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ON EQUIVARIANT MAPS BETWEEN STIEFEL MANIFOLDS

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Abstract. We use an index theory to prove the nonexistence of some equivariant maps between Stiefel manifolds. We also show how to construct equivariant maps between some Stiefel manifolds.

Introduction

We denote by $V_k(\mathbf{R}^n)$ the Stiefel manifold of all orthonormal k-frames in $\mathbf{R}^n$. There is an obvious action of the group $\mathbf{Z}/2$, on this manifold, where the generator of $\mathbf{Z}/2$ acts by sending a frame $[v_1, \dots, v_k]$ to $[-v_1, \dots, -v_k]$. In this paper we investigate the following question:

For a given n, k, m and l, is there a $\mathbf{Z}/2$-equivariant map between $V_k(\mathbf{R}^n)$ and $V_l(\mathbf{R}^m)$?

In general, this problem seems to be hard. For example, let us recall that $V_1(\mathbf{R}^n)$ is S^{n-1} and the question whether there exists an equivariant map from S^{n-1} to $V_n(\mathbf{R}^{n+k})$ or not, is equivalent to the question whether $\mathbf{R}P^{n-1}$ immerses in $\mathbf{R}^{n+k-1}$ or not. To find the smallest k such that $\mathbf{R}P^{n-1}$ immerses in $\mathbf{R}^{n+k-1}$ is a longstanding problem in topology and although much work has been done, the complete answer is still not known.

If $k = 1$ and $l = 1$ our problem reduces to the question of equivariant maps between spheres and in that case the answer is given by Borsuk-Ulam theorem.

In this paper we use the index theory of Fadell and Husseini (see [FH]) to answer our question in some cases.

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Index theory

Let us recall the basic definitions and methods of the index theory of Fadell and Husseini. We have a (compact Lie) group G acting on a (paracompact Hausdorff) space X. We may then compute $H_G^*(X)$ (Borel cohomology)

$$H_G^*(X) = H^*(EG \times_G X)$$

where $EG \longrightarrow BG$ represents the universal G-bundle. If $f : X \to Y$, is a G-equivariant map, then it induces a map in the equivariant cohomology in the opposite direction: $f^* : H_G^*(Y) \to H_G^*(X)$. Now, $H_G^*(pt) = H^*(BG)$ (by pt we denote a space consisting of one point only) and here is the main idea of that index theory. We define $\mathrm{Index}^G(X)$ as follows

$$\mathrm{Index}^G(X) = \mathrm{Ker}(c^* : H_G^*(pt) \longrightarrow H_G^*(X))$$

where by c we denote the unique map $X \longrightarrow pt$. So this index is an ideal in the ring $H^*(BG)$. If $f : X \to Y$ is a G-equivariant map between G-spaces we have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ pt & \xrightarrow{=} & pt \end{array}$$

and if we apply the functor $H_G^*(-)$ to this diagram we get

$$\begin{array}{ccc} H_G^*(X) & \xleftarrow{f^*} & H_G^*(Y) \\ {\scriptstyle c^*}\uparrow & & \uparrow{\scriptstyle c^*} \\ pt & \xleftarrow{=} & pt \end{array}$$

From this diagram one sees easily that if there exists an equivariant map from X to Y then

$$\mathrm{Index}^G(Y) \subset \mathrm{Index}^G(X).$$

In the case we are interested in, $G = \mathbf{Z}/2$ and $H^*(BG) = H^*(B\mathbf{Z}/2) = \mathbf{Z}/2[x]$ (we always use mod 2 coefficients). We proceed to show what is $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n))$.

Index of Stiefel manifolds

In [GH] (see also [G1], [G2]) one finds a definition of the projective Stiefel manifold, namely it is the orbit space of $V_k(\mathbf{R}^n)$ under the action of $\mathbf{Z}/2$ that we mentioned before (in the case $k = 1$ one of course gets real projective spaces). In the same paper, Gitler and Handel computed the cohomology of these spaces and here is the answer.

THEOREM 1. *Suppose $k < n$. Let*

$$N = \min\left\{j | n-k+1 \leq j \leq n, \binom{n}{j} \equiv 1 \pmod 2\right\}.$$

Then

$$H^*(Y_{n,k}) = \mathbf{Z}/2[y]/(y^N) \otimes V(y_{n-k}, \dots, y_{N-2}, y_N, \dots, y_{n-1})$$

as an algebra. Here, the degree of y is 1 and $p^*x = y$ where x is a generator of $H^1(\mathbf{R}P^\infty)$.

