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A PROOF OF AN ALJANČIĆ HYPOTHESIS ON $\mathcal{O}$–REGULARLY VARYING SEQUENCES

Dragan Ðurčić and **Vladimir Božin**

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Abstract. We prove a uniform convergence theorem and a representation theorem for $\mathcal{O}$-regularly varying sequences, and we answer positively an Aljančić hypothesis [**1**].

1. Introduction

The theory of regularly varying functions and sequences appeared about 1930 in the frame of Theory of Tauberian type theorems [**10**], [**11**], [**15**], [**16**], [**17**] etc. A full development of this theory occurred in the last three decads, when many applications were discovered. We only mention the monographs [**3**], [**5**], [**7**], [**9**], [**13**], [**18**], and the monographic paper [**2**]. One of the main notions in this theory is the notion of an $\mathcal{O}$-regularly varying sequences that appeared in the papers [**4**] and [**12**], and has been very much applied in several other fields (see for instance [**8**], [**14**], [**19**], [**20**] and others). In [**6**], Seneta and Bojanić have connected the theory of regularly varying sequences with the theory of regularly varying functions. In this paper we shall do a similar thing with $\mathcal{O}$-regularly varying functions and $\mathcal{O}$-regularly varying sequences, and answer affirmatively an Aljančić hypothesis. It is interesting to mention that this hypothesis has already been used in some papers without being proved, so that this paper makes all these results founded.

Definition 1. A positive function $F(x)$ defined on an interval $[a,+\infty)$ $(a>0)$ is called $\mathcal{O}$-regularly varying if it is measurable and

$$\overline{\lim_{x\to+\infty}}\frac{F(\lambda x)}{F(x)}=k_F(\lambda)<+\infty \tag{1}$$

for every $\lambda>0$. The class of all such functions is denoted ORV.

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Definition 2. A sequence of positive numbers (c_n) is called $\mathcal{O}$-regularly varying if

$$\overline{\lim_{n\to+\infty}} \frac{c_{[\lambda n]}}{c_n} = k_c(\lambda) < +\infty \tag{2}$$

for all $\lambda > 0$. The class of all such sequences is denoted ORV.

2. Results

We firstly prove two lemmas which will be necessary in the proof of the main Theorem 1.

Lemma 1. *If $\lambda > 0$ and $n \in N$ are fixed, then there is an interval $[\alpha,\beta]$ $(0<\alpha<\beta)$ such that $\lambda \in [\alpha,\beta]$ and $[xn]=[\lambda n]$ for each $x \in [\alpha,\beta]$.*

Proof. Since the function $f(x) = nx$ $(n \in N, x > 0)$ is continuous and increasing, in case $\lambda n \notin N$, we can take that $[\alpha,\beta]$ is a sufficiently small interval such that $\lambda \in (\alpha,\beta)$. In the remaining case, $\lambda n \in N$, we can take that $\alpha = \lambda$ and $\beta \in (\lambda, \lambda + 1/n)$. $\square$

Lemma 2. *If $[a,b]$ is a fixed interval, $\lambda > 0$ is fixed and $\eta = \dfrac{2\lambda}{a+b}$, then for all sufficiently large x there is a $t \in [a,b]$ so that $t \cdot \big[\eta\,[x]\big] = [\lambda x]$.*

Proof. Since

$$\frac{\lambda x - 1}{\eta\, x} \le \frac{[\lambda x]}{[\eta [x]]} \le \frac{\lambda x}{\eta(x-1)-1},$$

and

$$\frac{\lambda x - 1}{\eta\, x} = \frac{\lambda}{\eta} - \frac{1}{x\,\eta} = \frac{a+b}{2} + o(1),$$

we have

$$\frac{\lambda x}{\eta(x-1)-1} = \frac{\lambda}{\eta - (\eta+1)/x} = \frac{a+b}{2} + o(1),$$

as $x \to +\infty$. Thus $\big[\lambda x\big]/\big[\eta\,[x]\big] \in [a,b]$ for all sufficiently large x. $\square$

The next theorem is the affirmatively proved Aljančić hypothesis.

Theorem 1. *Let (c_n) be a sequence of positive numbers. Then the following assertions are equivalent:*

(a) $(c_n) \in ORV$; (b) $F(x) = c_{[x]} \in ORV$ on interval $[1,+\infty)$.

