PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 62 (76), 1997, 99–104

ON PRE-PARAREFLEXIVE OPERATOR ALGEBRAS

Pawan Bala

Communicated by Stevan Pilipović

Abstract. The notions of pre-parareflexivety and pre-algebraic reflexivity are introduced and studied. The conditions under which pre-parareflexivity coincides with pre-reflexivity or pre-algebraic reflexivity, are obtained.

Let H denote an infinite-dimensional separable complex Hilbert space and $B(H)$, the set of all bounded linear operators on H. An operator range is a linear manifold in H that is also the range of an operator in $B(H)$. For any set $S \subseteq B(H)$, $\operatorname{Lat} S$ denotes the lattice (with closed linear span as join and intersection as meet) of all the closed linear subspaces of H invariant under every operator in S, $\operatorname{Lat}_{1/2} S$ denotes the lattice (with algebraic sum of linear manifolds as join and intersection as meet) of all operator ranges invariant under every operator in S and $\operatorname{Lat}_0 S$ denotes the lattice of all invariant linear manifolds for S. If S is singleton say $\{T\}$, we write these notions as $\operatorname{Lat} T$, $\operatorname{Lat}_{1/2} T$ and $\operatorname{Lat}_0 T$ respectively. $\operatorname{Alg}\operatorname{Lat} S\,(\operatorname{Alg}\operatorname{Lat}_{1/2} S, \operatorname{Alg}\operatorname{Lat}_0 S)$ denotes the algebra of all operators leaving invariant every member of $\operatorname{Lat} S\,(\operatorname{Lat}_{1/2} S$, $\operatorname{Lat}_0 S$ respectively). A weakly closed algebra $\mathcal{A}$ containing identity is called reflexive if $\operatorname{Alg}\operatorname{Lat}\mathcal{A} = \mathcal{A}$ and pre-reflexive [**1**] if

$$(\operatorname{Alg}\operatorname{Lat}\mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*$$

where $\mathcal{A}^* = \{T^*: T \in \mathcal{A}\}$. Ong [**8**] considered an analogous notion of reflexivity with respect to the lattice of invariant operator ranges and the lattice of invariant linear manifolds.

Definition 1. [**8**]. An algebra $\mathcal{A} \subseteq B(H)$ (not necessarily closed in any topology) is parareflexive if for any $T \in B(H)$ with $\operatorname{Lat}_{1/2} T \supseteq \operatorname{Lat}_{1/2}\mathcal{A}$ we have $T \in \mathcal{A}$. Equivalently $\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A} = \mathcal{A}$.

AMS Subject Classification (1991): Primary 47D25

Key words and Phrases: Pre-reflexive algebra, parareflexive algebra, pre-parareflexive algebra, pre-algebraically reflexive algebra, strictly cyclic algebra, operator range.

Definition 2. [**8**]. An algebra $\mathcal{A}$ is algebraically reflexive if $\mathcal{A}$ contains all operators $T \in B(H)$ such that $\operatorname{Lat}_0 T \supseteq \operatorname{Lat}_0 \mathcal{A}$. Equivalently $\operatorname{Alg}\operatorname{Lat}_0 \mathcal{A} = \mathcal{A}$.

In this paper we define an operator range analogue of prereflexive algebras namely, pre-parareflexive algebras and discuss some properties of such algebras.

Definition 3. An algebra $\mathcal{A}$ is said to be pre-parareflexive if

$$\mathcal{A} \cap \mathcal{A}^* = (\operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{A})^*.$$

Definition 4. An algebra $\mathcal{A}$ is said to be prealgebraically reflexive if

$$\mathcal{A} \cap \mathcal{A}^* = (\operatorname{Alg}\operatorname{Lat}_0 \mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}_0 \mathcal{A})^*.$$

It is easily seen that every parareflexive algebra is pre-parareflexive and every algebraically reflexive algebra is prealgebraically reflexive. For more details on parareflexive algebras and algebraically reflexive algebras, one may refer to [**4**], [**6**], [**8**].