Some explanation seems to be in order here. Namely, we get the map p as follows. There is a well known fibration

$$V_k(\mathbf{R}^n) \longrightarrow BO(n-k) \longrightarrow BO(n)$$

(recall that $V_k(\mathbf{R}^n) = O(n)/O(n-k)$). If ξ is the Hopf bundle over $\mathbf{R}P^\infty$ and $f : \mathbf{R}P^\infty \to BO(n)$ classifies $n\xi$, this map induces the following fibration

$$V_k(\mathbf{R}^n) \xrightarrow{i} Y_{n,k} \xrightarrow{p} \mathbf{R}P^\infty.$$

Now, it is shown in [GH] that the space $Y_{n,k}$ may be identified with the corresponding projective Stiefel manifold $V_k(\mathbf{R}^n)/(\mathbf{Z}/2)$. In order to explain the remaining part of the cohomology of $Y_{n,k}$ we recall the following

THEOREM 2 (see [Bo]) *We have $H^*(V_k(\mathbf{R}^n)) = V(z_{n-k}, \dots, z_{n-1})$.*

Here the degree of z_q is q and $V(x_1, \dots x_m)$ denotes any commutative, associative algebra over $\mathbf{Z}/2$ with unit, generated by elements $x_1, \dots, x_m$, such that $x_1^{\epsilon_1} \cdots x_m^{\epsilon_m}$ where $\epsilon_i \in \{0,1\}$, form an additive basis for that algebra (the so-called simple system of generators). It is also shown in [GH] that $i^*(y_q) = z_q$. By now, it should be clear what is $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n))$. We have a fibration

$$V_k(\mathbf{R}^n) \longrightarrow E\mathbf{Z}/2 \times_{\mathbf{Z}/2} V_k(\mathbf{R}^n) \xrightarrow{p_1} B\mathbf{Z}/2.$$

But, since the action of $\mathbf{Z}/2$ on $V_k(\mathbf{R}^n)$ is free, $E\mathbf{Z}/2 \times_{\mathbf{Z}/2} V_k(\mathbf{R}^n)$ is homotopy equivalent to $V_k(\mathbf{R}^n)/(\mathbf{Z}/2)$. There are only two possibilities for $p_1^*(x)$ (where $H^*(B\mathbf{Z}/2) = \mathbf{Z}/2[x]$)—it is either y or 0. But, if it were zero, elementary examination of the spectral sequence associated to this fibration would show that one cannot get the correct cohomology for our total space (which is, as we have seen, a projective Stiefel manifold). So, $p_1^*(x) = y$ and we get

THEOREM 3. *One has $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = (x^N)$, where (x^N) is the ideal in $\mathbf{Z}/2[x]$ generated by (x^N), where N is as before.*

Since, in what follows, we will be using only indices of Stiefel manifolds, and they are characterized (as we have seen) by a number, we will, by abuse of language, say that the index of $V_k(\mathbf{R}^n)$ is N.

When one deals with binomial coefficients modulo 2, the following lemma is almost indispensable.

LEMMA 1. *If* $a = \sum_{i=0}^{m} a_i 2^i$ *and* $b = \sum_{i=0}^{m} b_i 2^i$, *are the dyadic expansions of* a *and* b *respectively, then*

$$\binom{b}{a} \equiv \prod_{i=0}^{m} \binom{b_i}{a_i} \pmod 2$$

As to the proof, the reader may supply it himself, or look it up in [SE].

Now we give some conditions that ensure that the index is very small and some others where it is very big.

PROPOSITION 1. *We have*

a) $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-1}(\mathbf{R}^n)) = \mathrm{Index}^{\mathbf{Z}/2}(SO(n)) = 2$ *if* $n \equiv 2,3 \pmod 4$;

b) $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-2}(\mathbf{R}^n)) = 3$, *for* $n \equiv 3 \pmod 4$ *and*
$\mathrm{Index}^{\mathbf{Z}/2}(SO(n)) \neq 3$, *for all* n;

c) $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-3}(\mathbf{R}^n)) = 4$, *for* $n \equiv 4,5,6,7 \pmod 8$;
$\mathrm{Index}^{\mathbf{Z}/2}(V_{n-2}(\mathbf{R}^n)) = 4$, *for* $n \equiv 4,5,7 \pmod 8$;
$\mathrm{Index}^{\mathbf{Z}/2}(SO(n)) = 4$, *for* $n \equiv 4,5 \pmod 8$;

d) $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-4}(\mathbf{R}^n)) = 5$, *for* $n \equiv 5,7 \pmod 8$, *while* $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-3}(\mathbf{R}^n))$, $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-2}(\mathbf{R}^n))$ *and* $\mathrm{Index}^{\mathbf{Z}/2}(SO(n))$ *are never equal to* 5.

