PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 62 (76), 1997, 53–62

ON POLYNOMIALS ASSOCIATED WITH HUMBERT'S POLYNOMIALS

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Communicated by Gradimir Milovanović

Abstract. The principal object of this paper is to provide a natural further step toward the unified presentation of a class of Humbert's polynomials which generalizes the well known class of Gegenbauer, Legendre, Pincherle, Horadam, Kinney, Horadam-Pethe, Gould and Milovanović-Đorđević polynomials and many not so well-known polynomials. We shall give some basic relations involving the generalized Humbert polynomials and then take up several generating functions, hypergeometric representations and expansions in series of some relatively more familiar polynomials of Legendre, Gegenbauer, Hermite and Laguerre. Some of these results may be looked upon as providing useful extensions of the known results of Dilcher, Horadam, Sinha, Shreshtha and Milovanović-Đorđević.

1. Introduction. Gould [**3**] presented a systematic study of an interesting generalization of Humbert, Gegenbauer and several other polynomial systems defined by

$$(c - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, c)t^n \tag{1.1}$$

where m is a positive integer and other parameters are unrestricted in general. For the table of main special cases of (1.1), including Gegenbauer, Legendre, Tchebycheff, Pincherle, Kinney and Humbert polynomials, see Gould [**3**]. In [**10**], Milovanović and Đorđević considered the polynomials $\{p_{n,m}^{\lambda}\}_{n=0}^{\infty}$ defined by the generating function

$$G_m^{\lambda}(x, t) = (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x)t^n, \tag{1.2}$$

AMS Subject Classification (1991): Primary 33C55

Key words and Phrases: Humbert's polynomials, Gegenbauer polynomials, Horadam polynomials, Kinney polynomials, Pincherle polynomials and Horadam-Pethe polynomials.

where $m \in N$ and $\lambda > -1/2$. Note that

$$\begin{aligned} p_{n,1}^{\lambda}(x) &= \frac{(\lambda)_n}{n!}(2x-1)^n && \text{(Horadam polynomials [4])} \\ p_{n,2}^{\lambda}(x) &= C_n^{\lambda}(x) && \text{(Gegenbauer polynomials)} \\ p_{n,3}^{\lambda}(x) &= p_{n+1}^{\lambda}(x) && \text{(Horadam-Pethe polynomials [5])} \end{aligned}$$

where $(\lambda)_0 = 1, (\lambda)_n = (\lambda+1)(\lambda+2)\cdots(\lambda+n-1),\ \ \lambda = 1,2,3,\ldots.$ The explicit form of the polynomial $p_{n,m}^{\lambda}(x)$ is

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} \frac{(-1)^k(\lambda)_{n-(m-1)k}(2x)^{n-mk}}{k|n-mk|} \tag{1.3}$$

The set of polynomials denoted by $S_n^{\nu}(x)$ considered by Sinha [**13**],

$$[1-2xt+t^2(2x-1)]^{-\nu} = \sum_{n=0}^{\infty} S_n^{\nu}(x)t^n \tag{1.4}$$

is precisely a generalization of $S_n(x)$ defined and studied by Shreshtha [**12**]. For $\nu = 1/2$, (1.4) gives associated Legendre polynomials $S_n(x)$.

A generalization (and unification) of various polynomials mentioned above is provided by the definition

$$(c-axt+bt^m(2x-1)^d)^{-\nu} = \sum_{n=0}^{\infty} p_{n,m,a,b,c,d}^{\nu}(x)t^n = \sum_{n=0}^{\infty} \Theta_n(x)t^n \tag{1.5}$$

In the present paper we shall give some basic relations involving the generalized Humbert polynomials $\Theta_n(x)$ and then take up several operational results, series representations, hypergeometric representations and expansions of $\Theta_n(x)$ in series of other polynomials which are best stated in terms of the generalized polynomials. Definition (1.5) of $\Theta_n(x)$ is general enough to account for many of polynomials involved in generalized potential problems [**6**], [**7**], [**8**]. This is interesting since, as will be shown, the polynomials $\Theta_n(x)$ contain [**3**], [**10**], [**13**]

$$P_n(m,x,y,p,c) = \sum_{k=0}^{[n/m]} \binom{p}{k}\binom{p-k}{n-mk} c^{p-n+(m-1)k} y^k(-mx)^{n-mk} \tag{1.6}$$

$$S_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}(2x-1)^k}{k!(n-2k)!} \tag{1.7}$$

$$= \sum_{k=0}^{[n/2]} \frac{(2\nu)_n(x-1)^{2k}x^{n-2k}}{2^{2k}(\nu+1/2)_k(n-2k)!k!} \tag{1.8}$$

