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ASYMPTOTIC BEHAVIOR OF SINGULAR VALUES OF CERTAIN INTEGRAL OPERATORS

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Abstract. The exact asymptotics of singular values of a fractional integral operator

$$I^\alpha \cdot = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} \cdot dy$$

for $1/2 < \alpha$ is found. The results related to asymptotic behavior of singular values of convolution operators similar to fractional integral operator are given. We also obtained a result about the asymptotic behavior of convolution operators with logarithm-singularity.

1. Introduction. In [**7**] Hille and Tamarkin obtained bounds for the eigenvalues of fractional integral operators (F.I.O.). Chang [**1**] extended these results to singular values of ordinary integral operators.

Faber and Wing [**3**] found an upper bound for the singular values of F.I.O. and some other similar operators. They stated as an open problem to find the precise asymptotics of the singular values of I^α for $0 < \alpha < 1$. Also, the following is conjectured:

If K_1 and K_2 are two convolution operators

$$K_i \cdot = \int_0^x K_i(x-y) \cdot dy; \qquad i = 1, 2$$

where K_i are smooth functions on $(0, 1]$ so that $\lim\limits_{x\to 0} \dfrac{K_1(x)}{K_2(x)} = 1$, then $\lim\limits_{n\to\infty} \dfrac{s_n(K_1)}{s_n(K_2)} = 1$. ($s_n(K_i)$ are the singular values of K_i).

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The conjecture is shown in the case

$$K_i(x) = x^n k_i(x), \tag{$*$}$$

where $k_i(0) = 1$ and n in a natural number.

In this paper we will prove the conjecture when K_i are of the form $(*)$ and n is not a natural number. We will also find the exact asymptotics of singular values of F.I.O. I^α for $\alpha > 1/2$. The case $0 < \alpha \le 1/2$ was treated in [**2**]. The conjecture with kernels K_i having logarithm-singularity in the point $x = 0$ will be proved.

Asymptotic behavior of singular values and singular functions of convolution operators with sufficiently smooth kernels can be found in [**5**].

2. The singular values of F.I.O. Let H be a complex Hilbert space and T a compact operator on H. The singular values of T $(s_n(T))$ are eigenvalues of the operator $(T^*T)^{1/2}$ (or $(TT^*)^{1/2}$).

We will consider the operator $I^\alpha : L^2(0,1) \to L^2(0,1)$ defined by

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)dy.$$

It is easy to prove that I^α is compact [**3**].

THEOREM 1. *If $\alpha > 0$, then $\lim_{n\to\infty} n^\alpha s_n(I^\alpha) = \pi^{-\alpha}$.*

The case $0 < \alpha \le 1/2$ is proved in [**2**]. Before proving Theorem 1, we will prove some lemmas.

LEMMA 1. *If $\alpha \in (n, n+1/2) \cup (n+1/2, n+1)$ $(n = 0, 1, 2, \ldots)$ and $B : L^2(0,1) \to L^2(0,1)$ is defined by*

$$Bf(x) = \int_0^1 \frac{|x-y|^{2\alpha-1}}{2\Gamma(2\alpha)\cos\alpha\pi} f(y)dy,$$

then $\lim\limits_{n\to\infty} n^{2\alpha} s_n(B) = \pi^{-2\alpha}$.

In [**11**] and [**13**] are given some results about eigenvalues of integral operators with kernels "close to" kernel of operator B. We give a new proof of Lemma 1.

Proof of Lemma 1. Let us consider the function

$$G_{2\alpha}(x) = \frac{2^{1/2-\alpha}}{\sqrt{\pi}\Gamma(\alpha)} K_{1/2-\alpha}(|x|) \cdot |x|^{\alpha-1/2},$$

where $K_\nu(\cdot)$ is McDonald function. It is known that $G_{2\alpha}(\cdot) \in L^1(\mathbf{R})$ and

$$\int_{\mathbf{R}} e^{itx} G_{2\alpha}(t)dt = (1+x^2)^{-\alpha}$$

(see [**12**]). By direct calculation we get

$$G_{2\alpha}(x) = \frac{2^{1/2-\alpha}\sqrt{\pi}}{2\Gamma(\alpha)\cos\alpha\pi}\cdot\frac{1}{2^{\alpha-1/2}\Gamma(\alpha+1/2)}|x|^{2\alpha-1} + \frac{2^{1/2-\alpha}\sqrt{\pi}}{2\Gamma(\alpha)\cos\alpha\pi}\sum_{k=1}^{\infty}\frac{|x|^{2\alpha+2k-1}}{k!2^{2k+\alpha-1/2}\Gamma(k+\alpha+1/2)} + \varphi_0(x) \tag{1}$$

where φ_0 is an even entire function.

