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SOME REMARKS ON POSSIBLE GENERALIZED INVERSES IN SEMIGROUPS

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Abstract. For a given element a of a semigroup S it is possible that the system of equations in x: $axa = a$, $ax = xa$ is inconsistent, and that one or both systems (S_k): $a^{k+1}x = a^k$, $ax = xa$ and (Σ_k): $axa = a$, $a^k x = xa^k$ are consistent for some positive integer k, in which case they have more than one solution. Some relations between those two systems are established. However, the chief aim of this note is to investigate the possibilities of extending (S_k), by adding new balanced equations, so that this new system has unique solution. It is proved that if the extended system has unique solution, then the generalized inverse of a, defined by it, must be the Drazin inverse. It is also shown that the system $(\Sigma_2) \wedge ax^2 = x^2 a \wedge xax = x$ cannot be extended into a system with unique solution.

1. Let S be a semigroup and let a be a fixed element of S. A term $t(a,x)$ made up from a and a variable $x \in S$ is called an (a,x)-term. Clearly, any (a,x)-term has the form

$$a^{n_1} x^{m_1} a^{n_2} x^{m_2} \cdots a^{n_s} x^{m_s}, \tag{1}$$

where $s \in N$, $n_i \in N$ for $i = 2, \dots, s$; $m_i \in N$ for $i = 1, \dots, s-1$, n_1, $m_s \in N_0$. If $n_1 = 0$ then the term (1) begins with x^{m_1}, and if $m_s = 0$ then it ends with a^{n_s}.

Let $t_1(a,x), \dots, t_r(a,x)$, $t'_1(a,x), \dots, t'_r(a,x)$ be (a,x)-terms. For a given $a \in S$ the system of equations in x:

$$t_1(a,x) = t'_1(a,x), \dots, t_r(a,x) = t'_r(a,x) \tag{2}$$

will be called an (a,x)-system.

Suppose that $\sigma(a,x)$ is an (a,x)-system. If there exists an (a,x)-term $t(a,x)$ such that

$$\sigma(a,u) \wedge \sigma(a,v) \Rightarrow t(a,u) = t(a,v)$$

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we say that $t(a,x)$ is an invariant of the system $\sigma(a,x)$. The following example will be needed later.

E.1. If $k \in N$, the term $a^k x^{k+1}$ is an invariant of the system

$$\sigma(a,x): \quad a^{k+1}x = a^k, \quad ax = xa.$$

Indeed,

$$\begin{aligned}\sigma(a,u)\wedge\sigma(a,v) &\Leftrightarrow a^{k+1}u = a^k,\ au = ua,\ a^{k+1}v = a^k,\ av = va\\ &\Rightarrow a^k u^{k+1} = a^{k+1}vu^{k+1} = va^{k+1}u^{k+1} = va^k u^k\\ &\qquad = a^k vu^k = a^k v^2 u^{k-1} = \cdots = a^k v^{k+1}.\end{aligned}$$

The following assertion is obvious.

A.1. *If $t(a,x)$ is an invariant of the system $\sigma(a,x)$, then the system $\sigma(a,x)\wedge t(a,x) = x$ can have at most one solution.*

If $t_1(a,x) = a^{n_1}x^{m_1}\ldots a^{n_s}x^{m_s}$, $t_2(a,x) = a^{n_1'}x^{m_1'}\cdots a^{n_t'}x^{m_t'}$ are two (a,x)-terms, we say that the formula $t_1(a,x) = t_2(a,x)$ is balanced if

$$n_1 + \cdots + n_s - (n_1' + \cdots + n_t') = m_1 + \cdots + m_s - (m_1' + \cdots + m_t').$$

If all the equations of the system (2) are balanced, we say that (2) is a balanced (a,x)-system.

A balanced (a,x)-system has the property that it reduces to a system of identities in the case when x is the true inverse of a (if S is a monoid). Hence, any balanced (a,x)-system can be taken to define a generalized (pseudo, quasi) inverse of a.

