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ON MOORE-PENROSE INVERSE OF BLOCK MATRICES AND FULL-RANK FACTORIZATION

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Dedicated to Professor Petar Madić on the occasion of his 75th birthday

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Abstract. We develop a few methods for computing the Moore-Penrose inverse, based on full-rank factorizations which arise from different block decompositions of rectangular matrices. In this way, the paper is a continuation of the previous works given by Noble [**5**] and Tewarson [**10**]. We compare the obtained results with the known block representations of the Moore-Penrose inverse. Moreover, efficient block representations of the weighted Moore-Penrose inverse are introduced using the same principles.

1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices, and $\mathbb{C}_r^{m\times n}=\{X\in\mathbb{C}^{m\times n}:\operatorname{rank}(X)=r\}$. With $A^{|r}$ and $A_{|r}$ we denote the submatrix of A which contains the first r columns of A and the first r rows of A, respectively. Similarly, $A^{r|}$ and $A_{r|}$ denote the last r columns and the last r rows of A, respectively. Finally, $A_{r|}^{r|}$ denotes the submatrix of A generated by the first r columns and the last r rows of A. The identity matrix of the order k is denoted by I_k, and $\mathbb{O}$ denotes the zero matrix of a convenient size.

Penrose [**6**], [**7**] has shown the existence and uniqueness of a solution $X\in\mathbb{C}^{n\times m}$ of the following four equations:

$$(1)\ AXA=A,\quad (2)\ XAX=X,\quad (3)\ (AX)^*=AX,\quad (4)\ (XA)^*=XA,$$

for any $A\in\mathbb{C}^{m\times n}$. For a sequence $\mathcal{S}$ of elements from the set $\{1,2,3,4\}$, the set of matrices which satisfy the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$.

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We use the following useful expansion for the Moore-Penrose generalized inverse $A^\dagger$ of A, based on the full-rank factorization $A = PQ$ of A [**1**], [**2**]:

$$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*.$$

The *weighted Moore-Penrose inverse* is investigated in [**3**] and [**8**]. The main results of these papers are:

PROPOSITION 1.1. [**8**] *Let given positive-definite matrices* $M \in \mathbb{C}^{m\times m}$ *and* $N \in \mathbb{C}^{n\times n}$. *For any matrix* $A \in \mathbb{C}^{m\times n}$ *there exists the unique solution* $X = A^\dagger_{M\circ,\circ N} \in A\{1,2\}$ *satisfying*

$$(5) \qquad (MAX)^* = MAX \qquad\qquad (6) \qquad (XAN)^* = XAN.$$

Similarly, we use the following notations:

$A^\dagger_{M\circ,N\circ}$ denotes the unique solution of the equations (1), (2), and

$$(7) \qquad (MAX)^* = MAX, \qquad\qquad (8) \qquad (NXA)^* = NXA;$$

$A^\dagger_{\circ M,N\circ}$ is the unique solution of the equations (1), (2) and

$$(9) \qquad (AXM)^* = AXM, \qquad\qquad (10) \qquad (NXA)^* = NXA;$$

$A^\dagger_{\circ M,\circ N}$ denotes the unique solution of the equations (1), (2) and

$$(11) \qquad (AXM)^* = AXM, \qquad\qquad (12) \qquad (XAN)^* = XAN.$$

PROPOSITION 1.2 *[**8**] The equation* (5) *is equivalent to* $(AXM^{-1})^* = AXM^{-1}$, *and* (6) *can be expressed in the form* $(N^{-1}XA)^* = N^{-1}XA$.

PROPOSITION 1.3 *[**8**] If* $A = PQ$ *is a full rank factorization of* A, *then:*

$$\begin{aligned} A^\dagger_{M\circ,\circ N} &= (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^* \\ &= NQ^*(P^*MANQ^*)^{-1}P^*M. \end{aligned}$$

Using these notations, the following fact can be easily verified.

PROPOSITION 1.4 (a) $A^\dagger_{M\circ,\circ N} = A^\dagger_{\circ M^{-1},\circ N} = A^\dagger_{M\circ,N^{-1}\circ} = A^\dagger_{\circ M^{-1},N^{-1}\circ}$,

(b) $A^\dagger_{M\circ,N\circ} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* = A_{M\circ,\circ N^{-1}}$,

(c) $A^\dagger_{\circ M,N\circ} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^* = A_{M^{-1}\circ,\circ N^{-1}}$,

(d) $A^{\dagger}_{\circ M,\circ N} = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*$.

