PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 62 (76), 1997, 13–25

ON MELLIN-BARNES TYPE OF INTEGRALS AND SUMS ASSOCIATED WITH THE RIEMANN ZETA-FUNCTION

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Communicated by Aleksandar Ivić

Abstract. Two types (binomial and exponential types) of power series, together with a related sum, associated with the Riemann zeta-function $\zeta(s)$ will be investigated by using Mellin-Barnes type integrals. As for generalizations of these sums we shall introduce hypergeometric type generating functions of $\zeta(s)$ and derive their basic properties.

1. Introduction

It is the main aim of this paper to study two types of power series, together with a related sum, associated with the Riemann zeta-function $\zeta(s)$. The first object is a binomial type series (2.1) given below, which will be studied in the next section, while the asymptotic behaviour of an exponential type series (3.2) will be investigated in Section 3. Section 4 will be devoted to the consideration of the sum (4.3). Mellin-Barnes type integrals such as (2.2), (3.3) and (4.4) will play essential roles in these investigations. Furthermore, as for generalizations of these sums we shall introduce hypergeometric type generating functions of $\zeta(s)$ and derive their basic properties in the final section. It should be remarked that functions of this type were recently introduced by Raina and Srivastava [RS], and the author [Ka3], independently of each other. Raina-Srivastava's function in the most general form [RS, (2.4)] includes our $\mathcal{F}_\nu(a,b;c;z)$ and $\mathcal{F}_\nu(a;c;z)$ (see Section 5) as the special cases.

It is worth while noting that efficient applications of Mellin-Barnes type integrals have recently been made by Motohashi [Mo1] and [Mo2] to study the fourth

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Key words and phrases. Riemann zeta-function, Hurwitz zeta-function, Mellin-Barnes integral, hypergeometric function.

AMS Subject Classification (1991): Primary 11M06; Secondary 11M41

The author was partially supported by Grant-in-Aid for Scientific Research (No. 08740032), the Ministry of Education, Science, Sports and Culture, Japan.

power mean of $\zeta(s)$ and Kuznetsov's spectral expansion of Kloosterman-sum zeta-function, respectively. Integrals of this type were also applied to deduce full asymptotic expansions for the mean squares of Dirichlet L-functions and Lerch zeta-functions (see [Ka1] and [Ka2]). Also in this paper, our main theorems result from the arguments of moving the path of integration, similarly to [Ka1], [Ka2], for Mellin-Barnes type integrals. Part of the results in this paper have been announced in [Ka3].

This work was initiated while the author was staying at the Department of Mathematics, Keio University in Yokohama. He would like to express his sincere gratitude to this institution, especially to Professor Iekata Shiokawa for warm hospitality and constant support. The author would also like to thank Professors Masayoshi Hata, Aleksandar Ivić, Kohji Matsumoto and Eiji Yoshida for valuable comments on this work.

2. Binomial type series

Let $\alpha > 0$ be a parameter, and $\zeta(s,\alpha)$ the Hurwitz zeta-function defined by

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s}\qquad(\operatorname{Re}s>1),$$

and its meromorphic continuation over the whole s-plane. Let $\Gamma(s)$ be the gamma-function and $(s)_n=\Gamma(s+n)/\Gamma(s)$ for any integer n Pochhammer's symbol.

The simple relation

$$\sum_{n=2}^{\infty}\{\zeta(n)-1\}=1$$

follows immediately from the inversion of the order of the double sum $\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}m^{-n}$, and was first mentioned (in a different but an equivalent form) by Christian Goldbach in 1729 (see [Sr3, Section 1]). This is in fact derived as a special case of Ramanujan's formula

$$\zeta(\nu,1+x)=\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\zeta(\nu+n)(-x)^n\qquad(|x|<1)\tag{2.1}$$

for any complex $\nu\neq 1$, which gave a base of his various evaluations of sums involving $\zeta(s)$ (see [Ram, Sections 5 and 6]). Noting the relations $\zeta(s,1)=\zeta(s)$ and $(\partial/\partial\alpha)^n\zeta(s,\alpha)=(-1)^n(s)_n\zeta(s+n,\alpha)$, we see that the right-hand side of (2.1) is actually the Taylor series expansion of $\zeta(\nu,1+x)$ as a function of x near $x=0$. Srivastava [Sr1], [Sr2], [Sr3] derived various interesting summation formulae related to (2.1), while Klusch [Kl] considered a generalization of (2.1) to the Lerch zeta-function. The latter direction was further pursued by Yoshimoto, Kanemitsu and the author [YKK]. On the other hand, Rane [Ran] recently applied (2.1) to study the mean square of Dirichlet L-functions. For related results and various generalizations of (2.1), we refer to [Kl], [Sr3] and their references.

