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POINT ESTIMATION AND THE CONVERGENCE OF THE EHRLICH-ABERTH METHOD

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Dedicated to Professor P. Madić on his 75th Birthday

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Abstract. The choice of starting points that provide a safe convergence of a given iterative method has an important role in solving nonlinear equations. In this paper we consider the Ehrlich-Aberth method for the simultaneous approximation of all simple zeros of a polynomial. For this method we state practically applicable initial conditions depending only on initial approximations, which enable the safe convergence.

1. Ehrlich-Aberth method

The choice of starting points that provide a safe convergence of a given iterative method has a very important role in solving nonlinear equations. Smale's point estimation theory, first introduced in [**14**] for Newton's method, treats convergence conditions and the domain of convergence in solving an equation $f(z) = 0$ using only the information of f at the initial point $\mathbf{z}^{(0)}$. X. Wang and Han [**15**] improved Smale's result. Their work was later extended by Curry [**4**] and Kim [**6**] to some higher-order iterative methods and generalized by Chen [**3**].

Let P be a monic polynomial with simple zeros $\zeta_1, \dots, \zeta_n$, and let $z_1^{(0)}, \dots, z_n^{(0)}$ be starting approximations to these zeros. In this paper we will use the Newton and Weierstrass correction given respectively by

$$N_i^{(m)} = \frac{P(z_i^{(m)})}{P'(z_i^{(m)})} \quad \text{and} \quad W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod\limits_{j\neq i}(z_i^{(m)} - z_j^{(m)})} \quad (i \in I_n;\ m = 0, 1, \dots),$$

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where $I_n := \{1, \dots, n\}$ is the index set. According to Smale [**14**], initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ which provide the safe convergence of a simultaneous method for finding polynomial zeros will be called *approximate zeros.*

In this paper we are concerned with one of the most efficient numerical methods for the simultaneous approximation of all zeros of a polynomial, given by the iterative formula

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\dfrac{1}{N_i^{(m)}} - \displaystyle\sum_{j \neq i} \frac{1}{z_i^{(m)} - z_j^{(m)}}} \quad (i \in I_n;\ m = 0, 1, \dots). \tag{1}$$

This formula was considered for the first time by Maehly [**7**] and Börsch-Supan [**2**], but a practical application and an analysis were presented by Ehrlich [**5**] and Aberth [**1**] so that this method is most frequently called the Ehrlich-Aberth method. The computational efficiency analysis of the iterative method (1) may be found in the book [**8**, Ch. 6].

Most of initial conditions considered in literature for simultaneous iterative methods are not of practical importance since they depend on unattainable data (for instance, on desired zeros). One of the approaches to the construction of initial approximations, providing a safe convergence of iterative methods for the simultaneous approximations of polynomial zeros, is of the form

$$w^{(0)} < c(n) \cdot d^{(0)}, \tag{2}$$

where

$$w^{(0)} = \max_{1 \le i \le n} |W_i^{(0)}|, \quad d^{(0)} = \min_{j \neq i} |z_i^{(0)} - z_j^{(0)}|,$$

and $c(n)$ is the so called inequality factor, for short i-factor, depending on the polynomial degree n. For more details see the recent papers [**9**]–[**13**], [**16**], [**17**]. During the last years a special attention has been directed to the increase of the i-factor $c(n)$ which multiplies the minimal distance d.

The aim of this paper is to state practically applicable initial conditions of the form (2) which enable a safe convergence of the Ehrlich-Aberth method, called for short the E-A method in the sequel. We establish initial conditions depending only on the vector $\mathbf{z}^{(0)} = (z_1^{(0)}, \dots, z_n^{(0)})$ of starting approximations and the values of P in the components of $\mathbf{z}^{(0)}$. For simplicity, in our analysis we will sometimes omit the iteration index m and new entries in the later $(m+1)$-st iteration will be additionally stressed by the symbol $\widehat{}$ (*hat*). For example, instead of $z_i^{(m)}, z_i^{(m+1)}, W_i^{(m)}, W_i^{(m+1)}, d^{(m)}, d^{(m+1)}, N_i^{(m)}, N_i^{(m+1)}$, etc. we will write $z_i, \hat{z}_i, W_i, \widehat{W}_i, d, \hat{d}, N_i, \hat{N}_i$. According to this we denote

$$w = \max_i |W_i|, \quad \hat{w} = \max_i |\widehat{W}_i|.$$

Throughout the paper a closed disk with center c and radius r will be denoted by the parametric notation $\{c; r\}$. If two disks $\{c_1; r_1\}$ and $\{c_2; r_2\}$ are disjoint, then

$$|c_1 - c_2| > r_1 + r_2. \tag{3}$$

2. Some necessary lemmas

Before establishing the point estimation theorem, we present a necessary localization theorem given in [**12**].

LEMMA 1. *Assume that the following condition*

$$w < \frac{d}{2n+3} \tag{4}$$

is satisfied. Then each disk $\{z_i - W_i; |W_i|\}$ $(i \in I_n)$ *contains one and only one zero of* P.