From the part (a) of Proposition 1.4 and Proposition 1.2, it is easy to conclude that each of the indices, used in notation of the weighted Moore-Penrose inverse, can be written in one of the following form:

$$\circ M, \circ N \qquad M\circ, \circ N \qquad \circ\, M, N\circ \qquad M\circ, N\circ$$

For the sake of clarity, we use the notation $\varphi(M,N)$ for an arbitrary of these indices. Following Proposition 1.3 and Proposition1.4, we conclude the following:

1. $\varphi(M,N) = \circ M, \circ N \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*$;
2. $\varphi(M,N) = M\circ, \circ N \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^*$;
3. $\varphi(M,N) = \circ M, N\circ \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*$;
4. $\varphi(M,N) = M\circ, N\circ \Rightarrow A^{\dagger}_{\varphi(M,N)} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*$.

The cases 1–4 can be written in the following way:

$$A^{\dagger}_{\varphi(M,N)} = (QN^{[-1]})^*(Q(QN^{[-1]})^*)^{-1}((M^{[-1]}P)^*P)^{-1}(M^{[-1]}P)^*$$

where $M^{[-1]}$ stands for one of the matrices M or M^{-1}, and $N^{[-1]}$ denotes N or N^{-1}, in view of one of the rules 1–4.

We restate the main block decompositions [**4**], [**11**–**13**]. For a given matrix $A \in \mathbb{C}_r^{m\times n}$ there exist the regular matrices R, G, the permutation matrices E, F and the unitary matrices U, V, such that:

$$(T_1) \quad RAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \qquad (T_2) \quad RAG = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2;$$

$$(T_3) \quad RAF = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3; \qquad (T_4) \quad EAG = \begin{bmatrix} I_r & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_4;$$

$$(T_5) \quad UAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \qquad (T_6) \quad RAV = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1;$$

$$(T_7) \quad UAV = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2; \qquad (T_8) \quad UAF = \begin{bmatrix} B & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_5;$$

$$(T_9) \quad EAV = \begin{bmatrix} B & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_6;$$

$$(T_{10a}) \quad EAF = \begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} = N_7;$$

where the multipliers S, T satisfy $T=A_{11}^{-1}A_{12}$, $\quad S=A_{21}A_{11}^{-1}$;

$$(T_{10b})\quad EAF=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{21}A_{11}^{-1}A_{12}\end{bmatrix}=N_7.$$

(T_{11}) Transformation of similarity for square matrices:

$$RAR^{-1}=RAFF^*R^{-1}=\begin{bmatrix}I_r & K\\ \mathbb{O} & \mathbb{O}\end{bmatrix}F^*R^{-1}=\begin{bmatrix}T_1 & T_2\\ \mathbb{O} & \mathbb{O}\end{bmatrix}.$$

For the sake of completeness and comparison with our results, we describe known block representations of the Moore-Penrose inverse.

In the begining, we restate the block representations of the Moore-Penrose inverse from [**11**], [**13**]. For $A\in\mathbb{C}_r^{m\times n}$, let

$$R=\begin{bmatrix}R_1\\ R_2\end{bmatrix},\quad G=[\,G_1,\quad G_2\,],\quad U=\begin{bmatrix}U_1\\ U_2\end{bmatrix},\quad V=[\,V_1,\quad V_2\,],$$

where R_1, U_1 are the first r rows of R and U, respectively, and G_1, V_1 denote the first r columns of G and V, respectively. Then the Moore-Penrose inverse can be represented in the following way, where the block representations (M_i) correspond to the block decompositions (T_i), $i\in\{1,\dots,9\}$.

$$\begin{aligned}(M_1)\quad A^\dagger&=G\begin{bmatrix}I_r & -R_1R_2^\dagger\\ -G_2^\dagger G_1 & G_2^\dagger G_1R_1R_2^\dagger\end{bmatrix}R=G\begin{bmatrix}I_r\\ -G_2^\dagger G_1\end{bmatrix}[\,I_r,\quad -R_1R_2^\dagger\,]R\\ &=(G_1-G_2G_2^\dagger G_1)(R_1-R_1R_2^\dagger R_2),\end{aligned}$$

$$\begin{aligned}(M_2)\quad A^\dagger&=G\begin{bmatrix}B^{-1} & -B^{-1}R_1R_2^\dagger\\ -G_2^\dagger G_1B^{-1} & G_2^\dagger G_1B^{-1}R_1R_2^\dagger\end{bmatrix}R\\ &=G\begin{bmatrix}I_r\\ -G_2^\dagger G_1\end{bmatrix}B^{-1}[\,I_r,\quad -R_1R_2^\dagger\,]R\\ &=(G_1-G_2G_2^\dagger G_1)B^{-1}(R_1-R_1R_2^\dagger R_2),\end{aligned}$$

$$\begin{aligned}(M_3)\quad A^\dagger&=F\begin{bmatrix}(I_r+KK^*)^{-1} & -(I_r+KK^*)^{-1}R_1R_2^\dagger\\ K^*(I_r+KK^*)^{-1} & -K^*(I_r+KK^*)^{-1}R_1R_2^\dagger\end{bmatrix}R\\ &=F\begin{bmatrix}I_r\\ K^*\end{bmatrix}(I_r+KK^*)^{-1}\left[\,I_r,\quad -R_1R_2^\dagger\right]R,\end{aligned}$$