2. Our starting point will be the (a,x)-system

$$axa = a, \qquad ax = xa \tag{3}$$

which defines a rather pleasant generalized inverse of a. Namely, it is well known that if the system (3) is consistent, then the system

$$axa = a, \qquad ax = xa, \qquad xax = x \tag{4}$$

has unique solution $a^{\#}$. This generalized inverse $a^{\#}$ has most of the properties of the true inverse, e.g. $a^{\#\#} = a$.

However, if the system (3) is inconsistent, then it is possible that for some integer $k(>1)$ one or both systems

$$a^{k+1}x = a^k, \qquad ax = xa \tag{S_k}$$

and

$$(\Sigma_k) \qquad axa = a, \qquad a^k x = xa^k,$$

which for $k = 1$ reduce to (3), are consistent.

The system (S_k) was considered by Drazin [**1**] who showed that the system

$$(D_k) \qquad a^{k+1}x = a^k, \qquad ax = xa, \qquad ax^2 = x$$

can have at most one solution. If the system (D_k) is consistent, its unique solution is called the Drazin inverse of a and is denoted by a^D. However, some of the properties of $a^\#$ are lost; in particular a^{DD} need not be equal to a.

The system (Σ_k) was considered, in connection with matrices, in a number of papers, e.g. in [**2**], [**3**], [**4**].

Clearly, if the system (S_k) is consistent for some $k \in N$, then the system (S_{k+1}) is also consistent, while the converse need not be true, and the same holds for the system (Σ_k). If k is the smallest positive integer such that (S_k) is consistent, following Drazin we say that the index of a is k and we write $i(a) = k$. Similarly, if k is the smallest positive integer such that (Σ_k) is consistent, we say that the Σ-index of a is k and we write $i^*(a) = k$. Besides, if S is a monoid and if a has its true inverse, we say that $i(a) = i^*(a) = 0$.

A.2. *If the system (Σ_k) is consistent, then the system (S_k) is also consistent. The converse need not be true.*

Proof. If a_0 is a solution of the system (Σ_k), then $a^k a_0^{k+1}$ is a solution of the system (S_k). That the converse need not be true is shown by the following example.

E.2. Consider the semigroup $S = \{a, b, c\}$, where $c^2 = a$ and $xy = b$ in all other cases. Then $i(a) = 2$, $i(c) = 3$, but $i^*(a)$ and $i^*(c)$ do not exist.

A.3. *We have:* $i^*(a) = 1 \Leftrightarrow i(a) = 1$, $i^*(a) = 2 \Rightarrow i(a) = 2$, *and*

$$(5) \qquad i^*(a) = k \Rightarrow i(a) \leq k, \qquad \text{for} \qquad k \geq 3.$$

Proof. This is an easy consequence of A.2.

The implication (5) suggests the following question:

Q.1. Is there a semigroup in which, for some $k > 2$, both systems (Σ_k) and (S_{k-1}) are consistent, while (Σ_{k-1}) is inconsistent?

An affirmative answer would imply that the inequality in (5) can be strict.

There exist semigroups in which every element a has both indices and $i(a) = i^*(a)$.

E.3. Let M_n be the semigroup of all real matrices of order n. If $a \in M_n$ and if a is nonsingular, then $i(a) = i^*(a) = 0$. If a is singular, then it has one of the following forms

$$(6) \qquad a = TNT^{-1} \qquad \text{or} \qquad a = T(N \oplus R)T^{-1},$$

where T and R are nonsingular and N is nilpotent.

Suppose that $i(a) = k$, i.e. that (S_k) has a solution x and that (S_{k-1}) does not have a solution. Depending on the form of a, write x as

$$x = TPT^{-1} \qquad \text{or} \qquad x = T \begin{Vmatrix} P & Q \\ U & V \end{Vmatrix} T^{-1}.$$

From the equation $a^{k+1}x = a^k$ we get, in both cases, $N^{k+1}P = N^k$. The equality $N^{k-1} = 0$ implies that (S_{k-1}) has a solution; hence, $N^{k-1} \neq 0$. We have

$$N^k = N^{k+1}P = NN^kP = N(N^{k+1}P)P = N^{k+2}P^2 = N^{k+3}P^3 = \cdots = 0,$$

since N is nilpotent. But then

$$x = TPT^{-1} \qquad \text{or} \qquad x = T(P \oplus R^{-1})T^{-1}$$

where $NPN = N$ (e.g. $P = N^+$, the Moore-Penrose inverse of N) is a solution of (Σ_k). Furthermore, (Σ_{k-1}) has no solution, for if this system had a solution, according to A.2 the system (S_{k-1}) would also have a solution, contrary to the hypothesis. Hence, $i(a) = k \Rightarrow i^*(a) = k$.