Remark 1. Since

$$\begin{aligned} |(z_i - W_i) - (z_j - W_j)| &\geq |z_i - z_j| - |W_i| - |W_j| > d - 2 \cdot \frac{d}{2n+3} \\ &> (2n+1)w \geq |W_i| + |W_j|, \end{aligned}$$

according to (3) it follows that the disks $\{z_1 - W_1; |W_1|\}, \ldots, \{z_n - W_n; |W_n|\}$ are mutually disjoint.

Remark 2. Since $\{z_i - W_i; |W_i|\} \subset \{z_i; 2|W_i|\}$, we may always take the disk $\{z_i; 2|W_i|\}$ as an inclusion disk instead of $\{z_i - W_i; |W_i|\}$. This substitution is convenient in the convergence analysis of the E-A method (1) (Section 3).

LEMMA 2. *Let* $z_1, \ldots, z_n$ *be disjoint approximations to the zeros* $\zeta_1, \ldots, \zeta_n$ *of a polynomial* P *of degree* n*, and let* $\hat{z}_1, \ldots, \hat{z}_n$ *be new respective approximations obtained by the E-A method* (1)*. Then the following formula is valid:*

$$\widehat{W}_i = -(\hat{z}_i - z_i)^2 \sum_{j \neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} \prod_{j \neq i} \Big(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\Big). \tag{5}$$

Proof. Using the Lagrange interpolation of P at $z_1, \ldots, z_n$ we represent P in terms of W_j's in the form

$$P(t) = \left(\sum_{j=1}^{n} \frac{W_j}{t - z_j} + 1 \right) \prod_{j=1}^{n} (t - z_j). \tag{6}$$

Hence, by applying the logarithmic derivative to (6) and putting $t = z_i$ in the obtained formula we find

$$\frac{P'(z_i)}{P(z_i)} = \sum_{j\neq i} \frac{1}{z_i - z_j} + \frac{1}{W_i}\Big(\sum_{j\neq i} \frac{W_j}{z_i - z_j} + 1\Big). \tag{7}$$

From the E-A method (1) one obtains

$$\frac{1}{\hat{z}_i - z_i} = \sum_{j\neq i} \frac{1}{z_i - z_j} - \frac{P'(z_i)}{P(z_i)},$$

so that, using (7),

$$\begin{aligned}\frac{W_i}{\hat{z}_i - z_i} &= W_i\left(\sum_{j\neq i} \frac{1}{z_i - z_j} - \frac{P'(z_i)}{P(z_i)}\right) = -W_i\left[\frac{1}{W_i}\Big(\sum_{j\neq i} \frac{W_j}{z_i - z_j} + 1\Big)\right]\\ &= -\sum_{j\neq i} \frac{W_j}{z_i - z_j} - 1.\end{aligned}$$

According to this we have

$$\begin{aligned}\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1 &= \frac{W_i}{\hat{z}_i - z_i} + \sum_{j\neq i} \frac{W_j}{\hat{z}_i - z_j} + 1\\ &= -\sum_{j\neq i} \frac{W_j}{z_i - z_j} - 1 + \sum_{j\neq i} \frac{W_j}{\hat{z}_i - z_j} + 1\\ &= -(\hat{z}_i - z_i)\sum_{j\neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}.\end{aligned}$$

Taking into account the last expression, returning to (6) we find for $t = \hat{z}_i$

$$\begin{aligned}P(\hat{z}_i) &= \Big(\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big)\prod_{j=1}^{n}(\hat{z}_i - z_j)\\ &= -(\hat{z}_i - z_i)^2 \sum_{j\neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} \prod_{j\neq i}(\hat{z}_i - z_j).\end{aligned}$$

After dividing by $\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)$ and rearranging, we obtain formula (5). $\square$

3. Convergence theorem

Now we give the convergence theorem for the E-A method (1) which involves only initial approximations to the zeros and the polynomial degree n.

Theorem 1. *Under the initial condition*

$$w^{(0)} < \frac{d^{(0)}}{2n+3} \tag{8}$$

the E-A method (1) *is convergent with the third order of convergence.*

Proof. The proof will be carried out by complete induction. The convergence analysis is based on the estimate procedure of the error $u_i^{(m)} = z_i^{(m)} - \zeta_i$.