$$\begin{aligned}(M_4)\quad A^\dagger&=G\begin{bmatrix}(I_r+K^*K)^{-1} & (I_r+K^*K)^{-1}K^*\\ -G_2^\dagger G_1(I_r+K^*K)^{-1} & -G_2^\dagger G_1(I_r+K^*K)^{-1}K^*\end{bmatrix}R\\ &=G\begin{bmatrix}I_r\\ -G_2^\dagger G_1\end{bmatrix}(I_r+K^*K)^{-1}[\,I_r,\quad K^*\,]E,\end{aligned}$$

$$(M_5)\quad A^\dagger = G\begin{bmatrix} I_r & \mathbb{O} \\ -G_2^\dagger G_1 & \mathbb{O}\end{bmatrix} U = G\begin{bmatrix} I_r \\ -G_2^\dagger G_1\end{bmatrix}[\,I_r, \quad \mathbb{O}\,]\,U$$
$$= (G_1 - G_2 G_2^\dagger G_1)U_1,$$

$$(M_6)\quad A^\dagger = V\begin{bmatrix} I_r & -R_1R_2^\dagger \\ \mathbb{O} & \mathbb{O}\end{bmatrix} R = V\begin{bmatrix} I_r \\ \mathbb{O}\end{bmatrix}[\,I_r, \quad -R_1R_2^\dagger\,]\,R$$
$$= V_1(R_1 - R_1R_2^\dagger R_2),$$

$$(M_7)\quad A^\dagger = V\begin{bmatrix} B^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}\end{bmatrix} U = V\begin{bmatrix} I_r \\ \mathbb{O}\end{bmatrix}[\,B^{-1}, \quad \mathbb{O}\,]\,U = V_1B^{-1}U_1,$$

$$(M_8)\quad A^\dagger = F\begin{bmatrix} B^*(BB^*+KK^*)^{-1} & \mathbb{O} \\ K^*(BB^*+KK^*)^{-1} & \mathbb{O}\end{bmatrix} U$$
$$= F\begin{bmatrix} B^* \\ K^*\end{bmatrix}(BB^*+KK^*)^{-1}[\,I_r, \quad \mathbb{O}\,]\,U,$$

$$(M_9)\quad A^\dagger = V\begin{bmatrix} (B^*B+K^*K)^{-1}B^* & (B^*B+K^*K)^{-1}K^* \\ \mathbb{O} & \mathbb{O}\end{bmatrix} E$$
$$= V\begin{bmatrix} I_r \\ \mathbb{O}\end{bmatrix}(B^*B+K^*K)^{-1}[\,B^*, \quad K^*\,]\,E.$$

These results are obtained by solving the equations (1)–(4).

Block decomposition (T_{10a}) is investigated in [**5**], [**13**], but in two different ways. In [**13**], the Moore-Penrose inverse is represented by solving the corresponding set of matrix equations. The results in [**5**] are obtained using a full-rank factorization, implied by the block decomposition (T_{10a}). The corresponding representation of the Moore-Penrose inverse is:

$$(M_{10a})\quad A^\dagger = \begin{bmatrix} I_r \\ T^*\end{bmatrix}\left(A_{11}^*[\,I_r, \quad S^*\,]\,EAF\begin{bmatrix} I_r \\ T^*\end{bmatrix}\right)^{-1} A_{11}^*[\,I_r, \quad S^*\,]$$
$$= F\begin{bmatrix} I_r \\ T^*\end{bmatrix}(I_r+TT^*)^{-1}A_{11}^{-1}(I_r+S^*S)^{-1}[\,I_r, \quad S^*\,].$$

The following block representation of the Moore-Penrose inverse was obtained in [**10**] and [**13**] for the block decomposition (T_{10b}):

$$(M_{10b})\quad A^\dagger = \begin{bmatrix} I_r \\ (A_{11}^{-1}A_{12})^*\end{bmatrix}\left(I_r + A_{11}^{-1}A_{12}(A_{11}^{-1}A_{12})^*\right)^{-1}$$
$$\times A_{11}^{-1}\left(I_r + (A_{21}A_{11}^{-1})^*A_{21}A_{11}^{-1}\right)^{-1}[\,I_r, \quad (A_{21}A_{11}^{-1})^*\,].$$

Also, in [**14**] is given the following block representation of the Moore-Penrose inverse, based on the formula $A^\dagger = A^*TA^*, \quad T \in A^*AA^*\{1\}$: If $A \in \mathbb{C}_r^{m\times n}$ has the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$, $A_{11} \in \mathbb{C}_r^{r\times r}$, then

$$(M'_{10b})\quad A^{\dagger} = [\,A_{11},\quad A_{12}\,]^{*}\, K_{11}^{*} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}^{*},$$

$$\text{where } K_{11} = \left([\,A_{11},\quad A_{12}\,]\, A^{*} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \right)^{-1}.$$

Block decomposition (T_{11}) is investigated in [**9**], but only for square matrices and the group inverse.

