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SURFACES OF CONSTANT CURVATURE AND GEOMETRIC INTERPRETATIONS OF THE KLEIN-GORDON, SINE-GORDON AND SINH-GORDON EQUATIONS

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Abstract. It is well known that the Sine-Gordon equation (SGE) $u_{xx} - u_{yy} = \sin u$ admits a geometric interpretation as the differential equation which determines surfaces of constant negative curvature in the Euclidean space R^3. This result can be generalized to the elliptic space S^3 and the hyperbolic space H^3. These results are analogous to the results of Chern that SGE also admits a geometric interpretation as the differential equation which determines spacelike surfaces of constant negative curvature in pseudo-Riemannian spaces V_1^3 of constant curvature, that is in the pseudo-Euclidean space R_1^3, in the pseudoelliptic space S_1^3, and in the pseudohyperbolic space H_1^3, and that the Sinh-Gordon equation (SHGE) $u_{xx} - u_{yy} = \sinh u$ admits geometric interpretations as the differential equation which determines timelike surfaces of constant positive curvature in the same spaces. In this paper it is proved also that the Klein-Gordon equation (KGE) $u_{xx} - u_{yy} = m^2 u$ admits analogous geometric interpretations in the *Galilean space* Γ^3, and in the *pseudo-Galilean space* Γ_1^3, that is, in the affine space E^3 whose plane at infinity is endowed with the geometry of the Euclidean plane R^2 and of the pseudo-Euclidean plane R_1^2, respectively, in the *quasielliptic space* $S^{1,3}$, in the *quasihyperbolic space* $H^{1,3}$, in the *quasipseudoelliptic space* $S_{01}^{1,3}$, and in the *quasipseudohyperbolic space* $H_{01}^{1,3}$, that is, in the projective space P^3 whose collineations preserve two conjugate imaginary planes and two conjugate imaginary points on the line of their intersection, two conjugate imaginary planes and two real points on the line of their intersection, two real planes and two conjugate imaginary points on the line of their intersection, and two conjugate imaginary planes and two real points on the line of their intersection, respectively.

1. The Klein-Gordon, Sine-Gordon and Sinh-Gordon equations. The relativistic wave equation of the motion of a free particle with zero spin, found by physicists O. Klein and V. Gordon is called *Klein-Gordon equation* (KGE). In the case when a particle is characterized by one space coordinate x the equation has the form

$$u_{tt} - u_{xx} = m^2 u, \tag{1}$$

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where t is time, $u = u(t, x)$ is a wave function, and m is the mass of the particle. Note that the equation (1) for $m = 0$ is a classical one-dimensional wave equation (the equation of a vibrating string) whose solution $u(t, x)$ has the form $f(x+t) + f(x-t)$, where $f(t)$ is an arbitrary function.

By analogy with KGE the equations

$$u_{tt} - u_{xx} = \sin u \tag{2}$$

$$u_{tt} - u_{xx} = \sinh u \tag{3}$$

are called *Sine-Gordon equation* (SGE) and *Sinh-Gordon equation* (SHGE), respectively. The equation (2) has also an important physical meaning: since the left-hand side of this equation coincides with the left-hand sides of the equation of a vibrating string and of the wave equation (1), this equation is also a wave equation, but, unlike the equation (1), it is nonlinear and describes physical processes related to the nonlinear waves, in particular solitary waves (solitons) which preserve their shape under interaction (this theory is very important for the theory of plasm).

2. Surfaces of constant curvature in Euclidean, elliptic, hyperbolic, pseudo-Euclidean, pseudoelliptic, and pseudohyperbolic spaces. The equations (2) and (3) have well known geometric meaning. In 1878 Chebyshev (Tchebycheff) in his lecture "The cutting out of clothes" [**1**] considered nets on surfaces in Euclidean space R^3 whose all net quadrilaterals have equal opposite sides (now such nets are called "Chebyshev nets"). If these curves are coordinate curves of such a surface, then the line element of the surface has the form

$$ds^2 = du^2 + 2\cos\varphi du dv + dv^2, \tag{4}$$

where φ is the angle between curves of the net, u and v are lengths of arcs of these curves. Chebyshev also found that the Gaussian curvature K of this surface satisfies the differential equation

$$\varphi_{uv} + K \sin\varphi = 0. \tag{5}$$

This equation for surfaces of constant negative Gaussian curvature is a SGE. In 1900 Hilbert in [**2**] proved that the Chebyshev net on this surface is formed by its asymptotic curves, and therefore the angle u between asymptotic curves on these surfaces satisfies a SGE.

Various authors found many concrete cases of surfaces of constant negative curvature which describe motions of various types of solitons (see Pozniak [**3**, **4**]).