In order to describe a prototype of the following discussions, we shall prove (2.1) as an application of Mellin-Barnes type integrals. Suppose first that $\operatorname{Re}\nu > 1$ and set

$$F_\nu(x) = \frac{1}{2\pi i}\int_{(b)} \frac{\Gamma(\nu+s)\Gamma(-s)}{\Gamma(\nu)}\zeta(\nu+s)x^s ds \tag{2.2}$$

for $x > 0$, where b is a constant fixed with $1 - \operatorname{Re}\nu < b < 0$ and (b) denotes the vertical straight line from $b - i\infty$ to $b + i\infty$. We can shift the path of integration in (2.2) to the right, provided $0 < x < 1$, since the order of the integrand is $O\{x^N(N + |\operatorname{Im} s|)^{\operatorname{Re}\nu-1}e^{-\pi|\operatorname{Im} s|}\}$ on the vertical line $\operatorname{Re} s = N + \frac{1}{2}$ with $N = 0, 1, 2, \ldots$. Collecting the residues of the poles at $s = n$ $(n = 0, 1, 2, \ldots)$, we see that $F_\nu(x)$ is equal to the right-hand infinite series in (2.1). On the other hand, since $\zeta(\nu+s) = \sum_{n=1}^\infty n^{-\nu-s}$ converges absolutely for $\operatorname{Re} s = b$, the term-by-term integration is permissible on the right-hand side of (2.2). Each term in the resulting expression can be evaluated by

$$(n+x)^{-\nu} = \frac{1}{2\pi i}\int_{(b)} \frac{\Gamma(-s)\Gamma(\nu+s)}{\Gamma(\nu)} n^{-\nu-s}x^s ds.$$

This is obtained by taking $-z = x/n$ in the formula

$$\Gamma(a)(1-z)^{-a} = \frac{1}{2\pi i}\int_{(\sigma)} \Gamma(-s)\Gamma(a+s)(-z)^s ds$$

for $|\arg(-z)| < \pi$ and $-\operatorname{Re} a < \sigma < 0$, which is a special case of Mellin-Barnes integral for Gauss' hypergeometric function $F(a, b; c; z)$ (cf. [WW, p. 289, 14.51, Corollary]). We therefore obtain

$$F_\nu(x) = \sum_{n=1}^\infty (n+x)^{-\nu} = \sum_{n=0}^\infty (n+1+x)^{-\nu} = \zeta(\nu, 1+x),$$

from which (2.1) follows immediately by analytic continuation.

3. Exponential type series

In 1962, Chowla and Hawkins [CH] found that the sum

$$G_0(x) = \sum_{n=2}^\infty \zeta(n)\frac{(-x)^n}{n!} \qquad (|x| < +\infty)$$

has the asymptotic formula

$$G_0(x) = x\log x + (2\gamma - 1)x + \frac{1}{2} + O(e^{-A\sqrt{x}}) \tag{3.1}$$

as $x \to +\infty$, where γ is Euler's constant and A is a certain positive constant. They conjectured at the same time that the error estimate in (3.1) cannot be essentially sharpened. Let a be an arbitrarily fixed real parameter. Buschman and Srivastava [BS] introduced a more general formulation

$$G_a(x) = \sum_{n>a+1} \zeta(n-a)\frac{(-x)^n}{n!},$$

where n runs through all nonnegative integers with $n > a+1$, and studied its asymptotic behaviour as $x \to +\infty$. The special cases where $a = -2, -1$ and 1 have been investigated by Verma [Ve], Tennenbaum [Te], and Verma and Prasad [VP], respectively.

Let ν be an arbitrary complex parameter. It is in fact possible to treat a slightly general series

$$G_\nu(x) = \sum_{n>\operatorname{Re}\nu+1} \zeta(n-\nu)\frac{(-x)^n}{n!}, \tag{3.2}$$

based on the formula

$$G_\nu(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma(-s)\zeta(s-\nu)x^s ds \tag{3.3}$$

for $x > 0$, where c is a constant fixed with $\operatorname{Re}\nu + 1 < c < [\operatorname{Re}\nu + 2]$. (Here $[\lambda]$ for real λ denotes the greatest integer not exceeding λ.) Formula (3.3) can be proved by shifting the path (c) to the right and collecting the residues of the poles at $s = n$ ($n = [\operatorname{Re}\nu + 2], [\operatorname{Re}\nu + 3], \dots$), since the order of the integrand is $O\{(ex/N)^N e^{-\frac{1}{2}\pi|\operatorname{Im} s|}\}$ on the vertical line $\operatorname{Re} s = N + \frac{1}{2}$ with $N = [\operatorname{Re}\nu + 1]$, $[\operatorname{Re}\nu + 2], \dots$. While the main method of [BS] is Euler-Maclaurin's summation device, our treatment of (3.2) is due to a refinement of the original argument of [CH].