Let $B'_\alpha : L^2(-1,1) \to L^2(-1,1)$ be the operator defined by

$$B'_\alpha f(x) = \int_{-1}^{1} G_{2\alpha}(x-y)f(y)dy.$$

According to Widom's result [**13**, Theorem 1] we get

$$s_n(B'_\alpha) \sim (2/n\pi)^{2\alpha}, \quad n \to \infty. \tag{2}$$

Let $D_\alpha : L^2(-1,1) \to L^2(-1,1)$ be the operator

$$D_\alpha f(x) = \frac{2^{1/2-\alpha}\sqrt{\pi}}{2\Gamma(\alpha)\cos\alpha\pi}\int_{-1}^{1}\frac{|x-y|^{2\alpha-1}}{2^{\alpha-1/2}\Gamma(\alpha+1/2)}f(y)dy.$$

Using the Legendre's formula $\Gamma(2\alpha) = \dfrac{2^{2\alpha-1}}{\sqrt{\pi}}\Gamma(\alpha)\Gamma(\alpha+1/2)$ we get

$$D_\alpha f(x) = \frac{1}{2\Gamma(2\alpha)\cos\alpha\pi}\int_{-1}^{1}|x-y|^{2\alpha-1}f(y)dy.$$

From (1), (2), Ky Fan's theorem [**6**] and a theorem of Krein [**6**, p. 157] it follows $s_n(D_\alpha) \sim s_n(B'_\alpha)$ and so

$$s_n(D_\alpha) \sim (2/n\pi)^{2\alpha}. \tag{3}$$

The operator D_α is selfadjoint, therefore from (3), using the substitution $x_1 = (1+x)/2$, $y_1 = (1+y)/2$, we get $s_n(B) \sim (\pi n)^{-2\alpha}$, $n \to \infty$. Lemma is proved. □

Let $0 < \beta < 1/2$, $\phi(t) = \int_t^{+\infty} s^{\beta-1}(1+s)^{\beta-1}ds$, $\alpha = n+\beta$,

$$M(x,y) = \begin{cases} |x-y|^{2\alpha-1}\phi(x/(y-x)); & y > x \\ |x-y|^{2\alpha-1}\phi(y/(x-y)); & x > y \end{cases} \qquad B_1 f(x) = \int_0^1 M(x,y)f(y)dy.$$

Lemma 2. *We have* $\lim_{m\to\infty} m^{2\alpha}s_m(B_1) = 0$.

Proof. Let $\varphi(t) = t^{2\alpha-1}\phi(1/t)\psi(t) = (1-t)^{2\alpha-1}\phi(t/(1-t))$. Expanding functions φ and ψ in series near the points $t = 0$ and $t = 1$ we get

$$\begin{aligned} \varphi^{(\nu)}(0_+) = \psi^{(\nu)}(1-0) = 0; \qquad \nu = 0, 1, \dots, 2n-1 \\ \varphi^{(2n)}(0_+) = \psi^{(2n)}(1-0), \qquad \varphi^{(2n+1)}(0_+) = \psi^{(2n+1)}(1-0). \end{aligned} \tag{4}$$

From

$$M(x,y) = \begin{cases} x^{2\alpha-1}\varphi(y/x-1); & y > x \\ x^{2\alpha-1}\psi(y/x); & x > y \end{cases}$$

and (4) it follows

$$\begin{aligned} &\partial^\nu M/\partial y^\nu|_{y=x} = 0 \quad && \text{for } \nu = 0, 1, \dots, 2n-1 \\ &\partial^\nu M/\partial y^\nu|_{y=x} && \text{exist for } \nu = 2n \text{ and } \nu = 2n+1. \end{aligned}$$

Let $0 < \delta < 1$ and let $P_\delta : L^2(0,1) \to L^2(0,1)$ be the linear operator defined by

$$P_\delta f(x) = \begin{cases} f(x), & 0 \le x < \delta \\ 0, & \delta \le x \le 1. \end{cases}$$

Then

$$B_1 = B_1(I - P_\delta) + (I - P_\delta)B_1 B_\delta + P_\delta B_1 P_\delta. \tag{5}$$

From

$$B_1(I - P_\delta)f(x) = \int_\delta^1 M(x,y)f(y)dy, \quad \text{and}$$

$$\left|\frac{\partial^{2n+1}M}{\partial y^{2n+1}}\right| \le M_\delta \quad \text{for} \quad \delta \le y \le 1,\ x \in [0,1],$$

according to Krein's theorem [**6**, p. 157] we conclude

$$s_m(B_1(I - P_\delta)) = o(m^{-2n-3/2}), \quad m \to \infty \tag{6}$$

and from this

$$s_m((I - P_\delta)B_1 P_\delta) = o(m^{-2n-3/2}), \quad m \to \infty. \tag{7}$$

We will show

$$m^{2\alpha}s_m(P_\delta B_1 P_\delta) \le C_0 \cdot \delta, \tag{8}$$

where C_0 is a constant independent on both m and δ.