Chern in 1978 in [**5**] proved that the angle between asymptotic curves on spacelike surfaces of constant negative curvature in pseudo-Riemannian spaces V_1^3 of constant curvature, that is, in the pseudo-Euclidean space R_1^3, in the pseudoelliptic space S_1^3, and in the pseudohyperbolic space H_1^3 (see [**6**, p. 67]), and on timelike surfaces of constant positive curvature in the same spaces satisfy equations SGE and SHGE, respectively.

On the surfaces in the spaces S^3, H^3, S^3_1 and H^3_1 there are two types of curvature: the inner or Riemannian curvature K, that is, the curvature of the surfaces regarded as the sectional curvature of a Riemannian space V^2 or a pseudo-Riemannian space V^2_1, and the exterior or Gaussian curvature K_e which is equal to the product of principal geodesic curvatures of the surfaces, that is, normal geodesic curvatures of the curvature lines on the surfaces). The curvatures K and K_e are related by

$$K = K_s + K_e,$$

where K_s is the curvature of the space equal to $1/r^2$ for S^3 and $-1/q^2$ for H^3 and H^3_1 (see [**7**, p. 422]). Note that in the expressions "surface of constant negative or positive curvature" the exterior curvature K_e is always meant, but in the equation (5), which is valid for S^3 and H^3 since this equation follows only from the formula (4) and can be written as $K = R_{1212}/\sin\varphi$, where R_{1212} is the unique coordinate of the Riemann curvature tensor which is expressed only through the coordinates of the metric tensor of a surface (in our case $g_{11} = g_{22} = g_{12} = \cos\varphi$), and their first and second derivatives, the curvature K is the inner curvature.

For instance, consider a Clifford surface in S^3, that is, an equidistant surface of a straight line in this space. This surface is a ruled quadric $[(x^0)^2+(x^1)^2]\cos^2 a/r - [(x^2)^2+(x^3)^2]\sin^2 a/r = 0$ (a is the distance of points on this surface from its axis and r is the radius of curvature of the space), whose rectilinear generators are paratactic to its axis. The asymptotic lines in this surface are its rectilinear generators, the angle u between rectilinear generators of different families is constant and equal to $2a/r$. Therefore u_{xx} and u_{yy} are equal to 0; this surface is isometric to a rhombus with acute angle $2a/r$ in Euclidean plane R^2 with clued opposite sides, and therefore the curvature K of this surface is also equal to 0. The principal geodesic curvatures of the curvature lines of this surface are equal to $1/r\cot a/r$ and $-1/r\tan a/r$, therefore the curvature K_e of this surface is equal to $-1/r^2$, and $K_s + K_e$ is also equal to 0.

3. Surfaces of constant curvature in Galilean and pseudo-Galilean spaces. Consider the surfaces of constant curvature in the spaces Γ^3 and Γ^3_1. This case was first considered by authors in [**8**]. The n-dimensional Galilean and pseudo-Galilean spaces Γ^n and Γ^n_1 can be defined as the affine space E^n whose hyperplane at infinity is endowed by the geometry of the Euclidean space R^{n-1} or the pseudo-Euclidean space R^{n-1}_1 (see [**7**, pp. 295–297]). If in the space E^n a system of affine coordinates is chosen such that the basis vectors $e_2, e_3, \dots, e_n$ are directed to the hyperplane at infinity of R^{n-1} or R^{n-1}_1, the distance d between two points $X(x^1, x^2, \dots, x^n)$ and $Y(y^1, y^2, \dots, y^n)$ is equal to $|y^1 - x^1|$, and if $x^1 = y^1$, when $d = 0$, then these points have another distance d' equal to the distance between the points $X'(x^2, x^3, \dots, x^n)$ and $Y'(y^2, y^3, \dots, y^n)$ in R^{n-1} or R^{n-1}_1, respectively. The motions in Γ^n and Γ^n_1 have the form

$$'x^1 = x^1 + a^1, \quad 'x^i = A^i_1 x^1 + A^i_j x^j + a^i \qquad (i, j = 2, 3, \dots, n),$$

where (A^i_j) is an orthogonal or pseudo-orthogonal $(n-1)\times(n-1)$-matrix, respectively (here the Einstein rule for summation is used). These formulas coincide with the formulas of transformation of orthogonal coordinates in the n-dimensional isotropic or pseudoisotropic spaces I^n or I^n_1, respectively, that is, the n-dimensional affine space E^n whose hyperplane at infinity is endowed with the geometry of the co-Euclidean space $(R^{n-1})^*$ or the copseudo-Euclidean space $(R^{n-1}_1)^*$ corresponding to R^{n-1} and R^{n-1}_1 in the duality principle of the projective space P^{n-1}. The motions in I^n and I^n_1 have the form

$$'x^1 = x^1 + A^1_i x^i + a^1, \quad 'x^i = A^i_j x^j + a^i \qquad (i,j = 2,3,\dots,n).$$