We first give a proof of

Theorem 3.1. *The following formulae hold for all $x \geq 2$.*

(i) *If $\nu \notin \{-1, 0, 1, 2, \dots\}$,*

$$G_\nu(x) = \Gamma(-\nu-1)x^{\nu+1} - \sum_{n=0}^{[\operatorname{Re}\nu+1]} \zeta(n-\nu)\frac{(-x)^n}{n!} + \mathcal{G}_\nu(x); \tag{3.4}$$

(ii) *If $\nu \in \{-1, 0, 1, 2, \dots\}$,*

$$G_\nu(x) = (-1)^\nu \frac{x^{\nu+1}}{(\nu+1)!}\left(\log x + 2\gamma - \sum_{n=1}^{\nu+1}\frac{1}{n}\right) - \sum_{n=0}^{\nu} \zeta(n-\nu)\frac{(-x)^n}{n!} + \mathcal{G}_\nu(x), \tag{3.5}$$

where the empty sum is to be regarded as null. Here $\mathcal{G}_\nu(x)$ is the error term satisfying the estimate

$$\mathcal{G}_\nu(x) = O(x^{-C}) \tag{3.6}$$

for any $C > 0$, where the implied O-constant depends only on C and ν.

Remark. This theorem refines the results in [BS].

Proof. We may restrict our consideration to the case of $\nu \notin \{-1, 0, 1, \dots\}$, since other cases can be treated by taking limits in (3.4). Let C be a constant fixed arbitrarily with $-C < \min(0, \operatorname{Re}\nu + 1)$. Then we can shift the path of integration in (3.3) from (c) to $(-C)$, since the order of the integrand is $O(|\operatorname{Im} s|^B e^{-\frac{1}{2}\pi|\operatorname{Im} s|})$ as $\operatorname{Im} s \to \pm\infty$ (with a positive constant B depending only on $\operatorname{Re} s$ and $\operatorname{Re}\nu$). Collecting the residues of the poles at $s = n$ $(n = 0, 1, \dots, [\operatorname{Re}\nu + 1])$ and $\nu + 1$, we obtain (3.4) with

$$\mathcal{G}_\nu(x) = \frac{1}{2\pi i} \int_{(-C)} \Gamma(-s)\zeta(s-\nu)x^s ds. \tag{3.7}$$

The estimate (3.6) follows immediately by noting that $|x^s| = x^{-C}$ holds on the path $\operatorname{Re} s = -C$. This completes the proof of Theorem 3.1. □

Chowla and Hawkins suggested in [CH] that the error term in (3.1) is expressible in terms of certain 'almost' Bessel functions; however, it seems that the functions in question have not been precisely determined. Let $K_\nu(z)$ be the modified Bessel function of the third kind defined by

$$K_\nu(z) = \frac{\pi}{2\sin \pi\nu} \left\{ I_{-\nu}(z) - I_\nu(z) \right\},$$

where

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}$$

is the Bessel function with purely imaginary argument (cf. [Er2, p. 5, 7.2.2 (12) and (13)]). We can indeed show that $\mathcal{G}_\nu(x)$ has the Voronoï type summation formula (cf. [Iv, Chapter 3]) involving $K_{\nu+1}(z)$.

Theorem 3.2. *For any $x \geq 2$ we have*

$$\begin{aligned} \mathcal{G}_\nu(x) = 2\left(\frac{x}{2\pi}\right)^{\frac{1}{2}(\nu+1)} \sum_{n=1}^{\infty} n^{-\frac{1}{2}(\nu+1)} \Big\{ & e^{-\frac{1}{4}(\nu+1)\pi i} K_{\nu+1}(2e^{\frac{1}{4}\pi i}\sqrt{2n\pi x}) \\ & + e^{\frac{1}{4}(\nu+1)\pi i} K_{\nu+1}(2e^{-\frac{1}{4}\pi i}\sqrt{2n\pi x}) \Big\}. \end{aligned}$$

Proof. For the proof we fix C such as $-C < \min(0, \operatorname{Re}\nu)$. Substituting the functional equation $\zeta(s-\nu) = \chi(s-\nu)\zeta(1-s+\nu)$ (cf. [Iv, Chapter 1, p. 9, 1.2 (1.24)]) into the right-hand side of (3.7), we get

$$\mathcal{G}_\nu(x) = \frac{x^{\nu+1}}{2\pi i}\int_{(-C)} \Gamma(-s)\Gamma(1-s+\nu)2\cos\left(\frac{\pi}{2}(s-\nu-1)\right) \times \zeta(1-s+\nu)(2\pi x)^{s-\nu-1}ds. \tag{3.8}$$