The operator $P_\delta B_1 P_\delta : L^2(0,\delta) \to L^2(0,\delta)$ is given by

$$P_\delta B_1 P_\delta f(x) = \int_0^\delta M(x,y) f(y) dy \qquad (0 < x < \delta).$$

Let us write $P_\delta B_1 P_\delta = C + C^*$ where

$$C : L^2(0,\delta) \to L^2(0,\delta); \qquad Cf(x) = \int_x^\delta M(x,y) f(y) dy$$
$$C^* : L^2(0,\delta) \to L^2(0,\delta); \qquad C^* f(x) = \int_0^x M(x,y) f(y) dy.$$

(C^* is the adjoint operator of C). Let $I : L^2(0,\delta) \to L^2(0,\delta)$ and $I^* : L^2(0,\delta) \to L^2(0,\delta)$ be the operators defined by $If(x) = \int_0^x f(s)\,ds$ and $I^* f(x) = \int_x^\delta f(s)\,ds$. Then

$$\begin{aligned} Cf(x) &= \int_x^\delta I^{*2n} f(y) \frac{\partial^{2n} M}{\partial y^{2n}} dy \\ &= \int_x^\delta I^{*2n} f(y) x^{2\alpha-2n-1} \varphi^{(2n)}\left(\frac{y}{x} - 1\right) dy \\ &= \int_x^\delta I^{*2n} f(y) x^{2\beta-1} \varphi^{(2n)}\left(\frac{y}{x} - 1\right) dy \\ &= D I^{*2n} f(x), \end{aligned}$$

where $D : L^2(0,\delta) \to L^2(0,\delta)$ is the linear operator defined by

$$Df(x) = \int_x^\delta x^{2\beta-1} \varphi^{(2n)}\left(\frac{y}{x} - 1\right) f(y) dy.$$

The fact $s_n(I^*) = s_n(I) = \delta/\pi(n - 1/2)$ [**6**, p. 155] implies that inequality (8) will be proved if we prove

$$m^{2\beta} s_m(D) \le C_1 \cdot \delta, \tag{9}$$

where the constant C_1 is independent on both m and δ.

Let D^* be the conjugate operator of the operator D in the space $L^2(0,\delta)$. Then

$$D^* f(x) = \int_0^x y^{2\beta-1} \varphi^{(2n)}\left(\frac{x}{y} - 1\right) f(y) dy.$$

From

$$\int_0^x y^{2\beta-1} \varphi^{(2n)}\left(\frac{x}{y} - 1\right) f(y) dy = \int_0^x (I^{2\beta} f)(y) \mathcal{A}(x,y) dy, \qquad [\mathbf{12}, \text{pp. } 42, 43],$$

where

$$\mathcal{A}(x,y) = -\frac{1}{\Gamma(1-2\beta)} \frac{d}{dy} \int_y^x \frac{t^{2\beta-1} \varphi^{(2n)}\left(\frac{x}{t} - 1\right)}{(t-y)^{2\beta}} dt.$$

it follows

$$D^* = F \cdot I^{2\beta}, \tag{10}$$

where

$$I^{2\beta} : L^2(0,\delta) \to L^2(0,\delta), \quad I^{2\beta} f(x) = \frac{1}{\Gamma(2\beta)} \int_0^x (x-y)^{2\beta-1} f(y)dy$$

and F is the linear operator on $L^2(0,\delta)$ defined by $Ff(x) = \int_0^x \mathcal{A}(x,y)f(y)dy$. We will show that the operator F is bounded. It is easy to check that $\mathcal{A}(\cdot,\cdot)$ is the homogeneous function of order -1. If

$$\theta(x) = \begin{cases} 1; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

then the function $\mathcal{A}(x,y)\theta(x-y)$ is also homogeneous of order -1.

According to the inequality of Hardy and Littlewood [**12**, p. 28] from

$$\int_0^\infty |\mathcal{A}(1,y)||\theta(1-y)|y^{-1/2}dy = \int_0^1 y^{-1/2}|\mathcal{A}(1,y)|dy = L(\alpha) < \infty$$

it follows that the operator

$$\int_0^\infty \mathcal{A}(x,y)\theta(x-y) \cdot dy : L^2(0,\infty) \to L^2(0,\infty)$$

is bounded with the norm not greater then $L(\alpha)$. But then the operator F is also bounded and $\|F\| \leq L(\alpha)$.

From $s_m(I^{2\beta}) \leq C_2 \cdot \delta^{2\beta}/m^{2\beta}$ (with the constant C_2 independent from both m and δ) and (10) we get $s_m(D) \leq C_2 L(\alpha) \cdot \delta^{2\beta}/m^{2\beta} < C_2 L(\alpha)\delta/m^{2\beta}$. This proves the inequality (9) and so (8).

From (6), (7), (8) and the properties of the singular numbers of the summ of operators it follows $\lim_{m\to\infty} m^{2\alpha} s_m(B_1) = 0$ and the lemma is proved. $\square$

Let $\alpha = n + 1/2 + \beta$, $0 < \beta < 1/2$, $\phi_0(x) = \int_x^\infty s^{\beta-1/2}(1+s)^{\beta-3/2}ds$,

$$\begin{aligned} R(x,y) = & |x-y|^{2n} \cdot \begin{cases} x^{\beta+1/2}y^{\beta-1/2}; & y \geq x \\ y^{\beta+1/2}x^{\beta-1/2}; & y \leq x \end{cases} \\ & + (1/2-\beta)|x-y|^{2\alpha-1} \cdot \begin{cases} \phi_0(x/(y-x)); & y > x \\ \phi_0(y/(x-y)); & x > y. \end{cases} \end{aligned}$$