THEOREM 5. *Every prereflexive algebra is pre-parareflexive and every pre-parareflexive algebra is prealgebraically reflexive.*

Proof. As $\operatorname{Lat}\mathcal{A} \subseteq \operatorname{Lat}_{1/2} \mathcal{A} \subseteq \operatorname{Lat}_0 \mathcal{A}$, we have

$$\operatorname{Alg}\operatorname{Lat}_0 \mathcal{A} \subseteq \operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{A} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{A}.$$

This implies that

$$\begin{aligned}(\operatorname{Alg}\operatorname{Lat}_0 \mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}_0 \mathcal{A})^* &\subseteq (\operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{A})^* \\ &\subseteq (\operatorname{Alg}\operatorname{Lat}\mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}\mathcal{A})^*\end{aligned}$$

The result follows.

THEOREM 6. *Any algebra unitarily equivalent to a pre-parareflexive algebra is pre-parareflexive.*

Proof. Let $\mathcal{A}$ be a pre-parareflexive algebra and S, a unitary operator. Let

$$\mathcal{B} = S\mathcal{A}S^* = \{SAS^* \colon A \in \mathcal{A}\}$$

be an algebra unitarily equivalent to $\mathcal{A}$. Let

$$T \in (\operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{B}) \cap (\operatorname{Alg}\operatorname{Lat}_{1/2} \mathcal{B})^*$$

Then

$$\operatorname{Lat}_{1/2} \mathcal{B} \subseteq \operatorname{Lat}_{1/2} T \quad \text{and} \quad \operatorname{Lat}_{1/2} \mathcal{B} \subseteq \operatorname{Lat}_{1/2} T^*$$

Let M be an operator range invariant under $\mathcal{A}$. Then

$$\mathcal{B}S(M) = (S\mathcal{A}S^*)S(M) \subseteq S(M)$$

Therefore $S(M)$ is an operator range invariant under $\mathcal{B}$ and thus under T and T^* both. This implies that for $M \in \operatorname{Lat}_{1/2}\mathcal{A}$

$$(S^*TS)M \subseteq M \quad \text{and} \quad S^*T^*S(M) \subseteq M.$$

Thus

$$S^*TS \in (\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*,$$

or

$$T \in (S\mathcal{A}S^*) \cap (S\mathcal{A}^*S^*) = \mathcal{B} \cap \mathcal{B}^*.$$

COROLLARY 7. *An algebra unitarily equivalent to a parareflexive algebra is parareflexive.*

For a Hilbert space H, the tensor product of H with itself denoted by $H \otimes H$ is the space $\sum_{n=1}^{\infty} \oplus H_n$ with $H_n = H$ for all n. For an algebra $\mathcal{A}$ the tensor product $\mathcal{A} \otimes B(H)$ is the set of all operators on $H \otimes H$ of the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

such that $A_{ij} \in \mathcal{A}$ for all i and j.

THEOREM 8. *If $\mathcal{A}$ is a preparareflexive algebra then, $\mathcal{A} \otimes B(H)$ is pre-parareflexive.*

Proof. Let $T \in (\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A} \otimes B(H)) \cap (\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A} \otimes B(H))^*$. Then

$$\operatorname{Lat}_{1/2}\mathcal{A} \otimes B(H) \subseteq \operatorname{Lat}_{1/2} T \quad \text{and} \quad \operatorname{Lat}_{1/2}\mathcal{A} \otimes B(H) \subseteq \operatorname{Lat}_{1/2} T^*.$$

Here T is an operator on $H \otimes H$ which is of the form

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots \\ B_{21} & B_{22} & B_{22} & \cdots \\ B_{31} & B_{32} & B_{33} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

where $B_{ij} \in B(H)$. Let M be an operator range invariant under $\mathcal{A}$. Let $N = \sum_{k=1}^{\infty} \oplus M_k$ where $M_k = M$ for each k. Then N is an operator range invariant under $\mathcal{A} \otimes B(H)$ and thus invariant under both T and T^*. This implies that M is invariant under every B_{ij} and B_{ij}^*. Thus

$$B_{ij} \in (\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A}) \cap (\operatorname{Alg}\operatorname{Lat}_{1/2}\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*$$

implying that $T \in (\mathcal{A} \otimes B(H)) \cap (\mathcal{A} \otimes B(H))^*$.