2. Finite series representations for $\Theta_n(x)$. Here we obtain the following two finite series representations for $\Theta_n(x)$, viz.

$$\Theta_n(x)=\sum_{k=0}^{[n/m]}\frac{(-1)^k c^{-\nu-n+(m-1)k}(\nu)_{n+(1-m)k}(ax)^{n-mk}[b(2x-1)^d]^k}{k!(n-mk)!} \tag{2.1}$$

$$\Theta_n(x)=\sum_{k=0}^{[(n-(m-2)s)/2]}\sum_{s=0}^{k}\frac{c^{-\nu-n+(m-1)s}(\nu)_k(-k)_s(2\nu+2k)_{n-2k-(m-2)s}}{(n-2k-(m-2)s)!k!s!} \times\left(\frac{ax}{2}\right)^{n-(m-2)s}\left[\frac{4bc(2x-1)^d}{a^2x^2}\right]^s. \tag{2.2}$$

Proof of (2.1):

(1.5) with the help of the result [**2**]

$$(1-z)^{-\alpha}={}_1F_0(\alpha;-;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n z^n}{n!} \tag{2.3}$$

yields

$$\sum_{n=0}^{\infty}\Theta_n(x)t^n=c^{-\nu}\sum_{n=0}^{\infty}\frac{(\nu)_n(\frac{a}{c}xt-\frac{b}{c}t^m(2x-1)^d)^n}{n!} \tag{2.4}$$

Also we know that

$$(t+v)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}t^k v^{n-k} \tag{2.5}$$

Using (2.5) in (2.4), we get

$$\sum_{n=0}^{\infty}\Theta_n(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k-\nu-n}(\nu)_n(axt)^{n-k}[bt^m(2x-1)^d]^k}{k!(n-k)!}$$

which on applying the result [**14**; pp. 100–101 eqn. (2) and (5)], gives

$$\sum_{n=0}^{\infty}\Theta_n(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-1)^{k-\nu-n+(m-1)k}(\nu)_{n+(1-m)k}(ax)^{n-mk}[b(2x-1)^d]^k t^n}{k!(n-mk)!}$$

On comparing the coefficients of t^n from both sides, we get the finite series representation (2.1) for $\Theta_n(x)$.

Proof of (2.2). From (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Theta_n(x)t^n&=[c-axt+bt^m(2x-1)^d]^{-\nu}\\&=c^{-\nu}\left[1-\frac{axt}{c}+\left(\frac{a}{2c}xt\right)^2-\left(\frac{a}{2c}xt\right)^2+\frac{b}{c}t^m(2x-1)^d\right]^{-\nu}\\&=c^{-\nu}\left(1-\frac{axt}{2c}\right)^{-2\nu}\left[1-\frac{a^2x^2t^2/4c^2-bt^m(2x-1)^d/c}{(1-axt/2c)^2}\right]^{-\nu}\end{aligned}$$

which with the help of the result (2.3), gives

$$= c^{-\nu}\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}\left[\frac{a^2x^2t^2}{4c^2}-\frac{bt^m(2x-1)^d}{c}\right]^k\left[1-\frac{axt}{2c}\right]^{-(2\nu+2k)}$$

$$= c^{-\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\nu)_k(2\nu+2k)_n}{k!n!}\left(\frac{ax}{2c}\right)^{n+2k}t^{n+2k}\left[1-\frac{4bct^{m-2}(2x-1)^d}{a^2x^2}\right]^k$$

$$= c^{-\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{(\nu)_k(2\nu+2k)_n(-k)_s}{n!k!s!}\left(\frac{ax}{2c}\right)^{n+2k}\left[\frac{4bc(2x-1)^d}{a^2x^2}\right]^2 t^{(m-2)s+n+2k}$$

Replacing n by $n-2k-(m-2)s$, we get

$$\sum_{n=0}^{\infty}\Theta_n(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{[(n-(m-2)s)/2]}\sum_{s=0}^{k}\frac{c^{-\nu(\nu)}k^{(2\nu+2k)}n-2k-(m-2)s^{(-k)}s}{(n-2k-(m-2)s)!k!s!}$$
$$\left(\frac{ax}{2c}\right)^{n-(m-2)s}\left[\frac{4bc(2x-1)^d}{a^2x^2}\right]^s t^n.$$

On comparing the coefficients of t^n from both the sides, we get the finite series representation (2.2) for $\Theta_n(x)$.