Let $B_2 : L^2(0,1) \to L^2(0,1)$ be the linear operator defined by

$$B_2 f(x) = \int_0^1 R(x-y)f(y)dy.$$

Let $0 < \delta < 1$ and $P_\delta : L^2(0,1) \to L^2(0,1)$ be the linear operator

$$P_\delta f(x) = \begin{cases} f(x); & 0 \leq x < \delta \\ 0; & \delta \leq x < 1. \end{cases}$$

Let $T_0 : L^2(0,\delta) \to L^2(0,\delta)$ and $S : L^2(0,\delta) \to L^2(0,\delta)$ be the linear operators given by

$$T_0 f(x) = \int_x^\delta \frac{\partial^{2n+1} R}{\partial y^{2n+1}} f(y)dy$$
$$Sf(x) = \int_0^x (y^{2\beta} - x^{2\beta})(x-y)^{2n} f(y)dy.$$

Lemma 3.

a) $s_m(T_0) \leq C_3 \cdot \delta / m^{2\beta}$, *where the constant* C_3 *is independent on both* m *and* δ.

b) $m^{2\alpha} s_m(s) \leq C_4 \cdot \delta$, *where the constant* C_4 *is independent on both* m *and* δ.

c) $\lim_{m\to\infty} m^{2\alpha} s_m(B_2) = 0$.

Proof. Let φ and ψ be the functions

$$\varphi(t) = (t-1)^{2n} t^{\beta-1/2} + (1/2-\beta)(t-1)^{2\alpha-1}\phi_0(1/(t-1)); \quad (t > 1)$$
$$\psi(t) = (1-t)^{2n} t^{\beta+1/2} + (1/2-\beta)(1-t)^{2\alpha-1}\phi_0(t/(1-t)); \quad (t < 1)$$

Then

$$R(x,y) = \begin{cases} x^{2\alpha-1}\varphi(y/x); & y \geq x \\ x^{2\alpha-1}\psi(y/x); & x \geq y. \end{cases}$$

It is easy to check that

$$\begin{cases} \varphi^{(\nu)}(1+0) = \psi^{(\nu)}(1-0) = 0, & \text{for } \nu = 0,1,\ldots,2n-1 \\ \varphi^{(\nu)}(1+0) = \psi^{(\nu)}(1-0), & \text{for } \nu = 2n,\ 2n+1,\ 2n+2. \end{cases} \tag{11}$$

Like in Lemma 2 we use the fact that

$$B_2 = B_2(I-P_\delta) + (I-P_\delta)B_2P_\delta + P_\delta B_2 P_\delta.$$

It follows from (11) that $\partial^\nu R/\partial y^\nu|_{y=x} = 0$ for $\nu = 0,1,\ldots,2n-1$ and that there exist $\partial^\nu R/\partial y^\nu|_{y=x}$ for $\nu = 2n,\ 2n+1,\ 2n+2$. From $|\partial^{2n+2}R/\partial y^{2n+2}| \leq M_\delta < \infty$ for $\delta \leq y \leq 1$, $0 \leq x \leq 1$ and

$$B_2(I-P_\delta)f(x) = \int_\delta^1 R(x,y)f(y)dy$$

it follows that

$$\begin{cases} s_m(B_2(I-P_\delta)) = o(m^{-(2n+2+1/2)}) \\ s_m((I-P_\delta)BP_\delta) = o(m^{-(2n+2+1/2)}) \end{cases}, \quad m \to \infty. \tag{12}$$

We write the operator

$$P_\delta B_2 P_\delta f(x) = \int_0^\delta R(x,y) f(y) dy : L^2(0,\delta) \to L^2(0,\delta)$$

in the form $P_\delta B_2 P_\delta = E + E^*$, where

$$Ef(x) = \int_x^\delta R(x,y) f(y) dy; \quad E^* f(x) = \int_0^x R(x,y) f(y) dy.$$

Using the partial integration $2n+1$ times and applying (11) we get

$$Ef(x) = I^{*2n+1} f(x) \cdot \partial^{2n} R / \partial y^{2n}|_{y=x+0} + T_0 I^{*2n+1} f(x),$$

and so

$$Ef(x) = \varphi^{(2n)}(1+0) \cdot x^{2\beta} I^{*2n+1} f(x) + T_0 I^{*2n+1} f(x).$$

Let

$$Vf(x) = \frac{x^{2\beta}}{(2n)!} \int_x^\delta (y-x)^{2n} f(y) dy : L^2(0,\delta) \to L^2(0,\delta).$$

Then $E = \varphi^{(2n)}(1+0) \cdot V + T_0 I^{*2n+1}$, and from this we get $E^* = \varphi^{(2n)}(1+0) \cdot V^* + I^{2n+1} T_0^*$. Therefore

$$P_\delta B_2 P_\delta = \varphi^{(2n)}(1+0)(V + V^*) + T_0 I^{*2n+1} + T_0^* I^{2n+1}. \tag{13}$$

Proof of part a) of Lemma 3 is the same as the proof of Lemma 2.