COROLLARY 9. *If $\mathcal{A}$ is a parareflexive algebra, then $\mathcal{A} \otimes B(H)$ is also parareflexive.*

Remark. We may likewise prove that for a prealgebraically reflexive algebra $\mathcal{A}$:

(i) $\mathcal{A} \otimes B(H)$ is prealgebraically reflexive,

(ii) $S\mathcal{A}S^*$ is prealgebraically reflexive for any unitary operator S.

THEOREM 10. *Let $\mathcal{A}$ be a weakly closed algebra of operators such that every operator range invariant under $\mathcal{A}$ is the range of an operator of the form*

$$\sum 2^{-k}(E_{k+1} - E_k),$$

where $\{E_k\}$ is an increasing sequence of projections whose ranges are in $\operatorname{Lat} \mathcal{A}$. *Then $\mathcal{A}$ is pre-parareflexive if and only if $\mathcal{A}$ is prereflexive.*

Proof. Let $\mathcal{A}$ satisfy the hypothesis and let

$$\mathcal{A} \cap \mathcal{A}^* = (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A}) \cap (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A})^*.$$

Let $T \in (\operatorname{Alg} \operatorname{Lat} \mathcal{A}) \cap (\operatorname{Alg} \operatorname{Lat} \mathcal{A})^*$. Let R be any operator range invariant under $\mathcal{A}$ and let R be the range of the operator $P = \sum 2^{-k}(E_{k+1} - E_k)$ for some increasing sequence $\{E_k\}$ of projections in $\operatorname{Lat} \mathcal{A}$. Let F be the spectral measure of P. Then

$$F(2^{-k}, 1] = E_k \in \operatorname{Lat} \mathcal{A}$$

which is contained in both $\operatorname{Lat} T$ and $\operatorname{Lat} T^*$. Thus the range of $F(2^{-k}, 1]$ is invariant under T and T^*, for $k = 1, 2, 3, \ldots$. By [**7**, Theorem B], the range R of P is invariant under both T and T^*. This implies that

$$\operatorname{Lat}_{1/2} \mathcal{A} \subseteq \operatorname{Lat}_{1/2} T \quad \text{and} \quad \operatorname{Lat}_{1/2} \mathcal{A} \subseteq \operatorname{Lat}_{1/2} T^*.$$

This gives

$$T \in (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A}) \cap (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*.$$

Hence $\mathcal{A}$ is prereflexive.

COROLLARY 11. *Let $\mathcal{A}$ be a weakly closed algebra containing a maximal abelian selfadjoint algebra. Then $\mathcal{A}$ is prereflexive if and only if it is pre-parareflexive.*

An algebra $\mathcal{A}$ is said to be strictly cyclic [**5**] on a Hilbert space H if there exists a vector x_0 in H such that $\mathcal{A}x_0 = H$. In the following we prove that abelian strictly cyclic prealgebraically reflexive algebras are pre-parareflexive.

THEOREM 12. *Let $\mathcal{A}$ be a strictly cyclic abelian algebra of operators. Then $\mathcal{A}$ is pre-parareflexive if and only if $\mathcal{A}$ is prealgebraically reflexive.*

Proof. For any $x \in H$, $\mathcal{A}x$ is an operator range invariant under $\mathcal{A}$ and any invariant linear manifold of $\mathcal{A}$ is the sum of spaces of the form $\mathcal{A}x$. Thus $\operatorname{Lat}_0 \mathcal{A} \subseteq \operatorname{Lat}_{1/2} \mathcal{A}$ and therefore

$$(\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A}) \cap (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A})^* = (\operatorname{Alg} \operatorname{Lat}_0 \mathcal{A}) \cap (\operatorname{Alg} \operatorname{Lat}_0 \mathcal{A})^*.$$

Definition 13. For an algebra $\mathcal{A}$, $\operatorname{Lat}_{1/2} \mathcal{A}$ is said to be commutative if there exists a commuting set of positive operators such that every element of $\operatorname{Lat}_{1/2} \mathcal{A}$ is the range of some operator in this set.