In (2.1) and (2.2), setting $a=m=2$, $b=c=1$ and $d=1$, we get the series representations (1.7) and (1.8) of Sinha [**13**, p. 439, (3 and 4)].

If in (2.1) and (2.2), we set $a=m$, $b=c=1$ and $d=0$, we get

$$h_{n,m}^{\nu}(x)=\sum_{k=0}^{[n/m]}\frac{(-1)^k(\nu)_{n+(1-m)k}(mx)^{n-mk}}{k!(n-mk)!} \tag{2.6}$$

$$h_{n,m}^{\nu}(x)=\sum_{k=0}^{[(n-(m-2)s)/2]}\sum_{s=0}^{k}\frac{(\nu)_k(-k)_s(2\nu+2k)_{n-2k-(m-2)s}(mx/2)^{n-ms}}{(n-2k-(m-2)s)!k!s!} \tag{2.7}$$

where $h_{n,m}^{\nu}(x)$ is Humbert polynomial [**6**].

For $m=a=3$ and $\nu=1/2$, (2.6) and (2.7) further reduce to

$$P_n(x)=\sum_{k=0}^{[n/3]}\frac{(-1)^k(1/2)_{n-2k}(3x)^{n-3k}}{k!(n-3k)!} \tag{2.8}$$

and

$$P_n(x)=\sum_{k=0}^{[(n-s)/2]}\sum_{s=0}^{k}\frac{(1/2)_k(-k)_s(1+2k)_{n-2k-s}(3x/2)^{n-3s}}{(n-2k-s)!k!s!} \tag{2.9}$$

respectively, where $P_n(x)$ is Pincherle polynomial [**6**].

For $a = m = 2$ and $\nu = 1/2$, (2.6) and (2.7) give finite series representations of Legendre polynomial [**11**, p. 164 (1)].

In (2.1) and (2.2), setting $m = a = 2$, $b = c = 1$ and $d = 0$, we get the following representations of Gegenbauer polynomial

$$C_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\nu)_{n-k} (2x)^{n-2k}}{k!(n-2k)!} \tag{2.10}$$

$$C_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \frac{(2\nu)_n x^{n-2k} (x^2-1)^k}{2^{2k}(\nu+1/2)_k k!(n-2k)!} \tag{2.11}$$

In (2.1) and (2.2), setting $a = c = 1$, $d = 0$ and $m = 2$ and replacing b and x by λz^2 and $1 + z + z^2$ respectively, we get

$$f_n^{\lambda,\nu}(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\nu)_{n-k} (1+z+z^2)^{n-2k} (\lambda z^2)^k}{k!(n-2k)!} \tag{2.12}$$

$$f_n^{\lambda,\nu}(z) = \sum_{k=0}^{[n/2]} \frac{(2\nu)_n \left(\frac{1+z+z^2}{2}\right)^n \left[1 - \frac{4\lambda z^2}{(1+z+z^2)}\right]^k}{2^{2k}(\nu+1/2)_k k!(n-2k)!} \tag{2.13}$$

where $\nu > 1/2$, λ is a real parameter. Note that $f_n^{\lambda,\nu}(z)$ is related to $C_n^{\nu}(z)$ by the relation [**1**, p. 474 (1.2)]

$$f_n^{\lambda,\nu}(z) = \lambda^{n/2} z^n C_n^{\nu}\left(\frac{1+z+z^2}{2\sqrt{\lambda}z}\right)$$

3. Hypergeometric representation for $\Theta_n(x)$. The finite representation (2.1) for $\Theta_n(x)$ is of particular interest to us in obtaining the following hypergeometric form for $\Theta_n(x)$,

$$\Theta_n(x) = \frac{(\nu)_n c^{\nu-n} (ax)^n}{n!} \tag{3.1}$$

$$ {}_mF_{m-1}\left[\begin{matrix} -\frac{n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; \\ \frac{-\nu-n+1}{m-1}, \frac{-\nu-n+2}{m-1}, \dots, \frac{-\nu-n+m-1}{m-1}; \end{matrix} \frac{m^m b c^{m-1} (2x-1)^d}{(m-1)^{m-1}(ax)^m}\right]$$

where $m \geq 2$.