It follows from this that

$$m^{2\alpha} s_m (T_0 I^{*2n+1} + T_0^* I^{2n+1}) \le C_5 \cdot \delta, \tag{14}$$

with C_5 independent from both m and δ.

Note that $V + V^* = S + W$, with $W : L^2(0,\delta) \to L^2(0,\delta)$ defined by

$$Wf(x) = \int_0^\delta x^{2\beta} (x-y)^{2n} f(y) dy.$$

The operator W is an operator of the rang $2n+1$ and if part b) of Lemma 3 is proved, we get

$$m^{2\alpha} s_m (V + V^*) \le C_6 \cdot \delta, \tag{15}$$

the constant C_6 is independent from both m and δ.

From (13), (14), (15) and from the properties of the singular values of the summ of operators it follows

$$m^{2\alpha} s_m (P_\delta B_2 P_\delta) \le C_7 \cdot \delta, \tag{16}$$

C_7 is independent from m and δ. But then from (12) and (16) it follows part c) of Lemma 3.

We will prove the statement b). For that it is sufficient to prove that $m^{2\alpha} s_m(S_1) \le C_4 \cdot \delta$ (C_4 is independent from m, δ), with

$$S_1 f(x) = \int_x^\delta (y^{2\beta} - x^{2\beta})(x-y)^{2n} f(y)dy : L^2(0,\delta) \to L^2(0,\delta).$$

Set $h(t) = (t^{2\beta} - 1)(t-1)^{2n}$. Then

$$S_1 f(x) = \int_x^\delta x^{2\alpha - 1} h\left(\frac{y}{x}\right) f(y)dy.$$

Note that

$$h^{(\nu)}(1+0) = 0, \text{ for } \nu = 0, 1, \dots, 2n \text{ and } h^{(2n+1)}(1+0) = -2\beta \tag{17}$$

Using (17), after $2n+1$ partial integrations we get $S_1 = DI^{*2n+1}$, where $D : L^2(0,\delta) \to L^2(0,\delta)$ is the linear operator defined by

$$Df(x) = \int_x^\delta x^{2\beta-1} h^{(2n+1)}\left(\frac{y}{x}\right) f(y)dy.$$

If we prove

$$m^{2\beta} s_m(D) \le C_8 \delta \qquad (C_8 \text{ is independent on } m,\, \delta), \tag{18}$$

the part b) of Lemma 3 will be proved.

Using [**12**, pp. 42, 43] we conclude

$$D^* f(x) = \int_0^x y^{2\beta-1} h^{(2n+1)}\left(\frac{x}{y}\right) f(y)dy = \int_0^x (I^{2\beta} f)(y) \mathcal{B}(x,y)dy$$

with

$$\mathcal{B}(x,y) = -\frac{1}{\Gamma(1-2\beta)} \frac{d}{dy} \int_y^x \frac{t^{2\beta-1} h^{(2n+1)}(x/t)}{(t-y)^{2\beta}} dt.$$

The operator

$$F_1 f(x) = \int_0^x \mathcal{B}(x,y) f(y)dy$$

is bounded and its norm has an upper bound independent of δ. The proof of this fact is as in Lemma 2. This implies (18) and completes the proof. $\square$

Lemma 4. *Let $L : L^2(0,1) \to L^2(0,1)$ be the linear operator defined by*

$$Lf(x) = \int_0^1 (x-y)^{2n} \ln(\sqrt{x} + \sqrt{y}) f(y)dy.$$

Then $\lim_{m\to\infty} m^{2n+1}s_m(L) = 0$.

Proof. Considered the operator L in the form

$$Lf(x) = \sum_{\nu=0}^{2n} \binom{2n}{\nu}(-1)^\nu x^{n-\nu/2} T_\nu f(x)$$

with

$$T_\nu f(x) = \int_0^1 y^{\nu/2}(\sqrt{x}+\sqrt{y})^{2n}\ln(\sqrt{x}+\sqrt{y})f(y)dy,$$

we conclude that it is enough to show that $\lim\limits_{m\to\infty} m^{2n+1}s_m(G_1) = 0$ for

$$G_1 f(x) = \int_0^1 (\sqrt{x}+\sqrt{y})^{2n}\ln(\sqrt{x}+\sqrt{y})f(y)dy.$$

To do this, it is sufficient to show that $\lim\limits_{m\to\infty} m^{2n+1}s_n(G) = 0$, for the operator

$$Gf(x) = \int_0^1 (x+y)^{2n}\ln(x+y)f(y)dy.$$

By partial integrations we get

$$G = \text{finite rank operator} + (2n)!H\cdot I^{2n}, \tag{19}$$

where

$$I^{2n}f(x) = \frac{1}{(2n-1)!}\int_0^x (x-y)^{2n-1}f(y)dy; \quad Hf(x) = \int_0^1 \ln(x+y)f(y)dy.$$

In [**2**] it was shown that $\lim\limits_{m\to\infty} ms_m(H) = 0$ and thus from (19) the conclusion of the lemma follows. $\square$

LEMMA 5. *Let* $P : L^2(0,1)\to L^2(0,1)$ *be the linear operator defined by*

$$Pf(x) = \int_0^1 \frac{(x-y)^{2n}}{\pi(2n)!}\ln|x-y|f(y)dy.$$

Then $\lim\limits_{m\to\infty} m^{2n+1}s_m(P) = \pi^{-2n-1}$.