Conversely, suppose that $i^*(a) = k$. We know that $i(a) \leq k$. However, if $i(a) = p < k$, then $i^*(a) = p < k$, contrary to the hypothesis, and so $i^*(a) = k \Leftrightarrow i(a) = k$.

Finally, since every singular matrix $a \in M_n$ has the form (6) where $N^{k-1} \neq 0$, $N^k = 0$ for some positive integer k, then for that k the systems (S_k) and (Σ_k) are consistent, the systems (S_{k-1}) and (Σ_{k-1}) are inconsistent, and hence a has both indices and $i(a) = i^*(a)$.

Remark. The index of a matrix a, $\operatorname{Ind} a$, is usually defined (see [**5**]) as the smallest positive integer k such that $\operatorname{rank} a^{k+1} = \operatorname{rank} a^k$. Clearly, $\operatorname{Ind} a = i(a) = i^*(a)$.

The above matrix example suggests the following question:

Q.2. Describe the semigroups in which $i^*(a) = k \Leftrightarrow i(a) = k$.

3. Starting with (S_k) we consider, in this section, all possible balanced (a, x)-systems which contain (S_k) as a subsystem. Having in mind the equations (S_k), all (a, x)-terms are

$$a^n \text{ and } a^m x^n \text{ where } m \in \{0, 1, \dots, k\},\ n \in N.$$

Therefore, all balanced (a, x)-equations are:

$$(B_{m,n,r}) \quad \begin{cases} a^{m+r}x^{n+r} = a^m x^n \\ m \in \{0, 1, \dots, k-1\},\ r \in \{1, 2, \dots, k\},\ 1 \leq m + r \leq k,\ n \in N \end{cases}$$

Then, trivially:

A.4. $(B_{m,n,r}) \Rightarrow (B_{m+1,n,r})$,

A.5. $(B_{m,n,r}) \Rightarrow (B_{m,n+1,r})$,

and the converse implications need not be true. Furthermore, we have:

A.6. $(S_k) \wedge (B_{m,n,p}) \Leftrightarrow (S_k) \wedge (B_{m,n,q})$.

Proof. This equivalence follows from:

$$\begin{aligned} a^{m+1}x^{n+1} = a^m x^n &\Rightarrow a^{m+2}x^{n+2} = a^{m+1}x^{n+1} \Rightarrow a^{m+2}x^{n+2} = a^m x^n \Rightarrow \cdots \\ &\Rightarrow a^{m+p}x^{n+p} = a^m x^n \Rightarrow \cdots \Rightarrow a^{m+q}x^{n+q} = a^m x^n \Rightarrow \cdots \\ &\Rightarrow a^k x^{k+n-m} = a^m x^n \Rightarrow a^{k+1}x^{k+n-m+1} = a^{m+1}x^{n+1} \\ &\Rightarrow a^k x^{k+n-m} = a^{m+1}x^{n+1} \Rightarrow a^{m+1}x^{n+1} = a^m x^n, \end{aligned}$$

where it was supposed that $p < q$.

A.7. *If a_0 is a solution of the system (S_k), then*

$$a^k a_0^{k+1},\ a^{k-1}a_0^k, \ldots, a^{k-m}a_0^{k-m+1} \tag{7}$$

are solutions of the system $(S_k) \wedge (B_{m,n,r})$.

Proof. Direct verification.

A.8. *The solutions* (7) *can be mutually different.*

This is shown by the following example.

E.4. If

$$a = \begin{Vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix} \in M_3, \qquad \text{then} \qquad a_0 = \begin{Vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Vmatrix}$$

is a solution of the system (S_3), and the matrices $a^3a_0^4$, $a^2a_0^3$, aa_0^2 are different from one another.