Note that the space I^4 can be regarded as the space-time of the classical mechanics of Galilei-Newton, if the distance between two events $E_1(x_1, y_1, z_1, t_1)$ and $E_2(x_2, y_2, z_2, t_2)$ is defined as the Euclidean distance between these events in the system of reference in which these events are simultaneous, and as $i|t_2 - t_1|$ in the system of reference in which these events have coinciding space coordinates x, y, z. The name of Galilean space is explained by the coincidence of the formulas of motions in Γ^4 with the formulas of transformation of orthogonal coordinates in I^4. Therefore Kotelnikov, who defined the space Γ^4, believed that it is the space-time of classical mechanics of Galilei-Newton, and this opinion was supported by many geometricians (see, for instance, [**7**, pp. 295–297]; see also [**9**]).

The hyperplane at infinity of R^{n-1} and R^{n-1}_1, that is, the $(n-2)$-plane $x^1 = 0$ in the hyperplane $x^0 = 0$, and the absolute imaginary or real hyperquadric in this $(n-2)$-plane, which is the intersection of all hyperspheres in R^{n-1} or R^{n-1}_1, form the *absolutes* of Γ^n and Γ^n_1. For Γ^3 and Γ^3_1 the absolutes consist of the plane at infinity, line $x^1 = 0$, and two imaginary conjugate or real points on this line. Depending on a position relative to the absolute of Γ^3 and Γ^3_1 the lines and planes in these spaces are divided into two classes: lines of general position which do not meet the line $x^1 = 0$, and *special lines* which meet this line: planes of general position which do not contain the line $x^1 = 0$ (the planes Γ^2 or Γ^2_1), and *special planes* which contain this line (the planes R^2 or R^2_1).

At each point X in Γ^3 or Γ^3_1 we determine *orthonormal frames* which consist of vectors e_1, e_2, e_3 of length 1 or i such that the line Xe_1 is a line of general position, and the lines Xe_2, Xe_3 are special lines which divide harmonically the lines joining X with two imaginary conjugate or real points of the absolute whose equations will be written as

$$g_{22}(x^2)^2 + g_{33}(x^3)^3 = 0$$

where $g_{22} = g_{33} = 1$ for Γ^3 and $g_{22} = -g_{33} = \pm 1$ for Γ^3_1.

If a point X is characterized by a position vector x, then the derivation formulas for these frames are

$$dx = \omega^i e_i, \quad de_1 = \omega^u_1 e_u, \quad de_2 = \omega^3_2 e_3, \quad de_3 = \omega^2_3 e_2, \quad \omega^2_3 = -\delta\omega^3_2$$

where $i, j = 1, 2, 3$, $u, v = 2, 3$, $\delta = 1$ for Γ^3 and $\delta = -1$ for Γ^3_1. Exterior differentiation of these equations gives

$$(6) \qquad d\omega^1 = 0, \quad d\omega^u = \omega^i \wedge \omega^u_i, \quad d\omega^u_1 = \omega^v_1 \wedge \omega^u_v, \quad d\omega^3_2 = 0.$$

Formulas (6) show that the linear forms ω^1 and ω^3_2 are locally exact differentials, therefore

$$(7) \qquad \omega^1 = du, \qquad \omega^3_2 = dv.$$

Consider a curve C of general position in Γ^3 or Γ^3_1. If X is a point on this curve, e_1 is tangent vector to this curve at X, e_2 is a special vector of the oscillating plane of this curve at X, and e_3 is the third vector of an orhonormal frame, the derivation equations of this curve are

$$(8) \qquad \frac{dx}{dt} = e_1, \quad \frac{de_1}{dt} = ke_2, \quad \frac{de_2}{dt} = \kappa e_3, \quad \frac{de_3}{dt} = -\delta\kappa e_2,$$

where t is the natural parameter, k and κ are the curvature and the torsion of this curve.

Consider a surface S of general position in Γ^3 or Γ^3_1. Let us suppose that the intersections of this surfaces with Euclidean or pseudo-Euclidean planes $x^1 = \text{const}$ are not straight lines, that is, this surface has no special rectilinear generators. We determine at a point X of this surface the orthonormal frame, whose vectors e_1 and e_2 are tangent vectors to S at X and e_3 is the normal vector to S at X, that is, this vector is orthogonal to e_2 (the vectors e_2 and e_3 of this are in a plane $x^1 = 0$). The differential equation of Pfaff of the surface S is

$$(9) \qquad \omega^3 = 0.$$

The exterior differentiation of this equation gives

$$(10) \qquad \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 = 0,$$

hence by means of the Cartan lemma we obtain

$$(11) \qquad \omega^3_1 = a\omega^1 + b\omega^2, \quad \omega^3_2 = b\omega^1 + c\omega^2.$$

The first fundamental forms of the surface S are

$$(12) \qquad I = ds^2 = (\omega^1)^2$$

for curves of general position and

$$(13) \qquad I_1 = (ds_1)^2 = g_{22}(\omega^2)^2$$

for special curves, that is, intersections of S with planes $x^1 = 0$.

