \|u_1\|_{V^3})$, whence the upper estimate

$$\|\bar{z}\|_{C_\tau(H_h^1)} \le C(h+\tau)^2(\|u_0\|_{V^4} + \|u_1\|_{V^3}). \tag{24}$$

5. Interpolation. Now we are going to apply the interpolation theory to our problem. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be two interpolation pairs (see [**1**]). Then, if L is a continuous linear operator from $A_1 + A_2$ into $B_1 + B_2$ such that its restrictions $L\colon A_1 \to B_1$ and $L\colon A_2 \to B_2$ are bounded, the inequality

$$\|L\|_{(A_1,A_2)_{\theta,q}\to(B_1,B_2)_{\theta,q}} \le \|L\|^{1-\theta}_{A_1\to B_1} \|L\|^{\theta}_{A_2\to B_2}, \tag{25}$$

holds for $0<\theta<1$, $1\le q\le\infty$, where $(A_1,A_2)_{\theta,q}$ denotes the interpolation space obtained by the K-method of real interpolation (see [**1**]).

THEOREM. *Suppose $a\in C^3$ satisfying* (1), *u is the weak solution of (IBVP), v is the corresponding discrete approximation and let one of the conditions in Lemma* 1 *is satisfied. Then for the error $z = u - v$ the following estimates hold:*

(i) $\|\bar{z}\|_{C_\tau(H_h^1)} \le C(h+\tau)^{2(s-1)/3}(\|u_0\|_{V^s} + \|u_1\|_{V^{s-1}}),\ \ 1\le s\le 4,$

(ii) $\|\bar{z}\|_{C_\tau(H_h^1)} \le C(h+\tau)^{2(s-1)/3}(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}),\ \ 1\le s\le 4,\ \ s\ne \text{integer}+1/2,$

Proof. (i) Let $z^{(0)}$ denote the error in the case when $u_1 = 0$ and $z^{(1)}$ in the case $u_0 = 0$. Define the linear operators R_0, R_1 by $R_0u_0 = \bar{z}^{(0)}$, $R_1u_1 = \bar{z}^{(1)}$.

From (11) and (24) it follows that R_0 is a bounded linear operator from V^1 into $D \equiv C_\tau(H_h^1)$ and from V^4 into D. Of course, the corresponding conclusion also holds for R_1. Therefore, the interpolation inequality (25) yields

$$\|R_0\|_{(V^4,V^1)_{\theta,2}\to D} \le \|R_0\|_{V^4\to D}^{1-\theta}\|R_0\|_{V^1\to D}^{\theta}, \tag{26}$$

$$\|R_1\|_{(V^3,V^0)_{\theta,2}\to D} \le \|R_1\|_{V^3\to D}^{1-\theta}\|R_1\|_{V^0\to D}^{\theta}, \tag{27}$$

Applying the interpolation relation $(V^\alpha, V^\beta)_{\theta,2} = V^{(1-\theta)\alpha+\theta\beta}$, $\alpha > \beta \ge 0$ (see Proposition 4 in [**9**]), we have $(V^4, V^1)_{\theta,2} = V^{4-3\theta}$ and $(V^3, V^0)_{\theta,2} = V^{3-3\theta}$. Setting $4 - 3\theta = s$, from (11), (24), (26)–(27) one obtains

$$\begin{gathered} \|\bar z^{(0)}\|_{C_\tau(H_h^1)} \le C(h+\tau)^{2(s-1)/3}\|u_0\|_{V^s} \quad \text{and} \\ \|\bar z\|^{(1)}\|_{C_\tau(H_h^1)} \le C(h+\tau)^{2(s-1)/3}\|u_1\|_{V^{s-1}}. \end{gathered} \tag{28}$$

Using $\bar z = \bar z^{(0)} + \bar z^{(1)}$ we finally obtain the desired estimate.

(ii) The continuous injection $H_0^s \subset V^s$, $1 \le s \le 4$, $s \ne$ integer $+1/2$ (see [**9**]) applied in (28) implies the estimate (ii). $\square$

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