Proof. (b) $\Longrightarrow$ (a) is trivial.

(a) $\Longrightarrow$ (b). If a sequence (c_n) satisfies (a), then the function $F(x) = c_{[x]}$ $(x \ge 1)$ is positive, measurable and piecewise continuous. We shall prove that it satisfies (1). We first prove that there is an interval $[a,b]$ $(0<a<b)$ and $M>0$ such that for every $\lambda \in [a,b]$ and every $n \in N$, $c_{[\lambda n]}/c_n < M$ holds true. On the contrary, assume that

(3) *For each M and every $a, b > 0$ $(a < b)$, there is an $\lambda \in [a, b]$ and $n \in N$ such that $c_{[\lambda n]}/c_n > M$.*

We shall prove that this implies that there is a $\lambda > 0$ such that

$$\overline{\lim_{n\to+\infty}} \left(c_{[\lambda n]}/c_n\right) = +\infty.$$

Let λ_1, n_1 be such that $c_{[\lambda_1 n_1]}/c_{n_1} > 1$. Then by Lemma 1 there is an interval $[\alpha_1, \beta_1]$ containing λ_1 such that $c_{[\lambda n_1]}/c_{n_1} = c_{[\lambda_1 n_1]}/c_{n_1} > 1$ for every $\lambda \in [\alpha_1, \beta_1]$.

Let $a_1 = \alpha_1$, $b_1 = \beta_1$, and consider the interval $\left[\dfrac{2a_1 + b_1}{3}, \dfrac{a_1 + 2b_1}{3}\right]$. By (3) there is a number $\lambda_2 \in \left[\dfrac{2a_1 + b_1}{3}, \dfrac{a_1 + 2b_1}{3}\right]$ and some $n_2 \in N$ such that $c_{[\lambda_2 n_2]}/c_{n_2} > 2$. By Lemma 1, there is an interval $[\alpha_2, \beta_2]$, $\alpha_2 < \beta_2$ containing λ_2, such that $c_{[\lambda n_2]}/c_{n_2} > 2$ for every $\lambda \in [\alpha_2, \beta_2]$. Denoting $[a_2, b_2] = [a_1, b_1] \cap [\alpha_2, \beta_2]$, we can easily see that $a_2 < b_2$.

Continuing this procedure infinitely, we obtain a sequence of intervals $[a_k, b_k]$ and real numbers n_k $(k \in N)$ such that $c_{[\lambda n_k]}/c_{n_k} > k$ for every $\lambda \in [a_k, b_k]$, and $[a_k, b_k] \supseteq [a_{k+1}, b_{k+1}]$ for every $k \in N$. It follows that there is a real number $\lambda \in \bigcap_{k=1}^{\infty} [a_k, b_k]$. For this λ and every $k \in N$ we have that $c_{[\lambda n_k]}/c_{n_k} > k$. Consequently, we obtain that $\overline{\lim}_{n\to+\infty} \left(c_{[\lambda n]}/c_n\right) = +\infty$. This contradiction shows that (3) is impossible.

Hence, there is an $M > 0$ and some interval $[a, b]$ $(0 < a < b)$ such that $c_{[\lambda n]}/c_n < M$ for all $n \in N$ and every $\lambda \in [a, b]$.

Next, let $\lambda > 0$ and $\eta = \dfrac{2\lambda}{a + b}$. Using Lemma 2 and the previous proof we find that for all sufficiently large x there is a $t \in [a, b]$ such that

$$\frac{c_{[\lambda x]}}{c_{[x]}} = \frac{c_{[t[\eta[x]]]}}{c_{[\eta[x]]}} \cdot \frac{c_{[\eta[x]]}}{c_{[x]}}.$$

Since $c_{[t[\eta[x]]]}/c_{[\eta[x]]} < M$ and by assumption (a) $c_{[\eta[x]]}/c_{[x]} < K$ for some $K > 0$ (depending on λ) and all $x \geq x_0$, we obtain that $c_{[\lambda x]}/c_{[x]} < K \cdot M$ for all sufficiently large x. Consequently, $\overline{\lim}_{x\to+\infty} \left(c_{[\lambda x]}/c_{[x]}\right) < +\infty$. This means that $F(x) = c_{[x]} \in ORV$ on the interval $[1, +\infty)$. $\square$

Theorem 1 gives as a consequence the following uniform convergence theorem for $\mathcal{O}$-regularly convergence sequences.

