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ADDITIVE f-BIMORPHISMS AND COSINE TYPE EQUATIONS

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Dedicated to Professor Ching-Xu Tseng on his retirement

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Abstract. We introduce the notion of an additive f-bimorphism, and study the relationship between the cosine functional equation and the additive f-bimorphism. Indeed, the former may be characterized in terms of the latter. Moreover, some generalized cosine functional equations are presented, which may be similarly characterized.

Kurepa [**3**] considered the quadratic form f on a group G (not necessarily abelian) into an abelian group A, viz.,

$$f(xy) + f(xy^{-1}) = 2[f(x) + f(y)],$$

for $x, y \in$G. Upon a closer look at the paper [**3**; p. 31-34] one should be able to verify with little effort the following assertion: If f is such a form satisfying the Kannappan condition $f(xyz) = f(yxz)$, and $4B(x,y) = f(xy) - f(xy^{-1})$ for all $x, y, z \in$G, then B is an additive bimorphism on $G \times G$ into A such that $B(x,x) = f(x)$. Conversely, if B is an additive bimorphism with $B(x,y) = B(y,x)$, and $B(x,x) = f(x)$ for all $x, y \in G$, then f is a quadratic form such that the Kannappan condition is satisfied and $4B(x,y) = f(xy) - f(xy^{-1})$. The question which we want to discuss in this paper is as follows: If the quadratic form above is replaced by a cosine functional equation, viz., equation (a) below, is the mapping B still an additive bimorphism? The answer is given in Theorem below, and the mapping should be an additive f-bimorphism instead.

Kannappan [**2**], Hosszú [**1**] and many others studied the cosine functional equation:

$$\text{(a)} \qquad f(xy) + f(xy^{-1}) = 2f(x)f(y),$$

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which satisfies the Kannappan condition that $f(xyz) = f(yxz)$ for all $x, y, z \in$G, with the range of f contained in either the field of complex number C, or a commutative ring of characteristic zero. In this article we shall introduce the notion of an additive f-bimorphism, and study the relationship between this and the cosine functional equation.

In what follows f will denote a mapping, not identically zero, from G into C, and e the identity of G. We first need some well-known properties of the equation (a) from [**2**]: $f(e) = 1$, $f(x^{-1}) = f(x)$, $f(xy) = f(yx)$, and $f(x^2) + 1 = 2f^2(x)$ for all $x, y \in$G. We are now ready to introduce the following

Definition. Let $f :$ G$\to C$ be a mapping such that $f(e) = 1$, and $f(x^{-1}) = f(x)$ for all $x \in$G. A mapping $B :$ G$\times$G$\to C$ is called an additive f-bimorphism if

$$B(xy, z) = f(x)B(y, z) + f(y)B(x, z),$$
$$B(z, xy) = f(x)B(z, y) + f(y)B(z, x)$$

for $x, y, z \in$G.

Clearly, $B(xy, z) = B(yx, z)$, $B(z, xy) = B(z, yx)$, and $B(e, z) = B(z, e) = 0$. Also, if $f(x) \neq 0$, then $B(x^{-1}, z) = -B(x, z)$, and $B(z, x^{-1}) = -B(z, x)$ for $x, z \in G$ (e.g., let $y = x^{-1}$ in Definition).

Instead of the identity $4B(x, y) = f(xy) - f(xy^{-1})$ mentioned before, let the mapping B be defined by, for $x, y \in G$,

(*) $$2B(x, y) = f(xy) - f(xy^{-1}).$$

We shall write

(b) $$f(xy)f(xy^{-1}) = f^2(x) + f^2(y) - 1,$$

which will be used in the characterization later. It follows from (b) that $f(e) = 1$, and $f(y^{-1}) = f(y)$, if $f(y) \neq 0$ for $y \in$G. The former is justifiable by letting $y = e$ in (b) so that $f^2(e) = 1$. The latter may be obtained by putting $x = e$ in (b).

We are now in position to state the next

THEOREM. *Let $f : G \to C$ and $B : G \times G \to C$ be two mappings, and $f \neq 0$; then the following are equivalent:*

(1) *Two equations (a) and (*) hold (i.e., $f(xy) + f(xy^{-1}) = 2f(x)f(y)$, and $2B(x, y) = f(xy) - f(xy^{-1})$, respectively), and $f(xyz) = f(yxz)$ for all $x, y, z \in G$.*

(2) *B is an additive f-bimorphism (i.e., $B(xy, z) = f(x)B(y, z) + f(y)B(x, z)$, and $B(z, xy) = f(x)B(z, y) + f(y)B(z, x)$), $B(x, x) = f^2(x) - 1$, $B^2(x, y) = B(x, x)B(y, y)$, and (b) holds (i.e., $f(xy)f(xy^{-1}) = f^2(x) + f^2(y) - 1$) for all $x, y, z \in G$.*

Proof. (1) $\Rightarrow$ (2). Replacing x by xy, and y by xy^{-1} in (a) we have

$$2f(xy)f(xy^{-1}) = f(xyxy^{-1}) + f(xyyx^{-1}) = f(x^2) + f(y^2).$$

Note that $f(e) = 1$ and $f(x^{-1}) = f(x)$ as explained in above. Replace y by x in (a) to get $f(x^2) + 1 = 2f^2(x)$, and x by y in (a) to get $f(y^2) + 1 = 2f^2(y)$. Hence, (b) holds by combining the three identities just obtained.