The second fundamental form of the surface S is

$$II = (d^2a, e_3) = g_{33}(\omega^1\omega_1^3 + \omega^2\omega_2^3) = g_{22}[a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2]. \tag{14}$$

We call a surface in Γ_1^3 *spacelike* if $g_{22} = +1$ and *timelike* if $g_{22} = -1$. The line of the absolute determines on the surface S the *Koenigs net* consisting of special curves which are intersections of S with the planes $x^1 = 0$, and of curves of tangency of cones with apices on the line of the absolute tangent to S. The curves of this net are curvature lines of S, since normal lines to S along curves of this net form developable surfaces.

At the points of the curvature lines of S of general position, vectors e_2 of moving frames have constant directions, since they are directed to the apices of cones, therefore for curvature curves of general position $\omega_2^3 = 0$ and, it follows from (7), that the equations of curvature curves of S are $u = \text{const}$, $v = \text{const}$. The coordinates u and v are called *canonical coordinates* on the surface S.

The principal curvatures of S, that is normal curvatures $k_n = II/I$ for curves $\omega^1 = 0$ and $\omega_2^3 = 0$ are, respectively,

$$k_1 = \delta c, \quad k_2 = g_{33}\frac{ac - b^2}{c}. \tag{15}$$

hence the curvature $K = K_e = k_1k_2$ of the surface S is

$$K = k_1k_2 = \delta g_{33}(ac - b^2). \tag{16}$$

Therefore the Gaussian curvature of a surface S in Γ^3 and of a timelike surface in Γ_1^3 is equal to $ac - b^2$ and of a spacelike surface in Γ_1^3 is equal to $b^2 - ac$. Note that the condition for the surfaces S in Γ^3 and Γ_1^3 have no special rectilinear generators is the equality $c = 0$.

The curves on a surface S which are determined by the equation $II = 0$ are asymptotic curves. Since the form II is expressed by the formula (14), the condition for finding asymptotic directions $\varphi = \omega^2/\omega^1$ of general position is

$$a + 2b\varphi + c\varphi^2 = 0. \tag{17}$$

In the case when vector e_1 of the moving frame is fixed at every point A on S, the moving frame is canonical and all other forms are principal, that is

$$\omega_1^2 = \alpha\omega_1 + \beta\omega^2. \tag{18}$$

The exterior differentiation of the forms (11) and (18) and the substitution of expressions (9), (11), (16) and (18) into (6) give

$$-\alpha_2 + \beta_1 + \beta = g_{33}K, \tag{19}$$

$$-a_2 + 2b\beta + b_1 - c\alpha = 0, \tag{20}$$

$$-b_2 + c_1 + c\beta = 0, \tag{21}$$

where the indices $_1$ and $_2$ mean Pfaffian derivatives determined by the formula

$$dp = p_1\omega^1 + p_2\omega^2.$$

Let the vector e_1 be tangent to a curvature curve of general position, that is, the coordinate system on the surface S is canonical system u, v, and let us find the corresponding differential forms of moving frame. Since the curvature of a special curvature curve is $k_1 = dv/ds_1$, where v is angle between tangent lines and s_1 is the length of special curve, formulas (13) and (15) imply that $dv = c\omega^2$. Let us denote the radius of curvature of this curve by

$$\gamma = k_1^{-1} = \delta c^{-1}; \tag{22}$$

then we have

$$\omega^2 = \delta\gamma dv. \tag{23}$$

By the substitution (23) into the second formula (11), and by formulas (7) and (29) we obtain

$$b = 0. \tag{24}$$

Therefore the formulas (15) have the form

$$k_1 = \delta c, \quad k_2 = g_{33}a. \tag{25}$$

By formulas (7) and (23) we can express the Pfaffian derivatives p_1 and p_2 through the partial derivatives p_u and p_v in the form $p_1 = p_u$, $p_2 = cp_v$. Therefore from the equations (20), (21), and (24) we obtain

$$\beta = -\gamma c_u - \frac{\gamma_u}{\gamma}, \quad \alpha = -a_v,$$

that is,

$$\omega_1^2 = -a_v\omega^1 + \gamma_u\gamma^{-1}\omega^2 = -a_v du + \delta\gamma_u dv.$$

In this case formula (19) can be written as

$$a_{vv} + \delta\gamma_{uu} = -\delta a. \tag{26}$$

This formula shows that the conditions of integrability of the differential equations of a surface S are reduced to single differential equation for its principal curvatures k_1 and k_2.