Theorem 2. *If (c_n) is an $\mathcal{O}$-regularly varying sequence and $[a, b]$ is a finite interval included in $(0, +\infty)$, then*

$$\overline{\lim_{n\to+\infty}} \sup_{\lambda\in[a,b]} \frac{c_{[\lambda n]}}{c_n} < +\infty. \tag{4}$$

Proof. If $(c_n) \in ORV$, then by Theorem 1, $F(x) = c_{[x]} \in ORV$ on the interval $[1, +\infty)$, so [**2**] provides that

$$\overline{\lim_{x \to +\infty}} \sup_{\lambda \in [a,b]} \frac{F(\lambda x)}{F(x)} < +\infty.$$

Since next

$$\sup_{\substack{x \geq t \\ x \in N}} \frac{F(\lambda x)}{F(x)} \leq \sup_{x \geq t} \frac{F(\lambda x)}{F(x)} \qquad (t \geq 1,\ \lambda \in [a,b]),$$

we find that

$$\inf_{t \geq 1} \sup_{\lambda \in [a,b]} \sup_{n \geq [t]+1} \frac{c_{[\lambda n]}}{c_n} \leq \inf_{t \geq 1} \sup_{\lambda \in [a,b]} \sup_{x \geq t} \frac{c_{[\lambda x]}}{c_{[x]}},$$

that is

$$\inf_{t \geq 1} \sup_{n \geq [t]+1} \sup_{\lambda \in [a,b]} \frac{c_{[\lambda n]}}{c_n} \leq \inf_{t \geq 1} \sup_{x \geq t} \sup_{\lambda \in [a,b]} \frac{c_{[\lambda x]}}{c_{[x]}}.$$

Since $F \in ORV$, we finally obtain relation (4). $\square$

Now we shall prove a representation theorem for the sequences from the class ORV.

THEOREM 3. *Let (c_n) be a sequence of positive numbers. Then the next assertions are equivalent:*

(*a*) $(c_n) \in ORV$*;*

(*b*) *The sequence (c_n) is represented as*

$$c_n = \exp\Big\{\mu_n + \sum_{k=1}^{n} \frac{\delta_k}{k}\Big\}, \tag{5}$$

where (μ_n) and (δ_n) are bounded sequences.

Proof. (a) $\Longrightarrow$ (b). If a sequence $(c_n) \in ORV$, then by Theorem 1 the function $F(x) = c_{[x]} \in ORV$ on the interval $[1, +\infty)$. By [**2**] for every $n \geq 1$ one has

$$c_n = F(n) = \exp\Big\{\mu(n) + \int_1^n \frac{\epsilon(t)}{t}\, dt\Big\},$$

where μ and ϵ are bounded and measurable functions on the interval $[1, +\infty)$. This means that $c_n = \exp\Big\{\mu_n + \sum_{k=1}^{n} \frac{\delta_k}{k}\Big\}$, where $\mu_n = \mu(n)$ is the general term of a bounded sequence, $\delta_k = k \int_{k-1}^{k} \epsilon(t)/t\, dt$ for all $k \geq 2$, and $\delta_1 = 0$. Finally, we have that

$$\begin{aligned}
|\delta_k| &= k \cdot \Big| \int_{k-1}^{k} \frac{\epsilon(t)}{t}\, dt \Big| \\
&\leq k \cdot \sup_{t \geq k-1} |\epsilon(t)| \cdot \log\Big(1 + \frac{1}{k-1}\Big) e \\
&\leq 2 \sup_{t \geq k-1} |\epsilon(t)| \leq M < +\infty,
\end{aligned}$$

for $k \geq 2$, since $\epsilon(t)$ is a bounded function on the interval $[1, +\infty)$.