Set $y = x$ in (*) and (a) to get $2B(x,x) = f(x^2) - f(e)$, and $f(x^2) + f(e) = 2f^2(x)$, respectively. It follows that $B(x,x) = f^2(x) - 1$.

The relation $B^2(x,y) = B(x,x)B(y,y)$ is valid since (*), (a) and (b) imply

$$\begin{aligned}
4B^2(x,y) &= f^2(xy) + f^2(xy^{-1}) - 2f(xy)f(xy^{-1}) \\
&= 4f^2(x)f^2(y) - 4f(xy)f(xy^{-1}) = 4[f^2(x)f^2(y) - f^2(x) - f^2(y) + 1] \\
&= 4[f^2(x) - 1][f^2(y) - 1] = 4B(x,x)B(y,y).
\end{aligned}$$

To show that B is an additive f-bimorphism we may use (a) to produce the following four equations:

$$\begin{aligned}
f(xyz) + f(x(yz)^{-1}) &= 2f(x)f(yz); \\
f(yxz) + f(y(xz)^{-1}) &= 2f(y)f(xz); \\
f(xyz^{-1}) + f(x(yz)^{-1}) &= 2f(x)f(yz^{-1}); \\
f(yxz^{-1}) + f(y(xz^{-1})^{-1}) &= 2f(y)f(xz^{-1}),
\end{aligned}$$

for all $x, y, z \in$G. Since $f(xyz) = f(yxz)$, we have

$$\begin{aligned}
&f(x(yz)^{-1}) - f(x(yz^{-1})^{-1}) + f(y(xz)^{-1}) - f(y(xz^{-1})^{-1}) \\
&\quad = f(xy^{-1}z^{-1}) - f(xy^{-1}z) + f(yx^{-1}z^{-1}) - f(yx^{-1}z) \\
&\quad = f(yx^{-1}z) - f(yx^{-1}z^{-1}) + f(yx^{-1}z^{-1}) + f(yx^{-1}z) = 0
\end{aligned}$$

It follows that if we add the first two of the above four equations and subtract the last two equations, we find

$$f(xyz) - f(xyz^{-1}) = f(x)[f(yz) - f(yz^{-1})] + f(y)[f(xz) - f(xz^{-1})].$$

This identity becomes $B(xy,z) = f(x)B(y,z) + f(y)B(x,z)$ due to $(*)$, which is the required first equality. The additivity in the second argument follows similarly and we shall omit the details.

(2)$\Rightarrow$(1). Let us show equation (a) first, which will be discussed in two cases.

Case 1. If $f(x) \neq 0$ and $f(y) \neq 0$ for $x, y \in G$, then

$$\begin{aligned}
f^2(xy) &= B(xy,xy) + 1 = f(x)B(y,xy) + f(y)B(x,xy) + 1 \\
&= f(x)[f(x)B(y,y) + f(y)B(y,x)] + f(y)[f(x)B(x,y) + f(y)B(x,x)] + 1 \\
&= f^2(x)[f^2(y) - 1] + f^2(y)[f^2(x) - 1] + f(x)f(y)[B(y,x) + B(x,y)] + 1 \\
&= 2f^2(x)f^2(y) - [f^2(x) + f^2(y) - 1] + f(x)f(y)[B(y,x) + B(x,y)].
\end{aligned}$$

Similarly,

$$f^2(xy^{-1}) = 2f^2(x)f^2(y) - [f^2(x) + f^2(y) - 1] - f(x)f(y)[B(y,x) + B(x,y)].$$

By adding the above two identities we have

$$\begin{aligned} f^2(xy) + f^2(xy^{-1}) &= 4f^2(x)f^2(y) - 2[f^2(x) + f^2(y) - 1] \\ &= 4f^2(x)f^2(y) - 2f(xy)f(xy^{-1}) \end{aligned}$$

by (b), i.e.,

$$\text{(c)} \qquad [f(xy) + f(xy^{-1})]^2 = [2f(x)f(y)]^2.$$

Thus, for any $x, y \in G$, it is either $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ which is (a), or $= -2f(x)f(y)$. The latter case is impossible. Because, in particular if $y = e$, then $2f(x) = -2f(x)$, and so $f(x) = 0$, which contradicts the assumption that $f(x) \neq 0$.

Case 2. If $f(x) = 0$ or $f(y) = 0$, then from (c) we conclude that $f(xy) + f(xy^{-1}) = 0$, and (a) is fulfilled. In any case we showed the validity of (a).