Since $\zeta(1-s+\nu) = \sum_{n=1}^{\infty} n^{s-\nu-1}$ converges absolutely for $\operatorname{Re}s = -C$, the term-by-term integration is permissible on the right-hand side of (3.8), and this gives

$$\mathcal{G}_\nu(x) = x^{\nu+1}\sum_{n=1}^{\infty}\left\{g_\nu(2n\pi x e^{\frac{1}{2}\pi i}) + g_\nu(2n\pi x e^{-\frac{1}{2}\pi i})\right\},$$

where

$$g_\nu(z) = \frac{1}{2\pi i}\int_{(-C)} \Gamma(-s)\Gamma(1-s+\nu)z^{s-\nu-1}ds \tag{3.9}$$

for $|\arg z| < \pi$. Noting that the pair

$$x^\nu K_\nu(x), \quad 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu) \qquad (\operatorname{Re}s > \max(0, -2\operatorname{Re}\nu))$$

is a pair of Mellin transforms (cf. [Ti, Chapter VII, p. 197, (7.9.12)]), we obtain

$$g_\nu(z) = 2z^{-\frac{1}{2}(\nu+1)}K_{\nu+1}(2z^{\frac{1}{2}})$$

for $|\arg z| < \pi$, by which the proof of Theorem 3.2 is complete. □

Let $(\nu, m) = \Gamma(\frac{1}{2}+\nu+m)/m!\Gamma(\frac{1}{2}+\nu-m)$ for any integer $m \geq 0$ be Hankel's symbol. Applying the asymptotic expansion

$$K_{\nu+1}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left\{\sum_{m=0}^{M-1}(\nu+1, m)(2z)^{-m} + O(|z|^{-M})\right\} \tag{3.10}$$

for $|\arg z| < 3\pi/2$, $|z| \geq 1$ and any integer $M \geq 0$ (cf. [Er2, p. 24, 7.4.1 (4)]) to Theorem 3.2, we can further prove

Corollary 3.1. *The asymptotic formula*

$$\mathcal{G}_\nu(x) = \sqrt{2}\left(\frac{x}{2\pi}\right)^{\frac{1}{2}\nu+\frac{1}{4}} e^{-2\sqrt{\pi x}} \times\left\{\sum_{m=0}^{M-1}(\nu+1, m)(32\pi x)^{-\frac{1}{2}m}\cos\left(2\sqrt{\pi x}+\frac{\pi}{4}\left(\nu+\frac{3}{2}+m\right)\right) + O(x^{-\frac{1}{2}M})\right\} \tag{3.11}$$

holds for all $x \geq 2$ and all integers $M \geq 0$, where the implied O-constant depends only on ν and M.

Remark. This corollary gives an affirmative answer to the conjecture of Chowla and Hawkins [CH] mentioned at the beginning of this section (see (3.1) and below).

Proof. From (3.10) with $M = 0$, we have

$$K_{\nu+1}(2e^{\pm\frac{1}{4}\pi i}\sqrt{2n\pi x}) = O\{(nx)^{-\frac{1}{4}}\exp(-2\sqrt{n\pi x})\} \tag{3.12}$$

for $n \geq 1$ and $x \geq 1$. Noting that the inequality $\sqrt{n} \geq \sqrt{2}\left(1 + \frac{1}{5}\sqrt{n-2}\right)$ holds for all $n \geq 2$, from (3.12) we obtain

$$\sum_{n\geq 2} n^{-\frac{1}{2}(\nu+1)}K_{\nu+1}(2e^{\pm\frac{1}{4}\pi i}\sqrt{2n\pi x}) = O\{x^{-\frac{1}{4}}\exp(-2\sqrt{2\pi x})\}.$$

This, together with Theorem 3.2, yields

$$\begin{aligned}\mathcal{G}_\nu(x) = 2\left(\frac{x}{2\pi}\right)^{\frac{1}{2}(\nu+1)} \Big\{&e^{-\frac{1}{4}(\nu+1)\pi i}K_{\nu+1}(2e^{\frac{1}{4}\pi i}\sqrt{2\pi x}) \\ &+ e^{\frac{1}{4}(\nu+1)\pi i}K_{\nu+1}(2e^{-\frac{1}{4}\pi i}\sqrt{2\pi x})\Big\} \\ &+O\{x^{\frac{1}{2}\operatorname{Re}\nu+\frac{1}{4}}\exp(-2\sqrt{2\pi x})\},\end{aligned} \tag{3.13}$$

where the implied O-constant depends only on ν. The corollary now follows by substituting (3.10) into the first term on the right-hand side of (3.13). □

4. A related sum

Let $\binom{s}{n} = \Gamma(s+1)/\Gamma(s-n+1)n!$ for a nonnegative integer n be the binomial coefficient. The second object of the study in Chowla and Hawkins [CH] is the sum

$$H_0(N) = \sum_{n=2}^{N}(-1)^n\binom{N}{n}\zeta(n), \tag{4.1}$$

where N is a positive integer. Based on the study of the ratio $\sum_{n\leq x}(x-n)^s(x/n - [x/n])/\sum_{n\leq x}(x-n)^s$, they showed the asymptotic formula

$$H_0(N) = N\log N + (2\gamma - 1)N + o(1) \tag{4.2}$$

as $N \to +\infty$. The error term in (4.2) was sharpened as $O(N^{-1})$ by Verma [Ve], who applied Euler-Maclaurin's summation formula to evaluate (4.1).