Let a surface S have a curvature $K = \text{const} = \epsilon m^2$ where $\epsilon = \pm 1$.

A) In the case when a surface S is referred to the canonical coordinates let us show that the equation (26) can be reduced to the form (1). In this case formulas (25) can be written as

$$k_1 = \delta c = \gamma^{-1}, \quad k_2 = K k_1^{-1} = \epsilon \delta m^2 \gamma$$

and formula (26) can be written as

$$\epsilon g_{33} m^2 \gamma_{vv} + \delta \gamma_{uu} = -\delta \epsilon g_{33} m^2 \gamma. \tag{27}$$

Let us set $\epsilon = -1$ for a surface in Γ^3 and for a spacelike surface in Γ^3_1; then we obtain

$$\gamma_{uu} - m^2 \gamma_{vv} = m^2 \gamma,$$

that is, if we denote u by t, v/m by x, and the function γ by u, we obtain an equation KGE (1).

Let us set $\epsilon = +1$ for a timelike surface in Γ^3_1; then the equation (27) gives $m^2 \gamma_{vv} - \gamma_{uu} = m^2 \gamma$, that is, if we set $v/m = x$ and $u = t$ we obtain a KGE (1).

B) In the case when a surface S is referred to the asymptotic curves let us show that the function γ satisfies the equation

$$u_{\xi\eta} = m^2 u. \tag{28}$$

This equation coincides with the equation (1) referred to its characteristics

$$\xi = \frac{x - t}{\sqrt{2}}, \quad \eta = \frac{x + t}{\sqrt{2}}.$$

We consider a surface in Γ^3 and a spacelike surface in Γ^3_1 of constant negative Gaussian curvature and a timelike surface in Γ^3_1 of constant positive Gaussian curvature. Formula (16) implies that in the cases $ac - b^2 < 0$. Since in the case $ac - b^2 < 0$ the equation (17) has two distinct real roots, then at every point A on S there are two distinct asymptotic directions. If the vector e_1 is the tangent vector at A in one of these directions, since $\varphi = 0$ is a root of the equation (17), we obtain that

$$a = 0. \tag{29}$$

Therefore formulas (11) and (16) give $\omega_1^3 = b\omega^1$ and $K = -\delta g_{33} b^2$, and, since $K = \text{const}$, we obtain that

$$b = \text{const}. \tag{30}$$

Formula (14) implies that the equations of the asymptotic curves can be written as

$$\omega^2 = 0, \quad 2b\omega^1 + c\omega^2 = 0.$$

Therefore formulas (6) imply that

$$d(c\omega^2) = d(2b\omega^1 + c\omega^2) = (c_1 + c\beta)\omega^1 \wedge \omega^2 = 0. \tag{30}$$

Therefore the condition (21) has the form $c_1 + c\beta = 0$ and the forms $c\omega^2$ and $2b\omega^1 + c\omega^2$ are locally exact differentials. We denote

$$c\omega^2/b = d\xi, \quad 2\omega^1 + c\omega^2/b = d\eta.$$

Now we refer the moving frames to the local coordinates ξ, η. The parametric curves are asymptotic curves of general position, therefore we call the coordinates ξ, η *Chebyshev coordinates*. The Pfaffian derivatives are expressed through partial derivatives as follows

$$p_1 = 2p_\eta, \quad p_2 = \frac{c}{b}(p_\xi + p_\eta).$$

The substitution of p_1 and p_2 into (20) and (21) gives

$$\beta = 2\gamma^{-1}\gamma_\eta, \quad \alpha = 4\delta b\gamma_\eta, \tag{31}$$

that is,

$$\omega_1^2 = \delta b\gamma_\eta. \tag{32}$$

Formulas (18) imply that

$$-4\gamma_{\xi\eta} = \delta b^2\gamma. \tag{33}$$

If we denote $b^2 = m^2$ and $\gamma = u$, we obtain the equation (28).

C) Now we will show that the principal curvature k_2 (of the curve $v = \text{const}$), the angle between asymptotic curves of general position, and the curvatures of asymptotic curves also satisfy a KGE in the form (28).

From (15), (28) and (22) we obtain $k_2 = -g_{33}\delta b^2\gamma$. Using (28) we find roots of the equation (17) for asymptotic directions of general position

$$\varphi_1 = 0, \quad \varphi_2 = -2\delta b\gamma.$$

Therefore the angle between asymptotic directions is equal to $\varphi = \varphi_2 - \varphi_1 = -2\delta b\gamma$. From (B) and (29) it is clear that the functions k_2 and φ satisfy the equation (33).