Since we have just shown that (2) implies identity (a), we may use the two identities (a) and (b) to prove $(*)$ as follows:

Case 1. If $f(x) \neq 0$ and $f(y) \neq 0$ for any $x, y \in G$, then

$$\begin{aligned} 4B^2(x,y) &= 4[f^2(x) - 1][f^2(y) - 1] = 4f^2(x)f^2(y) - 4[f^2(x) + f^2(y) - 1] \\ &= f^2(xy) + f^2(xy^{-1}) + 2f(xy)f(xy^{-1}) - 4f(xy)f(xy^{-1}) \\ &= [f(xy) - f(xy^{-1})]^2. \end{aligned}$$

Hence, for any $x, y \in G$, either (*) holds, or $2B(x,y) = f(xy^{-1}) - f(xy)$. The latter case is impossible. Since, in particular if $y = x$, then $2B(x,x) = 1 - f(x^2) = 1 - [2f^2(x) - 1] = 2[1 - f^2(x)]$; and so $B(x,x) = 1 - f^2(x)$ which contradicts the assumption that $B(x,x) = f^2(x) - 1$.

Case 2. For any $x, y \in G$ with $f(x) = 0$, we have

$f(xy) + f(xy^{-1}) = 0$, and $f(xy)f(xy^{-1}) = f^2(y) - 1$ by (a) and (b), and so, $f(xy) - f(xy^{-1}) = 2f(xy)$. It follows that

$$B^2(x,y) = B(x,x)B(y,y) = -B(y,y) = 1 - f^2(y) = -f(xy)f(xy^{-1}) = f^2(xy).$$

Hence, for any $x, y \in G$, we have either $B(x,y) = f(xy)$, i.e., $2B(x,y) = 2f(xy) = f(xy) - f(xy^{-1})$ which is $(*)$, or $B(x,y) = -f(xy)$. The latter case is impossible. Since, in particular, $B(x,xy) = -f(x^2y)$. But $B(x,xy) = f(x)B(x,y) + f(y)B(x,x) = -f(y)$ as B is an additive f-bimorphism, which implies that $f(y) = f(x^2y)$. In particular, if $y = e$, then $1 = f(x^2) = 2f^2(x) - 1 = -1$ which is absurd.

Case 3. When $f(y) = 0$, it can be argued similarly as in Case 2, and this concludes the proof of the equation $(*)$.

Finally, since

$$f(xyz) = 2B(xy,z) + f(xyz^{-1}) = 2B(xy,z) + 2f(xy)f(z) - f(xyz)$$

by (a), we find

$$f(xyz) = B(xy,z) + f(xy)f(z) = B(yx,z) + f(yx)f(z) = f(yxz).$$

The proof of the theorem is now complete.

In conclusion let us point out some remarks as follows:

(i) The two equations (a) and (b) correspond to the two formulas in trigonometry: $\cos(x+y) + \cos(x-y) = 2(\cos x)(\cos y)$, and $\cos(x+y)\cos(x-y) = \cos^2 x + \cos^2 y - 1$, respectively. Under the supposition $f(xyz) = f(yxz)$, we see that (a) implies (b) was shown in the proof of Theorem. It can be easily proved however that the converse holds if one more relation that $1 + f(x^2) = 2f^2(x)$ (corresponding to the formula $1 + \cos(2x) = 2\cos^2 x$ in trigonometry) is assumed. Consequently, (b) may be regarded as a cosine functional equation, too. It seems that this was not mentioned in the literature.

(ii) It is possible to have further generalizations of equation (a), of course. The following are cosine-type equations (as we may call them):

$$\text{(a}'\text{)}\qquad f(xyz) + f(zxy^{-1}) + f(zyx^{-1}) + f(xyz^{-1}) = 4f(x)f(y)f(z);$$

$$\begin{aligned} &f(xyzw) + f(zyx^{-1}w) + f(xy^{-1}zw) + f(xyz^{-1}w) \\ &\quad + f(xyzw^{-1}) + f(xy^{-1}z^{-1}w) + f(xy^{-1}zw^{-1}) + f(xyz^{-1}w^{-1}) \\ \text{(a}''\text{)}\qquad &\quad = 8f(x)f(y)f(z)f(w), \end{aligned}$$

for all $x, y, z, w \in G$. In other words, equation (a) in (1) of Theorem may be replaced by (a$'$), or by (a$''$) without changing the statements of Theorem. Indeed, to prove identity (a') by using (2) in Theorem for example, we may show first that $f(xyz) + f(xyz^{-1}) = 2f(xy)f(z)$ holds.

(iii) One might attempt to consider another cosine-type equation

$$f^2(xy^{-1}) + f^2(yz^{-1}) + f^2(zx^{-1}) = 1 + 2f(xy^{-1})f(yz^{-1})f(zx^{-1})$$

for all $x, y, z \in G$.

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