Proof of (3.1). Since we know that [**11**, p. 58(2)]

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}, \quad 0 \leq k \leq n, \tag{3.2}$$

replacing α and k by ν and $(m-1)k$ respectively and using

$$(n-mk)! = \frac{(-1)^{mk}n!}{(-n)_{mk}}, \quad 0 \le mk \le n, \tag{3.3}$$

$$(-n)_{mk} = m^{mk}\prod_{s=1}^{m}\left(\frac{-n+s-1}{m}\right)_k \tag{3.4}$$

and

$$(1-\nu-n)_{(m-1)k} = (m-1)^{(m-1)k}\prod_{p=1}^{(m-1)}\left(\frac{-\nu-n+p}{m-1}\right)_k, \quad k=0,1,2,\dots, \tag{3.5}$$

we arrive at (3.1).

If in (3.1), we set $a=m=2$ and $b=c=d=1$, then we get a known result [**13**; p. 442 (12)].

In (3.1), setting $a=m, b=c=1$ and $d=0$, we get the following hypergeometric representation of Humbert polynomial

$$h_{n,m}^{\nu}(x) = \frac{(\nu)_n(mx)^n}{n!} \, {}_mF_{m-1}\left[\begin{matrix}\frac{-n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; \\ \frac{-\nu-n+1}{m-1}, \frac{-\nu-n+2}{m-1}, \dots, \frac{-\nu-n+m-1}{m-1};\end{matrix} \frac{1}{(m-1)^{m-1}x^m}\right] \tag{3.6}$$

For $m=2$, (3.6) gives hypergeometric representation of Gegenbauer polynomial

$$C_n^{\nu}(x) = \frac{(\nu)_n(2x)^n}{n!}\,{}_2F_1\left[\begin{matrix}-\frac{n}{2}, \frac{-n+1}{2}; \\ 1-\nu-n;\end{matrix} \frac{1}{x^2}\right] \tag{3.7}$$

which is a generalization of a known result [**11**; p. 166 (4)].

In (3.1), setting $a=c=1$, $m=2$, $d=0$ and replacing b and x by λz^2 and $1+z+z^2$ respectively, we get the following representation

$$f_n^{\lambda,\nu}(z) = \frac{(\nu)_n(1+z+z^2)^n}{n!}\,{}_2F_1\left[\begin{matrix}-\frac{n}{2}, \frac{-n+1}{2}; \\ 1-\nu-n;\end{matrix} \frac{4\lambda z^2}{(1+z+z^2)}\right] \tag{3.8}$$

For $\lambda=2$ and $\nu=1$, $f_n^{\lambda,\nu}(z)$ reduces to $f_n(z)$. As remarked by Dilcher [**1**], $f_n(z)$ are polynomials of degree $2n$, and their coefficients are rows of the "Pascal type" triangle (after normalizing) (see, e.g., [**1**; p. 473 (1.1)]). In view of (3.7), equation (3.1) of Dilcher's result [**1**, p. 476] readily follows from (3.8).

4. Additional generating function for $\Theta_n(x)$. We now obtain the following four additional generating functions for $\Theta_n(x)$.

$$\sum_{n=0}^{\infty} \frac{\Theta_n(x)t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \frac{c^{-\nu-n}(axt)^n}{n!} \;{}_1F_m\left[\nu+n; \frac{\nu+n}{m}, \frac{\nu+n+1}{m}, \cdots, \frac{\nu+n+m-1}{m}; \frac{-bt^m(2x-1)^d}{m^m}\right], \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{(e)_n\Theta_n(x)t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \frac{(e)_n c^{-\nu-n}(axt)^n}{n!} \;{}_{m+1}F_m\left[\begin{matrix}\nu+n, \frac{e+n}{m}, \frac{e+n+1}{m}, \cdots, \frac{e+n+m-1}{m}; \\ \frac{\nu+n}{m}, \frac{\nu+n+1}{m}, \cdots, \frac{\nu+n+m-1}{m};\end{matrix} \; -\frac{b}{c}t^m(2x-1)^d\right], \tag{4.2}$$