The main idea of this paper is to continue the papers [**5**] and [**10**]. In other words, we use the following algorithm: from the presented block decompositions of matrices find the corresponding full-rank factorizatons $A = PQ$, and then apply described general representations for $A^{\dagger}$ and $A^{\dagger}_{\varphi(M,N)}$. Main advantages of the introduced block representations are their simply derivation, computation and possibility of natural generalization.

2. The Moore-Penrose inverse.

In the following theorem we derive a few representations of the Moore-Penrose inverse, using described block decompositions and full-rank factorizations.

Theorem 2.1 *The Moore-Penrose inverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ can be represented as follows, where each block representation (G_i) is derived from the block decomposition (T_i), $i \in \{1, \ldots, 9, 10a, 10b, 11\}$:*

$$\begin{aligned}(G_1)\quad A^{\dagger} &= \left(G^{-1}{}_{|r}\right)^{*} \left(\left(R^{-1|r}\right)^{*} A \left(G^{-1}{}_{|r}\right)^{*} \right)^{-1} \left(R^{-1|r}\right)^{*} \\ &= \left(G^{-1}{}_{|r}\right)^{*} \left((RR^{*})^{-1}{}^{|r}_{|r} (G^{*}G)^{-1}{}^{|r}_{|r} \right)^{-1} \left(R^{-1|r}\right)^{*},\end{aligned}$$

$$\begin{aligned}(G_2)\quad A^{\dagger} &= \left(G^{-1}{}_{|r}\right)^{*} \left(\left(R^{-1|r}B\right)^{*} A \left(G^{-1}{}_{|r}\right)^{*} \right)^{-1} \left(R^{-1|r}B\right)^{*} \\ &= \left(G^{-1}{}_{|r}\right)^{*} \left(B^{*}(RR^{*})^{-1}{}^{|r}_{|r} B (G^{*}G)^{-1}{}^{|r}_{|r} \right)^{-1} B^{*} \left(R^{-1|r}\right)^{*},\end{aligned}$$

$$\begin{aligned}(G_3)\quad A^{\dagger} &= F \begin{bmatrix} I_r \\ K^{*} \end{bmatrix} \left(\left(R^{-1|r}\right)^{*} AF \begin{bmatrix} I_r \\ K^{*} \end{bmatrix} \right)^{-1} \left(R^{-1|r}\right)^{*} \\ &= \left(F^{|r} + F^{n-r|} K^{*}\right) \left((RR^{*})^{-1}{}^{|r}_{|r} (I_r + KK^{*}) \right)^{-1} \left(R^{-1|r}\right)^{*},\end{aligned}$$

$$\begin{aligned}(G_4)\quad A^{\dagger} &= \left(G^{-1}{}_{|r}\right)^{*} \left([\,I_r,\quad K^{*}\,]\, EA \left(G^{-1}{}_{|r}\right)^{*} \right)^{-1} [\,I_r,\quad K^{*}\,]\, E \\ &= \left(G^{-1}{}_{|r}\right)^{*} \left((I_r + K^{*}K)(G^{*}G)^{-1}{}^{|r}_{|r} \right)^{-1} \left(E_{|r} + K^{*}E_{n-r|} \right),\end{aligned}$$

$$(G_5)\quad A^{\dagger} = \left(G^{-1}{}_{|r}\right)^{*} \left(U_{|r} A \left(G^{-1}{}_{|r}\right)^{*} \right)^{-1} U_{|r} = \left(G^{-1}{}_{|r}\right)^{*} \left((G^{*}G)^{-1}{}^{|r}_{|r} \right)^{-1} U_{|r},$$

$$(G_6)\quad A^\dagger = V^{|r}\left(\left(R^{-1|r}\right)^* A V^{|r}\right)^{-1}\left(R^{-1|r}\right)^* = V^{|r}\left((RR^*)^{-1}{}^{|r}_{|r}\right)^{-1}\left(R^{-1|r}\right)^*,$$

$$(G_7)\quad A^\dagger = V^{|r}\left(B^* U_{|r} A V^{|r}\right)^{-1} B^* U_{|r} = V^{|r} B^{-1} U_{|r},$$

$$(G_8)\quad A^\dagger = F\begin{bmatrix} B^* \\ K^* \end{bmatrix}\left(U_{|r} A F\begin{bmatrix} B^* \\ K^* \end{bmatrix}\right)^{-1} U_{|r} = F\begin{bmatrix} B^* \\ K^* \end{bmatrix}(BB^* + KK^*)^{-1} U_{|r},$$