Proof. Let us just recall that $V_{n-1}(\mathbf{R}^n) = SO(n)$ (we just add an additional vector to get an element of $SO(n)$). Since all the proofs are quite similar (and simple) we prove only part c). So,

$$\mathrm{Index}^{\mathbf{Z}/2}(V_{n-2}(\mathbf{R}^n)) = \min\left\{ j | 4 = n-(n-3)+1 \le j \le n, \binom{n}{j} = \mathrm{odd} \right\}$$

and one only needs to check whether $\binom{n}{4}$ is odd. Now, this can be checked separately for all the possibilities for n mentioned in the statement, but it follows easily from the previous lemma—what this lemma actually states is that the binomial coefficient is odd *iff* whenever 1 appears in the dyadic expansion for a, then on the same place 1 should also appear in the dyadic expansion for b, and this is certainly true in our case:

$$n = \cdots 1 \cdot\cdot$$
$$4 = 000100$$

Remark. One sees that in many cases mentioned above, the index is as low as it can be.

Let us now turn to the question when is the index as big as it can be, namely when is $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n$. The answer is given by the following

THEOREM 4. *We have* $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n$ *iff* $\nu_2(n) \geq -[-\log_2(k)]$, *where* $\nu_2(n)$ *is the exponent of the highest power of* 2 *dividing* n *(*$n = 2^{\nu_2(n)}\cdot$*odd).*

Proof. Let us first unravel the curious condition mentioned in the statement of the theorem. It means that $2^s \mid n$ where 2^s is the smallest power of 2 not less than k.

$\Rightarrow$: Assume that $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n$ and that $2^{s-1} < k \leq 2^s$. We have to prove that $2^s \mid n$. If 2^s does not divide n, then n is of the form

$$n = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_l}$$

where $p_1 > p_2 > \cdots > p_l$ and $p_l \leq s-1$. But this gives

$$2^{p_l} \leq 2^{s-1} \leq k-1$$

or

$$n > n - 2^{p_l} \geq n - k + 1$$

and

$$\binom{n}{2^{p_l}} \equiv 1 \pmod 2,$$

and that contradicts the fact that $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n$.

$\Leftarrow$: Assume that $2^s \mid n$ where 2^s is the smallest power of 2 not less that k. So

$$n = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_l}, \quad \text{where } p_l \geq 2.$$

Now, $2^{p_l} \geq 2^s \geq k > k-1$ and it clearly follows from the lemma that $\binom{n}{j'} \equiv 0 \pmod 2$, for $1 \leq j' \leq k-1$ and, consequently, we have: $\binom{n}{j} \equiv 0 \pmod 2$ for $n-k+1 \leq j \leq n-1$ ($j = n-j'$, $\binom{n}{j} = \binom{n}{j'}$). So, $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n$.

COROLLARY 1. *We have* $\mathrm{Index}^{\mathbf{Z}/2}(SO(n)) = n$ *iff* n *is a power of* 2.

Proof. $\mathrm{Index}^{\mathbf{Z}/2}(V_{n-1}(\mathbf{R}^n)) = \mathrm{Index}^{\mathbf{Z}/2}(SO(n)) = n$ iff $2^s \mid n$ where 2^s is the smallest power of 2 not less than $n-1$ (from the previous theorem). So, $2^s \geq n-1$, $2^s \mid n$, therefore 2^s must be n.

We see from Theorem 3 that

$$n = \mathrm{Index}^{\mathbf{Z}/2}(V_1(\mathbf{R}^n)) \geq \mathrm{Index}^{\mathbf{Z}/2}(V_2(\mathbf{R}^n)) \geq \cdots \geq \mathrm{Index}^{\mathbf{Z}/2}(V_{n-1}(\mathbf{R}^n))$$

and

$$\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) \geq n - k + 1.$$

The following question then arises: When is $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n-k+1$ for all $1 \leq k \leq n-1$? The answer is quite simple.

Proposition 2. *We have* $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) = n-k+1$ *for all* $1 \leq k \leq n-1$ *iff* $n = 2^m - 1$ *for some* m.

The proof is along similar lines as the previous results and so will be omitted. We now proceed to the original question of (non)-existence of equivariant maps between Stiefel manifolds.