Proof. The function $G_1(x) = \pi^{-1}K_0(|x|)\in L^1(\mathbf{R})$ (K_0 is McDonald function [**12**]) satisfies the relation

$$\int_{\mathbf{R}} G_1(t)e^{itx}dt = (1+x^2)^{-1/2}.$$

Diferentiating this relation $2n$ times we get

$$\int_{\mathbf{R}} G_1(t)t^{2n}e^{itx}dt = (-1)^n \frac{d^{2n}}{dx^{2n}}(1+x^2)^{-1/2}$$

Using Widom's result [**13**] and having in mind that

$$\left|\frac{d^{2n}}{dx^{2n}}(1+x^2)^{-1/2}\right| \sim \frac{(2n)!}{x^{2n+1}} \qquad (x\to\infty)$$

(because $(1+x^2)^{-1/2} = \sum_{k=0}^{\infty}\binom{-1/2}{k}x^{-2k-1}$, $x \gg 1$) we obtain

$$s_m\left(\int_{-1}^{1} G_1(x-y)(x-y)^{2n}\cdot dy\right) \sim \frac{(2n)!}{(m\pi/2)^{2n+1}} \qquad (m\to\infty). \tag{20}$$

But, on the other side

$$G_1(x) = -\frac{\ln|x|}{\pi}\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{(k!)^2} + \varphi_0(x)$$

(φ_0 is an even entire function). Therefore, using Ky-Fan's theorem [**6**] and (20) we conclude

$$s_m\left(\int_{-1}^{1}\frac{(x-y)^{2n}\ln|x-y|}{\pi}\cdot dy\right) \sim \frac{(2n)!}{(m\pi/2)^{2n+1}}$$

and thus

$$s_m\left(\int_{-1}^{1}\frac{(x-y)^{2n}\ln|x-y|}{\pi\cdot(2n)!}\cdot dy\right) \sim \frac{1}{(m\pi/2)^{2n+1}}$$

The last relation implies $s_m(P)\sim 1/(m\pi)^{2n+1}$. The lemma is proved. $\square$

Proof of Theorem 1. By direct calculation we find the kernel $K_0(x,y)$ of the operator $A_0 = (I^\alpha)^* I^\alpha$:

$$K_0(x,y) = \begin{cases} \Gamma^{-2}(\alpha)\int_0^{1-x}(t+x-y)^{\alpha-1}t^{\alpha-1}dt; & 1\ge x\ge y\ge 0 \\ \Gamma^{-2}(\alpha)\int_0^{1-y}(t+x-y)^{\alpha-1}t^{\alpha-1}dt; & 1\ge y\ge x\ge 0. \end{cases}$$

The eigenvalues of the operator A_0 are the same as the eigenvalues of the operator A with kernel

$$\mathcal{K}_\alpha = \begin{cases} \Gamma^{-2}(\alpha)\int_0^{x} t^{\alpha-1}(t+y-x)^{\alpha-1}dt; & 1\ge y\ge x\ge 0 \\ \Gamma^{-2}(\alpha)\int_0^{y} t^{\alpha-1}(t+x-y)^{\alpha-1}dt; & 1\ge x\ge y\ge 0. \end{cases}$$

We will use the formulae from [**10**]:

$$\begin{aligned} \int x^p(1+x)^q dx &= \frac{x^{p+1}(1+x)^q}{p+q+1} + \frac{p}{p+q+1}\int x^p(1+x)^{q-1}dx \\ \int x^p(1+x)^q dx &= \frac{x^p(1+x)^{q+1}}{p+q+1} - \frac{p}{p+q+1}\int x^{p-1}(1+x)^q dx. \end{aligned} \tag{21}$$

From (21) we obtain

$$\int x^{\alpha-1}(1+x)^{\alpha-1}dx = \frac{x^{\alpha}(1+x)^{\alpha-1}}{2\alpha-1} + \frac{x^{\alpha-1}(1+x)^{\alpha-1}}{2(2\alpha-1)} - \frac{\alpha-1}{2(2\alpha-1)}\int x^{\alpha-2}(1+x)^{\alpha-2}dx. \tag{22}$$

I case: $\alpha = n+\beta$, $0<\beta<1/2$, $n=0,1,2,\ldots$. If we apply (21) n times, we get

$$\mathcal{K}_{\alpha}(x,y) = \text{kernel of a finite } (2n) \text{ rank operator} + \frac{(-1)^n(x-y)^{2n}}{(2\alpha-1)\cdot\ldots\cdot(2\alpha-2n)}\mathcal{K}_{\beta}(x,y).$$