Let ν be an arbitrary complex parameter. Corresponding to (3.2), we introduce the series

$$H_\nu(x) = \sum_{n>\operatorname{Re}\nu+1}(-1)^n\binom{x}{n}\zeta(n-\nu), \tag{4.3}$$

which converges absolutely for $x > 0$, since $(-1)^n\binom{x}{n} = O(n^{-x-1})$ as $n \to +\infty$. Note that (4.1) is a terminating case of (4.3). The formula

$$H_\nu(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(x+1)\Gamma(-s)}{\Gamma(x+1-s)}\zeta(s-\nu)ds \tag{4.4}$$

for $x > 0$ (c is a constant fixed with $\operatorname{Re}\nu + 1 < c < [\operatorname{Re}\nu + 2]$) is essential in the following derivation. This can be proved by shifting the path (c) to the right and collecting the residues of the poles at $s = n$ ($n = [\operatorname{Re}\nu + 2], [\operatorname{Re}\nu + 3], \dots$), since the order of the integrand is $O(|N + i\operatorname{Im} s|^{-x-1})$ on the vertical line $\operatorname{Re} s = N + \frac{1}{2}$ with $N = [\operatorname{Re}\nu + 2], [\operatorname{Re}\nu + 3], \dots$.

Let $\Psi(a, c; z)$ be the confluent hypergeometric function defined by

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)}\int_0^{\infty e^{i\phi}} e^{-z\tau}\tau^{a-1}(1+\tau)^{c-a-1}d\tau \tag{4.5}$$

for $\operatorname{Re} a > 0$, $-\pi < \phi < \pi$ and $-\pi/2 < \phi + \arg z < \pi/2$, where the path of integration is taken as a half-line from the origin to $\infty e^{i\phi}$ (cf. [Er, p. 256, 6.5 (3)]). Then we can prove

Theorem 4.1. *The following formulae hold for all* $x \geq |\operatorname{Re}\nu| + 2$.

(i) *If* $\nu \notin \{-1, 0, 1, 2, \dots\}$,

$$H_\nu(x) = \frac{\Gamma(x+1)\Gamma(-\nu-1)}{\Gamma(x-\nu)} - \sum_{n=0}^{[\operatorname{Re}\nu+1]}(-1)^n\binom{x}{n}\zeta(n-\nu) + \mathcal{H}_\nu(x); \tag{4.6}$$

(ii) *If* $\nu \in \{-1, 0, 1, 2, \dots\}$,

$$\begin{aligned} H_\nu(x) = (-1)^\nu\binom{x}{\nu+1}&\left\{\frac{\Gamma'}{\Gamma}(x-\nu) + 2\gamma - \sum_{n=1}^{\nu+1}\frac{1}{n}\right\} \\ &- \sum_{n=0}^{\nu}(-1)^n\binom{x}{n}\zeta(n-\nu) + \mathcal{H}_\nu(x), \end{aligned} \tag{4.7}$$

where the empty sum is to be regarded as null. Here $\mathcal{H}_\nu(x)$ *is the error term which can be expressed as*

$$\begin{aligned} \mathcal{H}_\nu(x) = \Gamma(x+1)\sum_{n=1}^{\infty}&\Big\{\Psi(x+1, \nu+2; 2n\pi e^{\frac{1}{2}\pi i}) \\ &+ \Psi(x+1, \nu+2; 2n\pi e^{-\frac{1}{2}\pi i})\Big\}. \end{aligned} \tag{4.8}$$

Remark. Using (4.8), we shall prove an upper-bound estimate for $\mathcal{H}_\nu(x)$ in Corollary 4.1.

Proof. As in the proof of Theorem 3.1, we restrict our consideration to the case of $\nu \notin \{-1, 0, 1, \dots\}$. Let $\sigma_0 = \min(0, \operatorname{Re}\nu) - \frac{1}{2}$. Then we can shift the path of integration in (4.4) from (c) to (σ_0), since the order of the integrand for $\operatorname{Re} s \geq \sigma_0$ is $O(|\operatorname{Im} s|^{-x+|\operatorname{Re}\nu|})$ as $\operatorname{Im} s \to \pm\infty$. Collecting the residues of the poles at $s = n$ $(n = 0, 1, \dots, [\operatorname{Re}\nu + 1])$ and $\nu + 1$, we obtain (4.6) with

$$\mathcal{H}_\nu(x) = \frac{1}{2\pi i}\int_{(\sigma_0)} \frac{\Gamma(x+1)\Gamma(-s)}{\Gamma(x+1-s)}\zeta(s-\nu)ds.$$