$$\sum_{n=0}^{\infty} \frac{\Theta_n(n)t^n}{(2\nu)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{c^{-\nu-n-2k}(-k)_s\left(\frac{axt}{2}\right)^{n+2k}\left[\frac{4bc^2t^{m-2}(2x-1)^d}{a^2x^2}\right]^s}{2^{2k}(\nu+1/2)_k k!n!(2\nu+n+2k)_{(m-2)s}s!} \tag{4.3}$$

$$\sum_{n=0}^{\infty} \frac{(e)_n\Theta_n(x)t^n}{(2\nu)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{c^{-\nu-n-2k}(e)_{n+2k}(-k)_s(e+n+2k)_{(m-2)s}}{n!k!2^{2k}(\nu+1/2)_k(2\nu+n+2k)_{(m-2)s}s!} \left(\frac{axt}{2}\right)^{n+2k}\left[\frac{4bc^2t^{m-2}(2x-1)^d}{a^2x^2}\right]^s. \tag{4.4}$$

Proofs of (4.1) *to* (4.4). From (2.1), we have

$$\sum_{n=0}^{\infty} \frac{\Theta_n(x)t^n}{(\nu)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} \frac{(-1)^k(\nu)_{n+(1-m)k}(ax)^{n-mk}[b(2x-1)^d]^k t^n}{c^{\nu+n-(m-1)k}(\nu)_n k!(n-mk)!}$$

On using the results [**14**; p. 101 (6), p. 22 (20)] and Gauss's multiplication theorem [**14**; p. 23 (26)], we have

$$\sum_{n=0}^{\infty} \frac{\Theta_n(x)t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \frac{c^{-\nu-n}(axt)^n}{n!} \sum_{k=0}^{\infty} \frac{(\nu+n)_k\left[-\frac{bt^m(2x-1)^d}{cm^m}\right]^k}{\prod_{p=1}^{m}\left(\frac{\nu+n+p-1}{m}\right)_k k!}$$

which is equivalent to (4.1).

If e is an arbitrary number, maybe a complex number, then following the method of derivation of (4.1), we obtain (4.2) to (4.4).

For the purpose of illustration of the usefulness of our results (4.1) to (4.4), we choose to mention here some special cases.

For $a=2$, $b=c=1$ and $d=0$, (4.1) gives the generating function for $p^{\nu}_{n,m}(x)$ defined and considered in [**9**]

$$(4.5)\quad \sum_{n=0}^{\infty}\frac{p^{\nu}_{n,m}(x)t^n}{(\nu)_n}= \sum_{n=0}^{\infty}\frac{(2xt)^n}{n!}\,{}_1F_m\left[\nu+n;\frac{\nu+n}{m},\frac{\nu+n+1}{m},\dots,\frac{\nu+n+m-1}{m};\left(-\frac{t}{m}\right)^m\right]$$

which further for $m=2$ and $m=3$ lead naturally to generating functions for the polynomials of Legendre, Pincherle, Humbert, Sinha, Shreshtha, Kinney, Horadam, Gegenbauer and Horadam-Pethe polynomials (see also [**10**]).

In (4.2), setting $a=m=2$ and $b=c=d=1$, we get the generating function for $S^{\nu}_n(x)$
(4.6)

$$\sum_{n=0}^{\infty}\frac{(e)_n S^{\nu}_n(x)t^n}{(\nu)_n}=\sum_{n=0}^{\infty}\frac{(e)_n(2xt)^n}{n!}\,{}_3F_2\left[\begin{matrix}\nu+n,\frac{e+n}{2},\frac{e+n+1}{2};\\ \frac{\nu+n}{2},\frac{\nu+n+1}{2};\end{matrix}\;-t^2(2x-1)\right]$$

For $e=\nu$, (4.6) reduces to a known result of Sinha [**13**; p. 439 (2)].

If in (4.2), we set $a=m$, $b=c=1$ and $d=0$, we get the generating function for Humbert polynomial

$$(4.7)\quad \sum_{n=0}^{\infty}\frac{(e)_n h^{\nu}_{n,m}(x)t^n}{(\nu)_n}= \sum_{n=0}^{\infty}\frac{(e)_n(mxt)^n}{n!}\,{}_{m+1}F_m\left[\begin{matrix}\nu+n,\frac{e+n}{m},\dots,\frac{e+n+m-1}{m};\\ \frac{\nu+n}{m},\frac{\nu+n+1}{m},\dots,\frac{\nu+n+m-1}{m};\end{matrix}\;-t^m\right]$$

which further reduces to a known result [**14**; p. 86 (26)] for $e=\nu$.