$$\begin{aligned}(G_9)\quad A^\dagger &= V^{|r}\left([\,B^*, \quad K^*\,]\, E A V^{|r}\right)^{-1} [\,B^*, \quad K^*\,]\, E \\ &= V^{|r}\,(B^*B + K^*K)^{-1}\,[\,B^*, \quad K^*\,]\, E,\end{aligned}$$

$$\begin{aligned}(G_{10a})\ A^\dagger &= F\begin{bmatrix} I_r \\ T^* \end{bmatrix}\left(A_{11}^*\,[\,I_r, \quad S^*\,]\, E A F\begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1} A_{11}^*\,[\,I_r, \quad S^*\,]\, E \\ &= \left(F^{|r} + F^{n-r|} T^*\right)\left((I_r + S^*S) A_{11} (I_r + TT^*)\right)^{-1}\left(E_{|r} + S^* E_{n-r|}\right),\end{aligned}$$

$$\begin{aligned}(G_{10b})\ A^\dagger &= F\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\left((A_{11}^*)^{-1}[A_{11}^*, \quad A_{21}^*] E A F\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\right)^{-1}(A_{11}^*)^{-1}[A_{11}^*, \quad A_{21}^*] E \\ &= F\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1} A_{11} (A_{11}^* A_{11} + A_{21}^* A_{21})^{-1}[A_{11}^*, \quad A\,{}^*_{21}] E,\end{aligned}$$

$$\begin{aligned}(G_{11})\ A^\dagger &= R^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\left(\left(R^{-1|r} T_1\right)^* A R^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\right)^{-1}\left(R^{-1|r} T_1\right)^* \\ &= R^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\left(T_1^*(RR^*)^{-1}{}^{|r}_{|r}[T_1, \quad T_2] R R^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\right)^{-1}\left(R^{-1|r} T_1\right)^*.\end{aligned}$$

Proof. (G_1) Starting from (T_1), we obtain

$$A = R^{-1}\begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} G^{-1} = R^{-1}\begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix}[\,I_r, \quad \mathbb{O}\,]\, G^{-1},$$

which implies

$$P = R^{-1}\begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = R^{-1|r}, \qquad Q = [\,I_r, \quad \mathbb{O}\,]\, G^{-1} = G^{-1}{}_{|r}.$$

Now, we get

$$\begin{aligned} A^\dagger &= Q^*\,(P^* A Q^*)^{-1}\, P^* = \left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1|r}\right)^* A\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1|r}\right)^* \\ &= \left(G^{-1}{}_{|r}\right)^*\left((R^*)^{-1}{}_{|r} R^{-1|r} G^{-1}{}_{|r} (G^*)^{-1|r}\right)^{-1}\left(R^{-1|r}\right)^* \\ &= \left(G^{-1}{}_{|r}\right)^*\left((RR^*)^{-1}{}^{|r}_{|r} (G^*G)^{-1}{}^{|r}_{|r}\right)^{-1}\left(R^{-1|r}\right)^*. \end{aligned}$$

The other block representations of the Moore-Penrose inverse can be developed in a similar way.

(G_5) The block decomposition (T_5) implies

$$P = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = U^{*|r}, \qquad Q = [\, I_r, \quad \mathbb{O}\,]\, G^{-1} = G^{-1}{}_{|r},$$

which means

$$\begin{aligned} A^\dagger &= \left(G^{-1}{}_{|r}\right)^* \left(U_{|r} A \left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r} \\ &= \left(G^{-1}{}_{|r}\right)^* \left(U_{|r} U^{*|r} G^{-1}{}_{|r} (G^*)^{-1|r}\right)^{-1} U_{|r} = \left(G^{-1}{}_{|r}\right)^* \left((G^*G)^{-1}{}^{|r}_{|r}\right)^{-1} U_{|r}. \end{aligned}$$

(G_7) It is easy to see that (T_7) implies

$$P = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} = U^{*|r} B, \qquad Q = [\, I_r, \quad \mathbb{O}\,]\, V^* = V^*{}_{|r}.$$

Now,

$$A^\dagger = V^{|r} \left(B^* U_{|r} U^{*|r} B V^*{}_{|r} V^{|r}\right)^{-1} B^* U_{|r} = V^{|r} (B^*B)^{-1} B^* U_{|r} = V^{|r} B^{-1} U_{|r}.$$

(G_{10a}) From (T_{10a}) we obtain

$$A = E^* \begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} F^* = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}\, [\, I_r, \quad T\,]\, F^*,$$

which implies, for example, the following full rank factorization of A:

$$P = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}, \quad Q = [\, I_r, \quad T\,]\, F^*.$$

Consequently, the Moore-Penrose inverse of A is

$$A^\dagger = F \begin{bmatrix} I_r, \\ T^* \end{bmatrix} \left(A_{11}^* \,[\, I_r, \quad S^*\,]\, EAF \begin{bmatrix} I_r, \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* \,[\, I_r, \quad S^*\,]\, E.$$