Substituting the functional equation $\zeta(s-\nu) = \chi(s-\nu)\zeta(1-s+\nu)$ and integrating term-by-term as in the proof of Theorem 3.1, we get

$$\mathcal{H}_\nu(x) = \sum_{n=1}^{\infty}\left\{h_\nu(x; 2n\pi e^{\frac{1}{2}\pi i}) + h_\nu(x; 2n\pi e^{-\frac{1}{2}\pi i})\right\},$$

where

$$h_\nu(x; z) = \frac{1}{2\pi i}\int_{(\sigma_0)} \frac{\Gamma(x+1)\Gamma(-s)\Gamma(1-s+\nu)}{\Gamma(x+1-s)}z^{s-\nu-1}ds \tag{4.9}$$

for $|\arg z| \leq \pi/2$ and $z \neq 1$. Here the region of convergence of (4.9) is ensured by the fact that the order of the integrand is $O\{|\operatorname{Im} s|^{-x+|\operatorname{Re}\nu|}e^{-(\frac{1}{2}\pi - |\arg z|)|\operatorname{Im} s|}\}$ as $\operatorname{Im} s \to \pm\infty$. (Notice that $x \geq |\operatorname{Re}\nu| + 2$.) The proof of Theorem 4.1 is therefore complete by showing

Lemma 4.1. *We have*

$$h_\nu(x; z) = e^{-z}\Gamma(x+1)\Psi(x+1, \nu+2; z)$$

for $|\arg z| \leq \pi/2$ *and* $z \neq 0$.

Proof of Lemma 4.1. It is sufficient to prove the lemma for $|\arg z| < \pi/2$, since the remaining case follows by continuity. Substituting the formula

$$z^{s-\nu-1}\Gamma(1-s+\nu) = \int_0^{\infty} e^{-z\tau}\tau^{-s+\nu}d\tau$$

for $|\arg z| < \pi/2$ and $\operatorname{Re} s < \operatorname{Re}\nu + 1$ into the right-hand integral in (4.9), and changing the order of integrations (by Fubini's theorem), we find

$$h_\nu(x; z) = \int_1^{\infty} e^{-z\tau}\tau^{\nu}(1-\tau^{-1})^x d\tau = e^{-z}\int_0^{\infty} e^{-z\tau}\tau^x(1+\tau)^{\nu-x}d\tau.$$

Here we used the fact that the resulting inner s-integral is equal to 0 for $0 < \tau < 1$, and $(1-\tau^{-1})^x$ for $\tau > 1$, respectively (cf. [Er3, p. 349, (20)]). The lemma now follows by noting (4.5). □

We next prove

Corollary 4.1. *The estimate*

$$\mathcal{H}_\nu(x) = O\left\{x^{\frac{1}{2}\max(\operatorname{Re}\nu,0)+\frac{1}{2}}\exp(-Dx^{\frac{1}{3}})\right\}$$

holds for $x \geq |\operatorname{Re}\nu| + 2$, *where* $D = \frac{3}{2}(4\pi^2\log 2)^{\frac{1}{3}} = 4.5201\ldots$..

Remark 1. Theorem 4.1 with this corollary refines the result of Verma [Ve] mentioned at the beginning of this section (see (4.2) and below), since $(\Gamma'/\Gamma)(x) = \log x - (2x)^{-1} + O(x^{-2})$ as $x \to +\infty$.

Remark 2. It is known that the asymptotic behaviour of the confluent hypergeometric function $\Psi(a, c; z)$ for large a and large z is complicated (see [Er1, p. 280, 6.13.3]). Hence it seems difficult to deduce from (4.8) a full asymptotic expansion, such as (3.11), for $\mathcal{H}_\nu(x)$.

Proof. Taking $\phi = \pm\pi/2$ and $z = 2n\pi e^{\pm\pi i/2}$ in (4.5), and changing the variable τ into $e^{\mp\pi i/2}\tau$, we have

$$\Gamma(x+1)\Psi(x+1,\nu+2;2n\pi e^{\pm\frac{1}{2}\pi i}) \ll \int_0^\infty e^{-2n\pi\tau}\tau^x(1+\tau^2)^{\frac{1}{2}(\lambda-x)}d\tau,$$

where (also in what follows) the implied $\ll$-constant depends on ν, and we set $\lambda = \operatorname{Re}\nu$ for simplicity. Hence

$$\begin{aligned}
\Gamma(x+1)&\sum_{n=1}^\infty \Psi(x+1,\nu+2;2n\pi e^{\pm\frac{1}{2}\pi i}) \\
&\ll \int_0^\infty \frac{e^{-2\pi\tau}}{1-e^{-2\pi\tau}}\tau^x(1+\tau^2)^{\frac{1}{2}(\lambda-x)}d\tau \\
&\ll \int_0^1 \exp\{-(x-1)\varphi(\tau)\}d\tau + \int_1^\infty \tau^\lambda\exp\{-2\pi\tau - x\varphi(\tau)\}d\tau \\
&= I_1 + I_2,
\end{aligned}$$

say, where $\varphi(\tau) = \frac{1}{2}\log(1+\tau^2) - \log\tau$. Since the function $\varphi(\tau)$ is monotone decreasing for $\tau \in]0,1]$, the estimate