In [**2**] it is shown that

$$\mathcal{K}_{\beta}(x,y) = \frac{B(\beta,1-2\beta)}{\Gamma^2(\beta)}|x-y|^{2\beta-1} + G(x,y)$$

with

$$G(x,y) = \begin{cases} -\Gamma^{-2}(\beta)(y-x)^{2\beta-1}\phi\left(\frac{x}{y-x}\right); & y>x \\ -\Gamma^{-2}(\beta)(x-y)^{2\beta-1}\phi\left(\frac{y}{x-y}\right); & x>y. \end{cases}$$

(The function ϕ is introduced before Lemma 2). Therefore

$$\begin{aligned} K_{\alpha}(x,y) &= \frac{(-1)^n|x-y|^{2\alpha-1}B(\beta,1-2\beta)}{\Gamma^2(\beta)(2\alpha-1)\cdot\ldots\cdot(2\alpha-2n)} \\ &\quad + \text{kernel of a finite rank operator} + C(\alpha)\cdot M(x,y) \\ &= \frac{|x-y|^{2\alpha-1}}{2\Gamma(2\alpha)\cos\alpha\pi} + \text{kernel of a finite rank operator} + C(\alpha)M(x,y). \end{aligned}$$

From this it follows

$$A = B + \text{finite } (2n) \text{ rank operator} + C(\alpha)\cdot B_1 \tag{23}$$

From (23), Lemma 1, Lemma 2 and Ky-Fan's theorem [**6**] it follows

$$s_m(A) \sim (\pi m)^{-2\alpha},\ (m\to\infty) \quad \text{and thus} \quad s_m(I^{\alpha}) \sim (\pi m)^{-\alpha},\ (m\to\infty).$$

II case: $\alpha = n+1/2+\beta$, $0<\beta<1/2$. Similarly to the previous case we obtain

$$\mathcal{K}_{\alpha}(x,y) = \text{kernel of a finite rank } (2n) \text{ operator} + \frac{(-1)^n(x-y)^{2n}}{(2\alpha-1)\cdot\ldots\cdot(2\alpha-2n)}\mathcal{K}_{\beta+1/2}(x,y)$$

Using (21) we get

$$\mathcal{K}_{\beta+1/2}(x,y) = \frac{1}{\Gamma^2(\beta+1/2)} \begin{cases} \frac{x^{\beta+1/2} y^{\beta-1/2}}{2\beta}; & y \geq x \\ \frac{y^{\beta+1/2} x^{\beta-1/2}}{2\beta}; & x \geq y \end{cases}$$
$$+ |x-y|^{2\beta} \cdot \frac{\beta - 1/2}{2\beta\Gamma^2(\beta+1/2)} \int_0^\infty s^{\beta-1/2}(1+s)^{\beta-3/2} ds$$
$$+ \frac{\beta-1/2}{2\beta\Gamma^2(\beta+1/2)} |x-y|^{2\beta} \cdot \begin{cases} \phi_0\left(\frac{x}{y-x}\right); & y \geq x \\ \phi_0\left(\frac{y}{x-y}\right); & x \geq y. \end{cases}$$

Then

$$\mathcal{K}_\alpha(x,y) = \frac{|x-y|^{2\alpha-1}}{2\Gamma(\alpha)\cos\alpha\pi} + \text{kernel of a finite rank operator} + q(\alpha) R(x,y)$$

From this it follows

$$A = B + \text{finite rank operator} + q(\alpha) B_2. \tag{24}$$

From (24), Lemma 1, Lemma 3 and Ky-Fan's theorem [**6**] it follows

$$s_m(A) \sim (\pi m)^{-2\alpha} \quad \text{and thus} \quad s_m(I^\alpha) \sim (\pi m)^{-\alpha}, \quad m \to \infty.$$

III case: $\alpha = n + 1/2$. From

$$\mathcal{K}_\alpha(x,y) = \text{kernel of a finite rank operator} + \frac{(-1)^n (x-y)^{2n}}{(2n)!} \mathcal{K}_{1/2}(x,y)$$

and

$$\mathcal{K}_{1/2}(x,y) = -\frac{1}{\pi} \ln|x-y| + \frac{2}{\pi} \ln(\sqrt{x} + \sqrt{y})$$

we conclude

$$A = \text{finite rank operator} + \frac{2}{\pi} \frac{(-1)^n}{(2n)!} L + (-1)^{n+1} P$$

Using (25), Lemma 4, Lemma 5 and Ky-Fan's theorem [**6**] we obtain

$$s_m(A) \sim 1/(m\pi)^{2n+1}, \quad m \to \infty \quad \text{and thus} \quad s_m(I^\alpha) \sim 1/(m\pi)^\alpha, \quad m \to \infty.$$

IV case: $\alpha = n$, (n is a natural number). In this case the problem on asymptotic behavior of singular numbers reduces to the problem on asymptotic behavior of eigenvalues of a differential operator with regular boundary conditions.