Equivariant maps between Stiefel manifolds

Let us recall that, should there exist a $\mathbf{Z}/2$-equivariant map

$$f : V_k(\mathbf{R}^n) \longrightarrow V_l(\mathbf{R}^m)$$

then $\mathrm{Index}^{\mathbf{Z}/2}(V_k(\mathbf{R}^n)) \leq \mathrm{Index}^{\mathbf{Z}/2}(V_l(\mathbf{R}^m))$ (we look at the index as an integer as we mentioned above). From this we get the following

Proposition 3. *If* n *is a power of* 2*, then there is no* $\mathbf{Z}/2$*-equivariant map*

$$f : SO(n) \longrightarrow V_k(\mathbf{R}^n),$$

where $k < m < n$.

This follows immediately from Corollary 1. In addition to this proposition, we can, using the results from the previous section, get many other results concerning equivariant maps between Stiefel manifolds. Here are some

Proposition 4. *There is no* $\mathbf{Z}/2$*-equivariant map*

$$\begin{aligned} f &: V_{n-2}(\mathbf{R}^n) \longrightarrow V_{n-1}(\mathbf{R}^n), \quad \textit{for} \;\; n \equiv 3 \pmod 4. \\ f &: V_{n-3}(\mathbf{R}^n) \longrightarrow V_{n-2}(\mathbf{R}^n), \quad \textit{for} \;\; n \equiv 7 \pmod 8. \\ f &: V_{n-4}(\mathbf{R}^n) \longrightarrow V_{n-3}(\mathbf{R}^n), \quad \textit{for} \;\; n \equiv 7 \pmod 8. \end{aligned}$$

All of this follows from Proposition 1. And some examples for particular manifolds.

Example 1. There are no $\mathbf{Z}/2$-equivariant maps:

$$\begin{aligned} V_3(\mathbf{R}^6) \longrightarrow V_5(\mathbf{R}^6), &\qquad V_2(\mathbf{R}^8) \longrightarrow V_9(\mathbf{R}^{10}), \\ V_2(\mathbf{R}^8) \longrightarrow V_9(\mathbf{R}^{11}), &\qquad V_3(\mathbf{R}^{13}) \longrightarrow V_7(\mathbf{R}^{15}) \end{aligned}$$

etc.

As far as the construction of equivariant maps goes, we mention, in addition to the trivial observation that there is an equivariant map $V_k(\mathbf{R}^n) \longrightarrow V_{k-1}(\mathbf{R}^n)$ (simply forget, say, the last vector in a frame), the following

Lemma 2. *If there exists a* $\mathbf{Z}/2$*-equivariant map* $f : S^{n-1} \to V_l(\mathbf{R}^m)$, *then there exists a* $\mathbf{Z}/2$*-equivariant map* $g : V_k(\mathbf{R}^n) \longrightarrow V_l(\mathbf{R}^{mk})$.

Proof. We have

$$g([v_1, \dots, v_k]) = \{f(v_1), \dots, f(v_k)\}^t.$$

So, if we have enough supply of equivariant maps from spheres to Stiefel manifolds, we can construct many equivariant maps from Stiefel manifolds to Stiefel manifolds. But, we do have that—we mentioned that the existence of such maps is, for some particular values of l, m (l, m from the previous lemma), equivalent to the existence of an immersion of a real projective space into an Euclidean one (see [BR]). For the results concerning immersions, see [D1], [D2], [D3]. We mention here only the following (see [Mi], [La]).

Theorem 5. *Let* $d = 1, 2, 4$ *or* 8. *If* $n \neq 1, 3, 7$ *and* $n + 1 \equiv 0 \pmod{d}$, *then* $\mathbf{R}P^n$ *immerses in* $\mathbf{R}^{2n-\alpha(n)-\beta(d)}$, *where* $\alpha(n)$ *is the number of* 1*'s in the dyadic expansion of* n, *and* $\beta(d) = d - 1 - \alpha(d-1)$.

So, as mentioned in the Introduction, there exist an equivariant map

$$S^n \longrightarrow V_{n+1}(\mathbf{R}^{2n-\alpha(n)-\beta(d)+1})$$

and, consequently (using the previous lemma), equivariant maps

$$V_k(\mathbf{R}^n) \longrightarrow V_{n+1}(\mathbf{R}^{(2n-\alpha(n)-\beta(d)+1)m})$$

for $k < n$. For example, there is an equivariant map

$$S^{31} \longrightarrow V_{32}(\mathbf{R}^{53})$$

and so an equivariant map

$$V_2(\mathbf{R}^{32}) \longrightarrow V_{32}(\mathbf{R}^{106}).$$

Let us just say a few words for the conclusion. As we mentioned in the Introduction, the immersion problem is rather hard and therefore so is the more general problem we discussed in this paper. But it may be that the results related to this more general problem would give us more insight to the immersion problem. That was this author prime motivation to deal with this problem.

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