THEOREM 14. *Let $\mathcal{A}$ be pre-parareflexive algebra. The following are equivalent.*

(i) *$\mathcal{A}$ contains a maximal abelian selfadjoint algebra.*

(ii) $\operatorname{Lat}_{1/2} \mathcal{A}$ *is commutative.*

Proof. Let $\operatorname{Lat}_{1/2} \mathcal{A}$ be commutative. Let $S \subseteq B(H)$ be a commutative set of positive operators such that every element of $\operatorname{Lat}_{1/2} \mathcal{A}$ is the range of some operator in S. Then S', the commutant of S, and $\mathcal{A}(S)$, the weakly closed algebra generated by S, leave invariant all the elements of $\operatorname{Lat}_{1/2} \mathcal{A}$ and thus are contained in $\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A}$. Between $\mathcal{A}(S)$ and S', there is a maximal abelian selfadjoint algebra, say M. As M is selfadjoint, $M \subseteq (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A})^*$. This implies that

$$M \subseteq (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A}) \cap (\operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*.$$

If $\mathcal{A}$ contains a maximal abelian selfadjoint algebra, then every element of $\operatorname{Lat}_{1/2} \mathcal{A}$ is the range of some operator in the maximal abelian selfadjoint algebra [**3**]. Thus $\operatorname{Lat}_{1/2} \mathcal{A}$ is commutative.

Corollary 11 together with Theorem 15 imply the following.

COROLLARY 15. *If $\mathcal{A}$ is a weakly closed algebra with commutative* $\operatorname{Lat}_{1/2} \mathcal{A}$, *then $\mathcal{A}$ is pre-parareflexive if and only if $\mathcal{A}$ is prereflexive.*

Definition 16. An operator T on H is called a parareflexive operator if every operator S on H leaving invariant all invariant operator ranges of T, is an entire function of T. Equivalently, if $S \in \operatorname{Alg} \operatorname{Lat}_{1/2} T$, then S is an entire function of T.

THEOREM 17. *Let $\mathcal{A}$ be a commutative parareflexive algebra on a finite-dimensional space. Then each element of $\mathcal{A}$ is parareflexive.*

Proof. Let $T \in \mathcal{A}$. Let $B \in \operatorname{Alg} \operatorname{Lat}_{1/2} T$. Then

$$\operatorname{Lat}_{1/2} \mathcal{A} \subseteq \operatorname{Lat}_{1/2} T \subseteq \operatorname{Lat}_{1/2} B.$$

This implies that $B \in \operatorname{Alg} \operatorname{Lat}_{1/2} \mathcal{A} = \mathcal{A}$. As $\mathcal{A}$ is commutative, $BT = TB$. Also $B \in \operatorname{Alg} \operatorname{Lat} T$ and acts on a finite-dimensional space. By [**2**, Theorem 10], B is a polynomial in T. Thus T is a parareflexive operator.

Acknowledgement: The author is thankful to Dr. S.C. Arora for many helpful suggestions.

References

1. W. Arveson, *Operator algebras and invariant subspaces*, Annals of Math. **100** (1974), 433–532.
2. L. Brickman and P.A. Fillmore, *The invariant subspace lattice of a linear transformation*, Canad. J. Math. **19** (1967), 810–822.
3. C. Foias, *Invariant paraclosed subspaces*, Indiana Univ. Math. J. **21** (1972), 881–907.
4. D.W. Hadwin and S.C. Ong, *On algebraic and parareflexivity*, J. Operator Theory **17** (1987), 23–31.
5. A. Lambert, *Strictly cyclic operator algebras*, Pacific J. Math. **39** (1971), 717–726.
6. M. Omladič, *Parareflexive operators on Banach spaces*, Michigan Math. J. **37** (1990), 133–143.
7. S.C. Ong, *Converse of a theorem of Foias and reflexive lattices of operator ranges*, Indiana Univ. Math. J. **30** (1981), 57–63.
8. S.C. Ong, *A note on parareflexivity of algebras of operators*, Rev. Roumane Math. Pures Appl. **32** (1987), 551–553.

Department of Mathematics
Kirori Mal College
Delhi University
Delhi 110007
India

(Received 24 07 1996)