Asymptotic of eigenvalues of a differential operator with regular boundary condition is known (see [**8**]) and therefore

$$s_m(I^n) \sim 1/(m\pi)^n, \quad m \to \infty.$$

Thisf completes the proof. $\square$

THEOREM 2. *Let* $K_i : L^2(0,1) \to L^2(0,1)$ $(i = 1,2)$ *be the operators defined by*

$$K_i f(x) = \int_0^x \mathcal{K}_i(x-y) f(y) dy,$$

where

$$\mathcal{K}_i(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}(1 + r_i(x)), \quad r_i \in C^{1+[\alpha]}[0,1], \quad \alpha > 0, \quad r_i(0) = 0,$$

$[\alpha]$ *is the greatest integer wich is not greater than* α. *Then*

$$\lim_{n\to\infty} \frac{s_n(K_1)}{s_n(K_2)} = 1.$$

Proof. It is sufficient to cinsider the case

$$\mathcal{K}_1(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}(1 + r(x)), \quad \mathcal{K}_2(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$$
$$r \in C^{1+[\alpha]}[0,1], \quad r(0) = 0, \quad r'(0) = 0,$$

since from $r'(0) \neq 0$ it follows

$$s_n\left(\int_0^x r'(0)\frac{(x-y)^\alpha}{\Gamma(\alpha)} \cdot dy\right) \sim \frac{r'(0)}{(n\pi)^{\alpha+1}},$$

by virtue of Theorem 1.

We will use the Keldysh–Krein's result [**9**]: If A and B are compact operators such that $A = B(I+T)$ for compact operator T such that $-1 \in \rho(T)$, then

$$\lim_{n\to\infty} s_n(A)/s_n(B) = 1.$$

For $A = K_1$, $B = K_2$, using the fractional integral operator we get (see [**12**]):

$$Tf(x) = \frac{(-1)^{[\alpha]}}{\Gamma(\alpha)} \int_0^x f(y) \cdot S(x,y) dy,$$

with

$$S(x,y) = -\frac{1}{\Gamma(1-\alpha+[\alpha])}\frac{d}{dy}\int_y^x (t-y)^{-(\alpha-[\alpha])}\frac{\partial^{[\alpha]}}{\partial t^{[\alpha]}}((x-t)^{\alpha-1}r(x-t))dt$$

From the conditions $r \in C^{1+[\alpha]}[0,1]$, $r(0)=0$, $r'(0)=0$ it follows that the function S is bounded on the set $\Delta = \{(x,y) \in \mathbf{R}^2 : 0 \le y \le x, \quad 0 \le x \le 1\}$ and thus the operator T is compact and Volterra. Therefore, according to quoted theorem we have $\lim\limits_{n\to\infty} s_n(K_1)/s_n(K_2) = 1$. $\square$

Corollary. *If $\alpha > 0$, $r \in C^{[\alpha]+1}[0,1]$, $r(0) \neq 0$, $k(x) = x^{\alpha-1}r(x)$ and $K : L^2(0,1) \to L^2(0,1)$ is the linear operator defined by*

$$Kf(x) = \int_0^x k(x-y)f(y)dy,$$

then $s_n(K) \sim r(0)\Gamma(\alpha)(n\pi)^{-\alpha}$, $n \to \infty$.

Let us consider the kernels $k_i(x) = \ln^\beta x^{-1}(1+r_i(x))$; $i=1,2$ with $r_i(0)=0$ and the operators

$$K_i f(x) = \int_0^x k_i(x-t)f(t)dt.$$

We will prove the following.

Theorem 3. *If $1<\beta<2$, $r_i \in C^3[0,1]$, $d^k r_i/dx^k|_{x=0} = 0$ for $k \in \{0,1,2\}$ then $\lim_{n\to\infty} s_n(K_1)/s_n(K_2) = 1$.*

Proof. Like in Theorem 2, it is enough to consider the case

$$k_1(x) = \ln^\beta x^{-1}(1+r(x)), \quad k_2(x) = \ln^\beta x^{-1},$$

with $r \in C^2[0,1]$, $r(0) = r'(0) = r''(0) = 0$.

If K_1 and K_2 are operators with kernels k_1 and k_2, then, with $A = K_1$ and $B = K_2$ in Keldysh–Krein theorem quoted in the proof of Theorem 2, we have $A = B(I+T)$ with

$$Tf(x) = P\int_0^x S(x,y)f(y)dy$$

where P is a bounded operator and [**12**, p. 487]

$$S(x,y) = \frac{d}{dx}\int_y^x \mu_{0,\beta}(x-t)\ln^\beta \frac{1}{t-y}r(t-y)dt.$$

Changing the variable in the last integral we obtain

$$S(x,y) = -\frac{d}{dx}\int_{y-x}^0 \mu_{0,\beta}(t)\ln^\beta \frac{1}{x-y-t}r(x-y-t)dt.$$

It is easy to prove, using the asymptotic behavior of the function $\mu_{0,\beta}$ [**12**, p. 482], that the operator T is Hilbert-Schmidt, and hence compact.

Reasoning as in the proof of Theorem 2 we conclude

$$\lim_{n\to\infty} s_n(K_1)/s_n(K_2) = 1.$$

Theorem is proved. □

Remark. It remains as an open problem to find the exact asymptotic of singular values of the operator $K: L^2(0,1) \to L^2(0,1)$ defined by

$$Kf(x) = \int_0^x \ln^\beta \frac{1}{x-y} f(y)dy, \quad \beta > 0.$$

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