$$I_1 \ll \exp(-x\log\sqrt{2}) \tag{4.10}$$

follows. Next note that the inequality $\varphi(\tau) = \frac{1}{2}\log(1+\tau^{-2}) \geq (\log\sqrt{2})\tau^{-2}$ holds for all $\tau \in [1,+\infty[$. Defining $\psi(\tau) = 2\pi\tau + (\log\sqrt{2})x\tau^{-2}$ for $\tau \in [1,+\infty[$, we have

$$\begin{aligned}
I_2 &\leq \int_1^\infty \tau^\lambda\exp\{-\psi(\tau)\}d\tau \\
&= \left(\int_1^{dx^{1/2}} + \int_{dx^{1/2}}^\infty\right)\tau^\lambda\exp\{-\psi(\tau)\}d\tau = I_{2,1} + I_{2,2},
\end{aligned}$$

say, where $d = (\log 2/2\pi)^{\frac{1}{3}}$. Since the function $\psi(\tau)$ attains its minimum value $Dx^{\frac{1}{3}}$ (with the constant D given in the corollary) at $\tau = dx^{\frac{1}{3}}$, we obtain

$$I_{2,1} \ll x^{\frac{1}{2}\max(\lambda,0)+\frac{1}{2}} \exp(-Dx^{\frac{1}{3}}). \tag{4.11}$$

On the other hand, it can be seen the estimates

$$I_{2,2} \ll \int_{dx^{1/2}}^{\infty} \tau^{\lambda} e^{-2\pi\tau} d\tau \ll x^{\frac{1}{2}\lambda} \exp(-2\pi dx^{\frac{1}{2}}), \tag{4.12}$$

where the last inequality for $\lambda \leq 0$ follows immediately, and that for $\lambda > 0$ reduces to the preceding case by integrating by parts repeatedly. The proof of the corollary is now complete by summing up the estimates (4.10), (4.11) and (4.12). □

5. Generating functions of $\zeta(s)$

Let a and ν be complex parameters with $\nu \notin \{1, 0, -1, -2, \dots\}$. We define

$$f_\nu(a; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \zeta(\nu + n) z^n \quad (|z| < 1),$$
$$e_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \zeta(\nu + n) z^n \quad (|z| < +\infty).$$

Since $\zeta(\nu + n) \to 1$ uniformly for $n = 0, 1, 2, \dots,$ as $\operatorname{Re}\nu \to +\infty$, we see that $f_\nu(a; z) \to (1 - z)^{-a}$ and $e_\nu(z) \to e^z$ as $\operatorname{Re}\nu \to +\infty$. This suggests us to define the hypergeometric type generating functions of $\zeta(s)$ as

$$\mathcal{F}_\nu(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \zeta(\nu + n) z^n \quad (|z| < 1), \tag{5.1}$$

$$\mathcal{F}_\nu(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} \zeta(\nu + n) z^n \quad (|z| < +\infty), \tag{5.2}$$

where a, b and c are arbitrarily fixed complex parameters with $c \notin \{0, -1, -2, \dots\}$. Then we can observe, when $\operatorname{Re}\nu \to +\infty$, that

$$\mathcal{F}_\nu(a, b; c; z) \longrightarrow F(a, b; c; z),$$
$$\mathcal{F}_\nu(a; c; z) \longrightarrow F(a; c; z),$$

where $F(a, b; c; z)$ and $F(a; c; z)$ denote hypergeometric functions of Gauss and Kummer, respectively.

Substituting the series expression $\zeta(\nu + n) = \sum_{m=1}^{\infty} m^{-\nu-n}$ for $\operatorname{Re}\nu > 1$ and $n \geq 0$ into (5.1) and (5.2), and changing the order of summations, respectively, we get

Theorem 5.1. *The Dirichlet series expressions*

$$\mathcal{F}_\nu(a,b;c;z) = \sum_{m=1}^{\infty} F\left(a,b;c;\frac{z}{m}\right) m^{-\nu}, \tag{5.3}$$

and

$$\mathcal{F}_\nu(a;c;z) = \sum_{m=1}^{\infty} F\left(a;c;\frac{z}{m}\right) m^{-\nu} \tag{5.4}$$

hold for $\operatorname{Re}\nu > 1$, *respectively.*

Recall that the hypergeometric functions have Euler's integral formulae (cf. [Er1, p. 59, 2.1.3, (10) and p. 255, 6.5, (1)]). Corresponding to these, from the term-by-term integrations we can deduce

Theorem 5.2. *It follows that*

$$\mathcal{F}_\nu(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \tau^{b-1}(1-\tau)^{c-b-1} f_\nu(a;\tau z) d\tau \tag{5.5}$$

for $0 < \operatorname{Re} b < \operatorname{Re} c$ *and* $|z| < 1$, *and*

$$\mathcal{F}_\nu(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \tau^{a-1}(1-\tau)^{c-a-1} e_\nu(\tau z) d\tau \tag{5.6}$$

for $0 < \operatorname{Re} a < \operatorname{Re} c$ *and* $|z| < +\infty$.