This part of the proof can be completed using

$$EAF = \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}\, [\, I_r, \quad T\,]$$

and

$$F \begin{bmatrix} I_r \\ T^* \end{bmatrix} = F^{|r} + F^{n-r|} T^*, \quad [\, I_r, \quad S^*\,]\, E = E_{|r} + S^* E_{n-r|}.$$

(G_{10b}) Follows from

$$P = E^* \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} A_{11}^{-1}, \quad Q = [\, A_{11}, \quad A_{12}\,] F^*. \qquad \square$$

Remark 2.1. (i) A convenient method for finding the matrices S, T and A_{11}^{-1}, required in (T_{10a}), was introduced in [**5**], and it was based on the following extended Gauss-Jordan transformation:

$$\begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & \mathbb{O} \end{bmatrix} \longrightarrow \begin{bmatrix} I & T & A_{11}^{-1} \\ \mathbb{O} & \mathbb{O} & -S \end{bmatrix}.$$

(ii) In [**10**] it was used the following full-rank factorization of the matrix A, represented in the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$:

$$P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \qquad Q = [\, I_r, \quad A_{11}^{-1}A_{12}\,].$$

3. The weighted Moore-Penrose inverse.

Using the general representation of the weighted Moore-Penropse inverse presented in Proposition 1.3: $A^{\dagger}_{M\circ,\circ N} = NQ^*(P^*MANQ^*)^{-1}P^*M$, and the algorithm of Theorem 2.1, we obtain the following block representation of the weighted Moore-Penrose inverse $A^{\dagger}_{M\circ,\circ N}$.

THEOREM 3.1 *The weighted Moore-Penrose inverse* $A^{\dagger}_{M\circ,\circ N}$ *of* $A \in \mathbb{C}_r^{m\times n}$ *possesses the following block representations* (Z_i), *which correspond to the block decompositions* (T_i), $i \in \{1, \dots, 9, 10a, 10b, 11\}$*:*

(Z_1) $\quad N\left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1^{|r}}\right)^* MAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1^{|r}}\right)^* M,$

(Z_2) $\quad N\left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1^{|r}}B\right)^* MAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1^{|r}}B\right)^* M,$

(Z_3) $\quad NF\begin{bmatrix} I_r \\ K^* \end{bmatrix} \left(\left(R^{-1^{|r}}\right)^* MANF\begin{bmatrix} I_r \\ K^* \end{bmatrix}\right)^{-1} \left(R^{-1^{|r}}\right)^* M,$

(Z_4) $\quad N\left(G^{-1}{}_{|r}\right)^* \left([\, I_r, \quad K^*\,] EMAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} [\, I_r, \quad K^*\,] EM,$

(Z_5) $\quad N\left(G^{-1}{}_{|r}\right)^* \left(U_{|r}MAN\left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r}M,$

$(Z_6)\quad NV^{|r}\left(\left(R^{-1|r}\right)^* MANV^{|r}\right)^{-1}\left(R^{-1|r}\right)^* M,$

$(Z_7)\quad NV^{|r}\left(B^*U_{|r}MANV^{|r}\right)^{-1} B^*U_{|r}M,$

$(Z_8)\quad NF\begin{bmatrix} B^* \\ K^* \end{bmatrix}\left(U_{|r}MANF\begin{bmatrix} B^* \\ K^* \end{bmatrix}\right)^{-1} U_{|r}M,$

$(Z_9)\quad NV^{|r}\left([\,B^*, \quad K^*\,]EMANV^{|r}\right)^{-1}[\,B^*, \quad K^*\,]EM,$

$(Z_{10a})\ NF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\left(A_{11}^*[\,I_r, \quad S^*\,]EMANF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1} A_{11}^*[\,I_r, \quad S^*\,]EM,$

$(Z_{10b})\ NF\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\left((A_{11}^*)^{-1}[A_{11}^*, \quad A_{21}^*]EMANF\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\right)^{-1}(A_{11}^*)^{-1}[A_{11}^*, \quad A_{21}^*]EM,$

$(Z_{11})\quad NR^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\left(\left(R^{-1|r}T_1\right)^* MANR^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\right)^{-1}\left(R^{-1|r}T_1\right)^* M.$

The following representations can be obtained from the main properties of the weighted Moore-Penrose inverse and Theorem 3.1.