Let us consider the asymptotic curve $\xi = \text{const}$, that is $\omega^2 = 0$ from (30), and let us find the curvature $\tilde{k}_1$ and the torsion κ_1 of this curve by formulas (7). In this case we find

$$\omega_1^2 = \alpha\omega^1, \ \ \omega_1^3 = 0, \ \ \omega_2^3 = b\omega^1,$$

and hence

$$\tilde{k}_1 = \alpha, \ \ \kappa_1 = b.$$

Therefore from (31), (30), and (B) we obtain that the curvature $\tilde{k}_1$ satisfies equation (33). The tangent vector $\tilde{e}_1$ to the asymptotic curve $\eta = \text{const}$ is $\tilde{e}_1 = e_1 - 2\delta b\gamma e_2$. Using (29) we obtain from (11) and (32)

$$\omega_1^2 = 0, \ \ \omega_1^3 = -2\delta b^2\gamma\omega^1, \ \ \omega_2^3 = -b\omega^1.$$

Therefore for the curve $\eta = \text{const}$ we have

$$\frac{d\tilde{e}_1}{dt} = 4\delta b\gamma_\xi e_2, \ \ \frac{de_2}{dt} = -be_3.$$

Comparing these formulas with formulas (8) we find the curvature and the torsion of the curve $\eta = \text{const}$:

$$\tilde{k}_2 = 4\delta b\gamma_\xi, \ \ \kappa_2 = -b.$$

It is evident that the curvature of this curve also satisfies the equation (33).

Thus for a surface S in Γ^3 and for a spacelike surface S in Γ_1^3 of constant negative Gaussian curvature and for a timelike surface S in Γ_1^3 of constant positive Gaussian curvature which have no special rectilinear generators the radius of curvature γ of special curves on S, the principal curvature k_2 of S, the angle φ between asymptotic curves, and curvatures $\tilde{k}_1$ and $\tilde{k}_2$ of asymptotic curves satisfy equations KGE, and the torsions of asymptotic curves of the surface S with constant Gaussian curvature $-b^2$ in Γ^3 and $g_{33}b^2$ in Γ_1^3 are equal to b and $-b$.

4. Surfaces of constant curvature in quasielliptic, quasihyperbolic, quasipseudoelliptic, and quasipseudohyperbolic spaces. The quasielliptic space $S^{m,n}$, the quasihyperbolic space $H^{m,n}$, the quasipseudoelliptic space $S_{kl}^{m,n}$, and the quasipseudohyperbolic space $H_{kl}^{m,n}$ can be defined as projective space P^n with the absolute consisting of an imaginary or real cone of second order C with equation which can be reduced to the form $g_{aa}(x^a)^2 = 0$ $(a = 0, 1, \ldots, m)$ and of an imaginary or real nondegenerate quadric Q on the real $(n-m-1)$-plane $x^a = 0$, which plays the role of an apex A of this cone, and the equation of the quadric Q can be reduced to the form $g_{uu}(x^u)^2 = 0$ $(u = m+1, \ldots, n)$. If the projective coordinates of the points x and y in these spaces are normalized by the condition $g_{aa}(x^a)^2 = \pm 1$, the distance ω between these points is determined by the formula $\cos\omega/r = g_{aa}x^a y^a$ (for $H^{m,n}$ and $H_{kl}^{m,n}$ $r - qi$ and $\cos\omega/r = \cos h\omega/q$) and if $\omega = 0$ the points x and y are located in the space R^{n-m} or R_i^{n-m} and the distance d between them is equal to the distance between them in these $(n-m)$-spaces (see [**7**, pp. 283–288]).

In the spaces $S^{1,3}$, $H^{1,3}$, $S_{01}^{1,3}$ and $H_{01}^{1,3}$ the role of the cone C is played by the couple of conjugate imaginary or real planes

$$g_{00}(x^0)^2 + g_{11}(x^1)^2 = 0 \tag{34}$$

and the role of the quadric Q is played by the couple of conjugate imaginary or real points

$$g_{22}(x^2)^2 + g_{33}(x^3)^2 = 0 \tag{35}$$

on the line $x^0 = x^1 = 0$. For $S^{1,3}$: $g_{00} = g_{11}$, $g_{22} = g_{33}$; for $H^{1,3}$: $g_{00} = -g_{11}$, $g_{22} = g_{33}$, for $S^{1,3}_{01}$: $g_{00} = g_{11}$, $g_{22} = -g_{33}$; for $H^{1,3}$: $g_{00} = -g_{11}$, $g_{22} = -g_{33}$.