A.9. *If the system (S_k) is consistent, then the system $(S_k) \wedge (B_{m,n,r})$ is also consistent.*

A.10. *If $m \geq 1$, the system $(S_k) \wedge (B_{m,n,r})$ can have more than one solution.*

Assertions A.9 and A.10 are easy consequences of A.7 and A.8.

A.11. *If $n > 1$, the system $(S_k) \wedge (B_{m,n,r})$ can have more than one solution, as shown by the following example.*

E.5. If $a = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} \in M_2$, then for arbitrary $\alpha \in R$, the matrix $\begin{Vmatrix} 0 & \alpha \\ 0 & 0 \end{Vmatrix}$ is a solution of the system $(S_2) \wedge (B_{0,2,1})$.

A.12. *The system $(S_k) \wedge (B_{0,1,r})$ can have at most one solution.*

Proof. Since the term $a^k x^{k+1}$ is an invariant (see E.1) of the system (S_k), the system $(S_k) \wedge (B_{0,1,k})$ cannot have more than one solution. On the other hand, according to A.6 all the systems $(S_k) \wedge (B_{0,1,r})$ are equivalent.

We are now able to prove the following

THEOREM 1. *If the system (S_k) is consistent, the only possible unique generalized inverse of a, defined by a system containing (S_k) as a subsystem, is the Drazin inverse a^D.*

Proof. If (S_k) is consistent, according to A.6 any system $(S_k)\wedge(B_{m,n,r})$ is also consistent. According to A.10 and A.11 it can have more than one solution if $m\geq 1$ or $n>1$, while according to A.12 it has exactly one solution if $m=0$, $n=1$. Since all the systems $(S_k)\wedge(B_{0,1,r})$ are equivalent, we see that $(S_k)\wedge(B_{0,1,r})\Leftrightarrow(S_k)\wedge(B_{0,1,1})$ and this is the Drazin system (D_k).

4. In this section we briefly consider the following two systems

$$axa=a,\quad a^2x=xa^2,\quad ax^2=x^2a \tag{8}$$

and

$$axa=a,\quad a^2x=xa^2,\quad ax^2=x^2a,\quad xax=x \tag{9}$$

which contain (Σ_2) as a subsystem.

There exist semigroups in which

$$(\Sigma_2)\ \text{is consistent} \Leftrightarrow (8)\ \text{is consistent}. \tag{10}$$

E.6. Such a semigroup is M_n. Indeed, if $a\in M_n$ and if a is nonsingular then (10) is true. If a is singular, its minimum polynomial has the form

$$t^m+\alpha_{m-1}t^{m-1}+\cdots+\alpha_1 t\qquad (n\geq m).$$

If $\alpha_1\neq 0$, then $i(a)=i^*(a)=1$ and (10) is true. If $\alpha_1=0$, $\alpha_2\neq 0$, a has the form

$$a=T(N\oplus R)T^{-1},\qquad\text{where}\qquad N=\begin{Vmatrix}0&1\\0&0\end{Vmatrix}$$

while T and R are nonsingular. However, in that case

$$T(M\oplus R^{-1})T^{-1},\qquad\text{where}\qquad M=\begin{Vmatrix}0&0\\1&0\end{Vmatrix}$$

is a solution of both (Σ_2) and (8), and (10) is true.

If $\alpha_1=\alpha_2=0$, the system (Σ_2) is inconsistent, and (10) is again true.

The above example suggests the following question:

Q.3. Is there a semigroup in which (Σ_2) is consistent and (8) is inconsistent?

On the other hand, in any semigroup we have

A.13. *The system* (8) *is consistent if and only if the system* (9) *is consistent.*

Proof. If a_0 is a solution of the system (8), then $a_0 a a_0$ is a solution of the system (9).

A.14. *The system* (9) *can have more than one solution, as shown by the following example.*

E.7. In the semigroup M_2 let $a = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}$. Any matrix $\begin{Vmatrix} \alpha & -\alpha^2 \\ 1 & -\alpha \end{Vmatrix}$ where α is arbitrary, is a solution of (9).