COROLLARY 3.1. *The weighted Moore-Penrose inverse $A^\dagger_{\varphi(M,N)}$ of $A \in \mathbb{C}_r^{m\times n}$ can be represented as follows:*

$(W_1)\ N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1|r}\right)^* M^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1|r}\right)^* M^{[-1]},$

$(W_2)\ N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left(\left(R^{-1|r}B\right)^* M^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1}\left(R^{-1|r}B\right)^* M^{[-1]},$

$(W_3)\ N^{[-1]}F\begin{bmatrix} I_r \\ K^* \end{bmatrix}\left(\left(R^{-1|r}\right)^* M^{[-1]}AN^{[-1]}F\begin{bmatrix} I_r \\ K^* \end{bmatrix}\right)^{-1}\left(R^{-1|r}\right)^* M^{[-1]},$

$(W_4)\ N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left([I_r, \quad K^*]EM^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1}[I_r, \quad K^*]\,EM^{[-1]},$

$(W_5)\ N^{[-1]}\left(G^{-1}{}_{|r}\right)^*\left(U_{|r}M^{[-1]}AN^{[-1]}\left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r}M^{[-1]},$

$(W_6)\ N^{[-1]}V^{|r}\left(\left(R^{-1|r}\right)^* M^{[-1]}AN^{[-1]}V^{|r}\right)^{-1}\left(R^{-1|r}\right)^* M^{[-1]},$

$(W_7)\ N^{[-1]}V^{|r}\left(B^*U_{|r}M^{[-1]}AN^{[-1]}V^{|r}\right)^{-1} B^*U_{|r}M^{[-1]},$

(W_8) $N^{[-1]}F\begin{bmatrix}B^*\\K^*\end{bmatrix}\left(U_{|r}M^{[-1]}AN^{[-1]}F\begin{bmatrix}B^*\\K^*\end{bmatrix}\right)^{-1}U_{|r}M^{[-1]}$,

(W_9) $N^{[-1]}V^{|r}\left([\,B^*,\quad K^*\,]\,EM^{[-1]}AN^{[-1]}V^{|r}\right)^{-1}[\,B^*,\quad K^*\,]\,EM^{[-1]}$,

(W_{10a}) $N^{[-1]}F\begin{bmatrix}I_r\\T^*\end{bmatrix}\left(A_{11}^*[I_r,\quad S^*]EM^{[-1]}AN^{[-1]}F\begin{bmatrix}I_r\\T^*\end{bmatrix}\right)^{-1}A_{11}^*[I_r,\quad S^*]EM^{[-1]}$,

$$(W_{10b})\ N^{[-1]}F\begin{bmatrix}A_{11}^*\\A_{12}^*\end{bmatrix}\left((A_{11}^*)^{-1}[A_{11}^*,\quad A_{21}^*]EM^{[-1]}AN^{[-1]}F\begin{bmatrix}A_{11}^*\\A_{12}^*\end{bmatrix}\right)^{-1}$$
$$\times\,(A_{11}^*)^{-1}\,[\,A_{11}^*,\quad A_{21}^*\,]\,EM^{[-1]},$$

(W_{11}) $N^{[-1]}R^*\begin{bmatrix}I_r\\\left(T_1^{-1}T_2\right)^*\end{bmatrix}\left(\left(R^{-1^{|r}}T_1\right)^*M^{[-1]}AN^{[-1]}R^*\begin{bmatrix}I_r\\\left(T_1^{-1}T_2\right)^*\end{bmatrix}\right)^{-1}\left(R^{-1^{|r}}T_1\right)^*M^{[-1]}$.

4. Examples.

Example 4.1. Consider $A=\begin{bmatrix}-1&0&1&2\\-1&1&0&-1\\0&-1&1&3\\0&1&-1&-3\\1&-1&0&1\\1&0&-1&-2\end{bmatrix}$. Using the Gauss-Jordan transformation, we get the following reduced row-echelon form of the matrix A:

$$RAF=R_A=\begin{bmatrix}I_r&K\\\mathbb{O}&\mathbb{O}\end{bmatrix}=\begin{bmatrix}1&0&-1&-2\\0&1&-1&-3\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

The matrix R_A is obtained using the permutation matrix $F=I_4$, and the following regular matrix:

$$R=\begin{bmatrix}-1&0&0&0&0&0\\-1&1&0&0&0&0\\-1&1&1&0&0&0\\1&-1&0&1&0&0\\0&1&0&0&1&0\\1&0&0&0&0&1\end{bmatrix}.$$

Using the method (G_3) we obtain

$$R^{-1^{|r}}=\begin{bmatrix}-1&0\\-1&1\\0&-1\\0&1\\1&-1\\1&0\end{bmatrix},\quad F\begin{bmatrix}I_r\\K^*\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\-1&-1\\-2&-3\end{bmatrix},$$

and the following Moore-Penrose inverse of A:

$$A^{\dagger} = \begin{bmatrix} -\frac{5}{34} & -\frac{3}{17} & \frac{1}{34} & -\frac{1}{34} & \frac{3}{17} & \frac{5}{34} \\ \frac{4}{51} & \frac{13}{102} & -\frac{5}{102} & \frac{5}{102} & -\frac{13}{102} & -\frac{4}{51} \\ \frac{7}{102} & \frac{5}{102} & \frac{1}{51} & -\frac{1}{51} & -\frac{5}{102} & -\frac{7}{102} \\ \frac{1}{17} & -\frac{1}{34} & \frac{3}{34} & -\frac{3}{34} & \frac{1}{34} & -\frac{1}{17} \end{bmatrix}.$$

Example 4.2. For the matrix A used in Example 3.1 we obtain (see [**5**])

$$A_{11}^{-1} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}.$$

Using (G_{10a}) we obtain the same Moore-Penrose inverse of A.