Recall further that the hypergeometric functions have Mellin-Barnes integral formula (cf. [Er1, p. 62, 2.1.3, (15) and p. 256, 6.5, (4)]). By the similar argument of moving the path of integratrion as in Section 2, we can show

Theorem 5.3. *For* $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ *and* $\operatorname{Re}\nu > 1$ *we have*

$$\mathcal{F}_\nu(a,b;c;z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{(\sigma_1)} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} \zeta(\nu+s)(-z)^s ds, \tag{5.7}$$

for $|\arg(-z)| < \pi$, *where* σ_1 *is fixed with* $\max(-\operatorname{Re} a, -\operatorname{Re} b, 1-\operatorname{Re}\nu) < \sigma_1 < 0$; *and*

$$\mathcal{F}_\nu(a;c;z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)} \int_{(\sigma_2)} \frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(c+s)} \zeta(\nu+s)(-z)^s ds \tag{5.8}$$

for $|\arg(-z)| < \pi/2$, *where* σ_2 *is fixed with* $\max(-\operatorname{Re} a, 1-\operatorname{Re}\nu) < \sigma_2 < 0$.

Formulae (5.1)–(5.8) are fundamental in deriving various properties of $\mathcal{F}_\nu(a,b;c;z)$ and $\mathcal{F}_\nu(a;c;z)$. Further investigations will be given in forthcoming papers.

References

[BS] Buschman, R. G. and Srivastava, H. M., *Asymptotic behavior of some power series with ζ-functions in the coefficients*, Mh. Math. **115** (1993), 291–298.

[CH] Chowla, S. and Hawkins, D., *Asymptotic expansions of some series involving the Riemann zeta function*, J. Indian Math. Soc. (N.S.) **26** (1962), 115–124.

[Er1] Erdélyi, A. et al., *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[Er2] ———, *ibid*, Vol. II, McGraw-Hill, New York, 1953.

[Er3] ———, *Tables of Integral Transforms*, Vol. I, McGraw-Hill, New York, 1954.

[Iv] Ivić, A., *The Riemann Zeta-function*, John Wiley & Sons, New York, 1985.

[Ka1] Katsurada, M., *An application of Mellin-Barnes type integrals to the mean square of L-functions*, (submitted for publication).

[Ka2] ———, *An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions*, Collect. Math. **48** (1997), 137-153.

[Ka3] ———, *Power series with the Riemann zeta-function in the coefficients*, Proc. Japan Acad. **72, Ser. A** (1996), 61–63.

[Kl] Klusch, D, *On the Taylor expansion of the Lerch zeta-function*, J. Math. Anal. Appl. **170** (1992), 513–523.

[Mo1] Motohashi, Y, *An explicit formula for the fourth power mean of the Riemann zeta-function*, Acta Math. **170** (1993), 181–220.

[Mo2] ———, *On the Kloosterman-sum zeta-function*, Proc. Japan Acad. **71, Ser. A** (1995), 69–71.

[Ram] Ramanujan, S., *A series for Euler's constant γ*, Messenger Math. **46** (1916–17), 73–80.

[Ran] Rane, V. V., *Dirichlet L-function and power series for Hurwitz zeta function*, Proc. Indian Acad. Sci. (Math. Sci.) **103** (1993), 27–39.

[RS] Raina, R. K. and Srivastava, H. M., *Certain results associated with the generalized Riemann zeta functions*, Rev. Téc. Ing. Univ. Zulia. **18** (1995), 301–304.

[Sr1] Srivastava, H. M., *Summation of a class of series involving the Riemann zeta function*, Rev. Téc. Ing. Univ. Zulia (Edición Especial) **9** (1986), 79–82.

[Sr2] ———, *Some infinite series associated with the Riemann zeta-function*, Yokohama Math. J. **35** (1986), 47–50.

[Sr3] ———, *Sums of certain series of the Riemann zeta function*, J. Math. Anal. Appl. **134** (1988), 129–140.

[Te] Tennenbaum, J., *On the function $\sum_{n=0}^{\infty} \frac{\zeta(n+2)}{n!} x^n$*, Math. Scand. **41** (1977), 242–248.

[Ti] Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, (2nd ed.), Oxford University Press, Oxford, 1948.

[Ve] Verma, D. P., *Asymptotic expansion of some series involving the Riemann zeta-function*, Indian J. Math. **6** (1964), 121–127.

[VP] Verma, D. P. and Prasad, S. N., *A power series with zeta function coefficients*, J. Bihar Math. Soc. **7** (1983), 37–40.

[WW] Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis*, (4th ed.), Cambridge University Press, Cambridge, 1927.

[YKK] Yoshimoto, M., Katsurada, M. and Kanemitsu, S., *On the Hurwitz-Lerch zeta-function*, (submitted for publication).

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(Received 24 04 1997)