A.15. *The term x^2 is an invariant of the system* (9)*; in other words, if u and v are solutions of* (9)*, then $u^2 = v^2$.*

Proof. If u and v are solutions of (9) then

$$\begin{aligned} &u^2 = uau^2 = u^3a \Rightarrow u^3 = u^4a \Rightarrow u^2 = u^3a = (u^4a)a = u^4a^2, \\ &v^2 = v^2av = av^3 \Rightarrow v^3 = av^4 \Rightarrow v^2 = a^2v^4. \end{aligned}$$

Furthermore,

$$\begin{aligned} u^2 &= u^4a^2 = u^4(ava)(ava) = u^4a^3v^2a = u^4a^4v^2 = u^4a^2a^2v = u^2a^2v^2, \\ v^2 &= a^2v^4 = aua^2uav^4 = au^2a^3v^4 = u^2a^4v^4 = u^2a^2a^2v^4 = u^2a^2v^2, \end{aligned}$$

implying $u^2 = v^2$.

Suppose now that $i^*(a) = 2$. This means that the systems (9) and (D_2) cannot have a common solution. On the other hand, we have

A.16. *If u is any solution of* (9) *and if v is (the unique) solution of* (D_2)*, then $u^2 = v^2$.*

Proof. Let u be a solution of (9) and let v be the unique solution of (D_2). Then

$$\begin{aligned} u^2 &= uauuau = u^3a^2u = u^3(a^3v)u = u^2ua^2avu = u^2(a^2ua)vu \\ &= u^2a^2vu = u^2(a^3v)vu = u^2a^3v^2u = ua^2uav^2u = ua^2v^2u = ua^3v^3u \\ &= a^2uav^3u = a^2v^3u = v^3a^2u = v^3ua^2 = v^4aua^2 = v^4a^2 = v^2. \end{aligned}$$

If the system (8) is consistent, then by A.13 the system (9) is also consistent, but by A.14 it can have more than one solution. We therefore look for a balanced (a,x)-system which contains (9) as a subsystem and which has a unique solution.

Having in mind the equations (9) the only possible (a,x)-terms are

$$a^n, \quad x^n \quad (n \in N), \quad ax, \quad ax^2, \quad a^2x, \quad a^2x^2, \quad xa,$$

and so the only possible balanced equations are

$$(11) \qquad ax = xa, \quad ax^2 = x, \quad a^2x = a, \quad a^2x^2 = ax, \quad a^2x^2 = xa.$$

If we add to the system (9) any one of the equations (11) we obtain a system equivalent to (4). Indeed, it is obvious that $(4) \Rightarrow (9) \wedge (E)$, where (E) is any one of the equations (11), and it is also obvious that $(9) \wedge ax = xa \Rightarrow (4)$. We also have

$$\begin{aligned}
&(9) \wedge ax^2 = x \Rightarrow ax = a^2x^2 \qquad \text{and} \qquad xa = ax^2a = a^2x^2 \Rightarrow ax = xa,\\
&(9) \wedge a^2x = a \Rightarrow ax = a^2x^2 \qquad \text{and} \qquad xa = xa^2x = a^2x^2 \Rightarrow ax = xa,\\
&(9) \wedge a^2x^2 = ax \Rightarrow a^2x^2a = a \Rightarrow xa^2xa = a \Rightarrow xa^2 = a \Rightarrow a^2x = a,\\
&(9) \wedge a^2x^2 = xa \Rightarrow a^3x^2 = a \Rightarrow axa^2x = a \Rightarrow a^2x = a.
\end{aligned}$$

Hence, we have

THEOREM 2. *If $i^*(a) = 2$, there is no balanced (a, x)-system, containing* (9) *as a subsystem, with unique solution.*

Finally, we pose the following question:

Q.4. If $k \in N$, do the assertions analogous to those displayed in this section hold for the systems

$$axa = a, \qquad a^kx = xa^k, \qquad ax^k = x^ka$$

and

$$axa = a, \qquad a^kx = xa^k, \qquad ax^k = x^ka, \qquad xax = x$$

which contain (Σ_k) as a subsystem?

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