Example 4.3. For the matrix $A = \begin{bmatrix} 4 & -1 & 1 & 2 \\ -2 & 2 & 0 & -1 \\ 6 & -3 & 1 & 3 \\ -10 & 4 & -2 & -5 \end{bmatrix}$ we obtain

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}, \quad F = I_4.$$

Then, one can verify the following:

$$\left(R^{-1|_r} T_1\right)^* = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \quad R^* \begin{bmatrix} I_r \\ \left(T_1^{-1} T_2\right)^* \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 1 & 3 \\ -1 & -1 \\ -2 & -3 \end{bmatrix}.$$

Finally, using (G_{11}), we get

$$A^{\dagger} = \begin{bmatrix} \frac{8}{81} & \frac{10}{81} & -\frac{2}{81} & -\frac{2}{27} \\ \frac{47}{162} & \frac{79}{162} & -\frac{16}{81} & -\frac{5}{54} \\ \frac{7}{54} & \frac{11}{54} & -\frac{2}{27} & -\frac{1}{18} \\ \frac{4}{81} & \frac{5}{81} & -\frac{1}{81} & -\frac{1}{27} \end{bmatrix}.$$

Example 4.4. Consider $A=\begin{bmatrix}1&-5&1&4\\-2&7&0&1\\0&-3&2&9\end{bmatrix}$ and positive definite matrices

$$M=\begin{bmatrix}5&-1&3\\-1&2&-2\\3&-2&3\end{bmatrix},\quad N=\begin{bmatrix}4&0&-1&0\\0&3&2&1\\-1&2&5&-2\\0&1&-2&6\end{bmatrix}.$$

Block decomposition (T_1) can be obtained by applying transformation (T_3) two times:

$$R_1AF_1=\begin{bmatrix}I_r&K\\\mathbb{O}&\mathbb{O}\end{bmatrix}=N_3,\qquad R_2N_3^TF_2=\begin{bmatrix}I_r&\mathbb{O}\\\mathbb{O}&\mathbb{O}\end{bmatrix}=N_1.$$

Then, the regular matrices R, G can be computed as follows:

$$N_1=N_1^T=F_2^TN_3R_2^T=F_2^TR_1AF_1R_2^T\Rightarrow R=F_2^TR_1,\quad G=F_1R_2^T.$$

For given matrix the following can be obtained:

$$N_3=\begin{bmatrix}1&0&-\frac{7}{3}&-11\\0&1&-\frac{2}{3}&-3\\0&0&0&0\end{bmatrix},\quad R_1=\begin{bmatrix}-\frac{7}{3}&-\frac{5}{3}&0\\-\frac{2}{3}&-\frac{1}{3}&0\\-2&-1&1\end{bmatrix},\quad F_1=I_4,$$

$$N_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&0\end{bmatrix},\quad R_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\\frac{7}{3}&\frac{2}{3}&1&0\\11&3&0&1\end{bmatrix},\quad F_2=I_3.$$

From $R=R_1$, $G=R_2^T$, we get

$$R^{-1^{|2}}=\begin{bmatrix}1&-5\\-2&7\\0&-3\end{bmatrix},\quad G^{-1}{}_{|2}=\begin{bmatrix}1&0&-\frac{7}{3}&-11\\0&1&-\frac{2}{3}&-3\end{bmatrix}.$$

Using formula (Z_1), we obtain the following representation for the weighted Moore-Penrose inverse of A:

$$A^\dagger_{M\circ,\circ N}=\begin{bmatrix}-\frac{8841}{207506}&-\frac{13865}{207506}&-\frac{13865}{207506}\\\frac{23355}{207506}&\frac{38035}{207506}&-\frac{14947}{207506}\\\frac{2149}{103753}&\frac{11585}{103753}&-\frac{7035}{103753}\\\frac{42301}{207506}&\frac{25265}{207506}&\frac{3465}{207506}\end{bmatrix}.$$

Example 4.5. Similarly, block decomposition (T_1) of the matrix A, considered in Example 4.1 can be obtained by transformation (T_3) two times, by means of the following matrices:

$$R_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_1 = I_4,$$

$$R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \quad F_2 = I_6.$$

From $R = R_1$, $G = R_2^T$, we get

$$R^{-1^{|2}} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad G^{-1}{}_{|2} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -3 \end{bmatrix}.$$

Using formula (G_1), we obtain the Moore-Penrose inverse $A^\dagger$, as in Example 4.1.

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