(b) $\Longrightarrow$ (a). Assume (b), and choose $\lambda > 1$. Then by (5)

$$\frac{c_{[\lambda n]}}{c_n} = \exp\left\{\mu_{[\lambda n]} - \mu_n\right\} \cdot \exp\Big\{\sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k}\Big\}.$$

Since (μ_n) is a bounded sequence, we have that

$$\overline{\lim_{n\to+\infty}} \exp\left\{\mu_{[\lambda n]} - \mu_n\right\} < +\infty.$$

Besides, we have that

$$\Big|\sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k}\Big| \leq \sup_{k\geq n+1} |\delta_k| \int_{n+1}^{[\lambda n]+1} \frac{dt}{t-1} = \sup_{k\geq n+1} |\delta_k| \log\Big(\frac{[\lambda n]}{n}\Big).$$

Hence

$$\overline{\lim_{n\to+\infty}} \Big|\sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k}\Big| \leq M \cdot \log\lambda = K < +\infty,$$

where K is a constant depending on λ.

Therefore we have that $\overline{\lim}_{n\to+\infty}(c_{[\lambda n]}/c_n) < +\infty$ if $\lambda \geq 1$. A similar proof holds when $\lambda \in (0,1)$. Hence $(c_n) \in ORV$. $\square$

Theorem 4. *Let $(c_n) \in ORV$. Then its index function k_c is in ORV.*

Proof. If $(c_n) \in ORV$, then by Theorem 1 $F(x) = c_{[x]} \in ORV$ on the interval $[1, +\infty)$. By formulas (1) and (2) we immediately find that $k_c(\lambda) \leq k_F(\lambda)$ for every $\lambda > 0$. On the other hand, for arbitrary fixed $\lambda > 0$ and $\delta > 1$ we find $(\lambda x)/[\lambda[x]] \in [1, \delta]$ for all sufficiently large x. Thus by Theorem 2

$$1 \leq M(\delta) = \overline{\lim_{x\to+\infty}} \sup_{\lambda\in[1,\delta]} \frac{c_{[\lambda x]}}{c_{[x]}} < +\infty.$$

So, for any $\delta > 1$ and $\lambda > 0$ we have

$$\begin{aligned} k_F(\lambda) = \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda x]}}{c_{[x]}} &\leq \overline{\lim_{x\to+\infty}} \frac{c_{[\lambda[x]]}}{c_{[x]}} \cdot \overline{\lim_{x\to+\infty}} \frac{c_{\left[\frac{\lambda x}{[\lambda[x]]}[\lambda[x]]\right]}}{c_{[\lambda[x]]}} \leq \\ &\leq k_c(\lambda) \cdot M(\delta). \end{aligned}$$

Since $M(\delta)$ is an increasing function on interval $[1, +\infty)$, we find that $1 \leq M = \lim_{\delta\to1+} M(\delta)$. Hence

$$k_c(\lambda) \leq k_F(\lambda) \leq k_c(\lambda) \cdot M \qquad (\lambda > 0).$$

Next observe that the function k_c is measurable on the interval $(0,+\infty)$ and

$$k_c(\lambda) \geq \frac{k_F(\lambda)}{M} \geq \frac{1}{Mk_F(1/\lambda)} > 0$$

(because $F \in ORV$), thus $k_c(\lambda)$ is positive on that interval.

Since besides

$$k_F(\lambda t) \leq k_F(\lambda)\, k_F(t) \qquad (\lambda, t > 0),$$

we find that

$$\begin{aligned} k_{k_c}(t) &= \overline{\lim_{\lambda\to+\infty}} \frac{k_c(\lambda t)}{k_c(\lambda)} \leq \overline{\lim_{\lambda\to+\infty}} \frac{k_F(\lambda t)}{\frac{1}{M}k_F(\lambda)} = \\ &= M\cdot k_{k_F}(t) \leq M\cdot k_F(t) < +\infty \qquad (t>0), \end{aligned}$$

hence we finally find that $k_c \in ORV$. $\square$

Remark. On the basis of the theory of $\mathcal{O}$-regularly varying functions [**5**] and by applying the previous four theorems, we can develop the theory and applications of $\mathcal{O}$-regularly varying sequences in a very close connection with the theory and applications of $\mathcal{O}$-regularly varying functions.

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Tehnički fakultet
Svetog Save 65
32000 Čačak
Yugoslavia

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Matematički institut
Kneza Mihaila 35/I
11001 Beograd
Yugoslavia