In these spaces we will consider such frames $\{E_i\}$ $(I, J = 0, 1, 2, 3)$ such that the points E_0 and E_1 do not belong to the absolute and divide harmonically the points of meeting of the line E_0E_1 with the planes (34) and the points E_2 and E_3 lie on the line A and divide harmonically the points (35). These frames are orthogonal, that is, the vectors e_i in quasi-Euclidean and quasipseudo-Euclidean 4-spaces $R^{2,4}$, $R^{2,4}_{10}$, $R^{2,4}_{01}$ and $R^{2,4}_{11}$, which represent the points E_i are orthogonal i.e. $g_{ij} = e_ie_j = 0$ for $i \neq j$. We normalize these vectors by the conditions

$$g_{00} = e_0^2 = 1,\ \ g_{11} = e_1^2 = \epsilon = \pm 1,\ \ g_{22} \cdot g_{33} = e_2^2 \cdot e_3^2 = \delta = \pm 1,$$

that is, for $S^{1,3}$: $g_{00} = g_{11} = g_{22} = g_{33} = \epsilon = \delta = 1$; for $H^{1,3}$: $g_{00} = 1$, $\epsilon = g_{11} = -1$, $g_{22} = g_{33} = \delta = 1$; for $S^{1,3}_{01}$: $\epsilon = g_{00} = g_{11} = 1$, $g_{22} = -g_{33} = \pm 1$, $\delta = -1$; for $H^{1,3}_{01}$: $g_{00} = 1$, $\epsilon = g_{11} = -1$, $g_{22} = -g_{33} = \pm 1$, $\delta = -1$.

The derivation formulas for these frames are

$$de_a = \omega_a^i e_i, \quad de_u = \omega_u^v \qquad (a, b = 0, 1;\ u, v = 2, 3;\ I, J = 0, 1, 2, 3), \tag{36}$$

where the differential forms ω_i^j satisfy the conditions

$$\omega_i^i = 0,\ \ \omega_1^0 = \epsilon\omega_0^1,\ \ \omega_3^2 = \delta\omega_2^3.$$

Let us denote the curvature radius of $S^{1,3}$ and $S^{1,3}_{01}$ by ρ and the curvature radius of $H^{1,3}$ and $H^{1,3}_{01}$ by $i\rho$, and the curvature ϵ/ρ^2 of these spaces by K_s. Exterior differentiation of formulas (36) gives the structure equations of these spaces

$$d\omega_0^1 = d\omega_2^3 = 0, \quad d\omega_a^u = -\frac{\epsilon}{\rho^2}\omega_0^a \wedge \omega_0^u + \omega_a^i \wedge \omega_i^u \qquad (i, j = 1, 2, 3). \tag{37}$$

We consider a surface S of arbitrary position (the lines and planes of arbitrary position are defined as in Section 3). We suppose that all sections of S by planes which do not contain the line A are not straight lines. At any point X on S we define an orthonormal frame so that its point E_0 coincides with X, the lines E_0E_1 and E_0E_2 are tangent to S at X and the line E_0E_3 is normal to S at X (that is, E_0E_3 meets the line A at a point which together with the meeting point of A with the plane $E_0E_1E_2$ divides harmonically the points Q). The differential equation of Pfaff of the surface S is

$$\omega_0^3 = 0. \tag{38}$$

Let us denote the forms ω_0^i by ω^i. Then the equation (37) coincides with (9). Exterior differentiation of this equation, as in Section 3, gives formulas (10) and (11). Now, the formula (12) has the form

$$I = ds^2 = (\omega^1)^2 e_1^2 = \epsilon(\omega^1)^2$$

and the formula (13) is valid for special lines $\omega^1 = 0$. The second fundamental form of S coincides with (14).

Since the absolutes of all four spaces $S^{1,3}$, $H^{1,3}$, $S_{01}^{1,3}$ and $H_{01}^{1,3}$ contain a straight line A, this line determines the *Koenigs net* on these surfaces. As in Γ^3 and Γ_1^3 we can prove that the equations of lines of this set are $\omega^1 = 0$ and $\omega_2^3 = 0$ and the first two formulas (36) imply that the forms ω^1 and ω_2^3 are exact differentials, that is formulas (7) are valid. As in Section 3 we call u and v *canonical coordinates* on S. Let us relate the surface S to the canonical coordinate frame; it is equivalent to the condition (24), that is $b = 0$. Since the lines of Koenigs net are curvature lines of the surface, the principal curvatures of S are the values for normal curvature $k_n = II/I$ for these lines $k_1 = \delta c, \quad k_2 = \epsilon g_{33} a$. Therefore the Gaussian curvature K_e of the surface is

$$K_e = k_1 k_2 = \epsilon\delta g_{33} ac = \epsilon g_{22} ac.$$

Let us introduce the notation (16) for the curvature radius of special lines on S and write the form ω_1^2 as (18). Then exterior differentiation of relations (24), $\omega_1^3 = a\omega^1$, and $\omega_2^3 - c\omega^2$ in terms of these relations and formulas (24) and (36) gives the following relations analogous to (19–21):

(39) $$-\alpha_2 + \beta_1 + \beta^2 = -\epsilon(1/\rho^2 + g_{22}K_e)$$

(40) $$\alpha c + a_2 = 0,$$

(41) $$c_1 + c\beta = 0.$$

If we find α and β from (38) and (39) and substitute them in (37) and replace Pfaff derivatives by partial ones, we obtain

(41) $$\delta a_{vv} + \gamma_{uu} = -\epsilon\left(\frac{1}{\rho^2} + g_{22}K_e\right)\gamma$$

analogous to (26).