In (4.7), setting $m=3$ and $\nu=1/2$, we get the generating function for Pincherle polynomial $P_n(x)$

$$(4.8)\quad \sum_{n=0}^{\infty}\frac{(e)_n P_n(x)t^n}{(1/2)_n}= \sum_{n=0}^{\infty}\frac{(e)_n(3xt)^n}{n!}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}+n,\frac{e+n}{3},\frac{e+n+1}{3},\frac{3+n+2}{3};\\ \frac{1/2+n}{3},\frac{3/2+n}{3},\frac{5/2+n}{3};\end{matrix}\;-t^3\right]$$

which further reduces to a known result [**3**; p. 697] for $e = 1/2$.

5. Expansions of $\Theta_n(x)$ in series of polynomials. Expansions $\Theta_n(x)$ in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials relevent to our present investigation are given by

$$(5.1)\quad \Theta_n(x) = \sum_{k=0}^{[\frac{n-(m-2)s}{m}]} \sum_{s=0}^{k} \frac{(-1)^k c^{-\nu-n+(m-1)(k-s)}(\nu)_{n-(1-m)k+(1-m)s}(-k)_s}{k!s!(3/2)_{n-mk-(1-m)s}} \\ (2n-2mk-2(2-m)s+1)P_{n-mk-(2-m)s}(ax/2)[b(2x-1)^d]^{k-s},$$

$$(5.2)\quad \Theta_n(x) = \sum_{k=0}^{[\frac{n-(m-2)s}{m}]} \sum_{s=0}^{k} \frac{(-1)^k c^{-\nu-n+(1-m)s+(m-1)k}(\nu)_{n+(m-1)s+(1-m)k}}{(\nu)_{n+1-mk-(1-m)s}k!s!} \\ (\nu+n-2s-m(k-s))(-k)_s[b(2x-1)^d]^{k-s}C^{\nu}_{n-2s-m(k-s)}(ax/2)$$

$$(5.3)\quad \Theta_n(x) = \sum_{k=0}^{[\frac{n-(m-2)s}{m}]} \sum_{s=0}^{k} \frac{(-1)^k c^{-\nu-n+(1-m)(s-k)}(\nu)_{n+(m-1)(s-k)}(-k)_s}{k!s!(n-2s-m(k-1))!} \\ [b(2x-1)^d]^{k-s}H_{n-2s-m(k-s)}\left(\frac{ax}{2}\right)$$

and

$$(5.4)\quad \Theta_n(x) = \sum_{s=0}^{[n-(m-2)k]} \sum_{k=0}^{[n/2]} \frac{(-1)^{k+s} c^{-\nu-n_(m-1)k}(\nu)_{n-(m-1)k}(1+\alpha)_n}{k!(n-s-mk)!(1+\alpha)_s} \\ 2^{n-mk}[b(2x-1)^d]^k L_s^{(\alpha)}(ax/2)$$

Proofs of (5.1) *to* (5.4). By using the relations (2.1), we have

$$\sum_{n=0}^{\infty} \Theta_n(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-1)^m c^{-\nu-n+(m-1)k}(\nu)_{n+(1-m)k}(ax)^{n-mk}}{k!(n-mk)!} \\ \times [b(2x-1)^d]^k t^n.$$

On using the results [**14**; p. 101 (6)] and [**11**; p. 181 (theorem 65)]

$$(5.5)\qquad \frac{(ax)^n}{n!} = \sum_{s=0}^{[n/2]} \frac{(2n-4s+1)P_{n-2s}(ax/2)}{s!(3/2)_{n-s}}$$

we get

$$\sum_{n=0}^{\infty}\Theta_n(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[n/2]}\frac{(-1)^k c^{-\nu-n-k}(\nu)_{n+k}(2n-4s+1)}{k!s!(3/2)_{n-s}} P_{n-2s}(ax/2)[b(2x-1)^d]^k t^{n+mk}$$

which on using the results [**14**; pp. 100 (1) and 101 (4)] and (3.3) and comparing the coefficients of t yields (5.1).

In a similar manner, results (5.2) to (5.4) are obtained by using [**11**, p. 283 (36), p. 194 (4) and p. 207 (2)] instead of (5.5).

The authors acknowledge their thanks to the referee for the helpful suggestions.

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(Received 17 05 1996)
(Revised 05 02 1997)