Let the surface S be of constant Gaussian curvature $K_e = \epsilon_1 m^2$ ($\epsilon_1 = \pm 1$). Then the relation (41) is reduced to

(42) $$\delta\epsilon\epsilon_1 g_{33} m^2 \gamma_{vv} + \gamma_{uu} = -\epsilon(1/\rho^2 + g_{33}\epsilon_1 m^2)\gamma.$$

In the case of the space $S^{1,3}$ for a surface S with Gaussian curvature $K_e = -m^2 < 0$, we set $\varepsilon_1 = -1$, $x = v/m$, $t = u$. Then the equation (41) is reduced to

(43) $$\gamma_{xx} - \gamma_{tt} = (1/\rho^2 - m^2)\gamma,$$

that is, a KGE of the form (1). Note, that for $K_e = -m^2 = 1/\rho^2 = -K_s$ the right-hand side of (43) is equal to 0, and the surfaces in $S^{1,3}$ with Gaussian curvature $K_e = -K_s = -1/\rho^2$ are analogous to Clifford surfaces in S^3.

In the case of the space $H^{1,3}$ for a surface S with $K_e = m^2 > 0$, we set $\epsilon_1 = 1$, $x = v/m$, $t = u$ and obtain a KGE

$$\gamma_{xx} - \gamma_{tt} = (1/\rho^2 + m^2)\gamma. \tag{44}$$

In $H^{1,3}$ also the surfaces with $K_e = -K_s$ and $K = K_s + K_e = 0$, analogous to Clifford surfaces in S^3, are possible.

In the case of $S^{1,3}_{01}$ for a spacelike surface S ($g_{22} = 1$, $g_{33} = -1$) with $K_e = m^2 < 0$, we set $\epsilon_1 = -1$, $x = v/m$, $T = u$ and for a timelike surface S ($g_{22} = -1$, $g_{33} = 1$) with $K_e = m^2 > 0$ we set $\epsilon_1 = 1$, $x = v/m$, $t = u$ and for both these surfaces we obtain a KGE (44). In $S^{1,3}_{01}$ spacelike surfaces with $K_e = -K_s$ and $K = K_s + K_e = 0$, analogous to Clifford surfaces in S^3, are possible.

In the case of $H^{1,3}_{01}$ for a spacelike surface S ($g_{22} = -1$, $g_{33} = 1$) with $K_e = -m^2 < 0$ we set $\epsilon_1 = -1$, $x = v/m$, $t = u$ and for a timelike surface S ($g_{22} = 1$, $g_{33} = -1$) with $K_e = m^2 > 0$ we set $\epsilon_1 = 1$, $x = v/m$, $t = u$ and for both cases we obtain a KGE (42). In $H^{1,3}_{01}$ timelike surfaces with $K_e = -K_s$ and $K = K_s + K_e = 0$ analogous to Clifford surfaces in S^3 are possible.

5. Surfaces of constant curvature in isotropic, pseudoisotropic, and flag spaces. In the *co-Euclidean and copseudo-Euclidean spaces* $(R^3)^*$ and $(R^3_1)^*$ dual to R^3 and R^3_1, respectively, in the *isotropic and pseudoisotropic spaces* I^3 and I^3_1 which can be defined as affine space E^3 whose plane at infinity is endowed with the geometry of the planes $(R^2)^*$ and $(R^2_1)^*$ dual to R^2 and R^2_1, respectively, and in the *flag space* F^3 which can be defined as E^3 whose plane at infinity is endowed with the geometry of the plane Γ^2, all surfaces are isometric to the planes of these spaces (therefore, in I^3 regarded as an isotropic hyperplane in R^4_1, its intersection with the sheets of a hypersphere of imaginary radius isometric to H^3, which can interpreted as horospheres in H^3, are isometric to the plane R^2, see [**7**, pp. 156–158]); that is, all surfaces of these spaces are surfaces of constant curvature; the spaces I^3, I^3_1, and F^3 are cases of the isotropic spaces I^n, pseudoisotropic spaces I^n_i, and flag spaces F^n (see [**7**, pp. 297–312]).

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