We first consider the typical step for $m = 0$ (omitting this index as mentioned above), which is the part of the proof with respect to $m = 1$. According to the initial condition (8) and Remark 2 we have

$$|u_i| = |z_i - \zeta_i| \le 2|W_i| \le 2w < 2 \cdot \frac{d}{2n+3}. \tag{9}$$

In view of this and the definition of the minimal distance d we find

$$|z_j - \zeta_i| \ge |z_j - z_i| - |z_i - \zeta_i| > d - \frac{2d}{2n+3} = \frac{2n+1}{2n+3} d. \tag{10}$$

Using the identity

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - \zeta_j},$$

from (1) we get

$$\begin{aligned} \hat{u}_i = \hat{z}_i - \zeta_i &= z_i - \zeta_i - \left(\frac{1}{u_i} + \sum_{j \ne i} \frac{1}{z_i - \zeta_j} - \sum_{j \ne i} \frac{1}{z_i - z_j} \right)^{-1} \\ &= u_i - \frac{u_i}{1 - u_i S_i} = -\frac{u_i^2 S_i}{1 - u_i S_i}, \end{aligned} \tag{11}$$

where $S_i = \sum_{j \ne i} \dfrac{u_j}{(z_i - \zeta_j)(z_i - z_j)}$.

Using the definition for d and the bounds (9) and (10), we estimate

$$\begin{aligned} |u_i S_i| &\le |u_i| \sum_{j \ne i} \frac{|u_j|}{|z_i - \zeta_j||z_i - z_j|} \\ &< \frac{2d}{2n+3} \cdot \frac{(n-1)\dfrac{2d}{2n+3}}{\dfrac{2n+1}{2n+3} d \cdot d} = \frac{4(n-1)}{(2n+1)(2n+3)} \le \frac{8}{63}. \end{aligned} \tag{12}$$

Now, by (9) and (12), we find from (1)

$$\begin{aligned} |\hat{z}_i - z_i| &= \Big|\frac{u_i}{1-u_iS_i}\Big| \le \frac{|u_i|}{1-|u_iS_i|} < \frac{|u_i|}{1-8/63} \\ &< \frac{63}{55}\cdot\frac{2d}{2n+3} < \frac{2.3d}{2n+3}, \end{aligned} \tag{13}$$

and also

$$|\hat{z}_i - z_i| < \frac{63}{55}|u_i| < \frac{63}{55}\cdot 2|W_i| < 2.3|W_i|. \tag{14}$$

According to this and having in mind that $|z_i - z_j| \ge d$ we have

$$|\hat{z}_i - z_j| \ge |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{2.3d}{2n+3} = \frac{2n+0.7}{2n+3}d, \tag{15}$$

and

$$\begin{aligned} |\hat{z}_i - \hat{z}_j| &\ge |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| \\ &> d - 2\cdot\frac{2.3d}{2n+3} = \frac{2n-1.6}{2n+3}d. \end{aligned} \tag{16}$$

Therefore

$$\hat{d} > \frac{2n-1.6}{2n+3}d, \quad \text{wherefrom} \quad d < \frac{2n+3}{2n-1.6}\hat{d}. \tag{17}$$

From the last inequality we estimate

$$\frac{d}{\hat{d}} < \frac{2n+3}{2n-1.6} < 2.1 \quad \text{for every } n \ge 3. \tag{18}$$

Using the starting inequality $w/d < 1/(2n+3)$ and the bounds (13), (14), (15) and (16), we estimate the quantities involved in (5):

$$\begin{aligned} |\widehat{W}_i| &\le |\hat{z}_i - z_i|^2 \sum_{j\ne i} \frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|} \prod_{j\ne i}\Big(1 + \frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\Big) \\ &< (2.3|W_i|)^2 \frac{(n-1)w}{\frac{2n+0.7}{2n+3}d\cdot d}\left(1 + \frac{\frac{2.3d}{2n+3}}{\frac{(2n-1.6)d}{2n+3}}\right)^{n-1} \\ &< 5.29|W_i|\frac{(n-1)(2n+3)}{2n+0.7}\Big(\frac{w}{d}\Big)^2\Big(1+\frac{2.3}{2n-1.6}\Big)^{n-1} \\ &< \frac{5.29(n-1)}{(2n+0.7)(2n+3)}\Big(1+\frac{2.3}{2n-1.6}\Big)^{n-1}|W_i| = f(n)|W_i|, \end{aligned}$$

where $f(n)$ is the term in front of $|W_i|$ depending on n.

The function

$$f(x) = \frac{5.29(x-1)}{(2x+0.7)(2x+3)}\Big(1+\frac{2.3}{2x-1.6}\Big)^{x-1}$$

has its maximum for $x_0 \approx 3.71$ equals $f(x_0) \approx 0.418$. We can take $f(n) < 0.45$ for $n \geq 3$ which yields

$$|\widehat{W}_i| < 0.45|W_i| < 0.45w.$$

Therefore we have proved

$$\hat{w} < 0.45w \tag{19}$$

so that, by (8), (17) and (19), we estimate

$$\hat{w} < 0.45w < \frac{0.45d}{2n+3} < \frac{0.45}{2n+3}\cdot\frac{2n+3}{2n-1.6}\hat{d},$$

wherefrom

$$\hat{w} < \frac{\hat{d}}{2n+3}. \tag{20}$$

Using the already derived bounds we find

$$\begin{aligned}|\hat{u}_i| &\leq \frac{|u_i|^2|S_i|}{1-|u_iS_i|} < \frac{63}{55}|u_i|^2\sum_{j\neq i}\frac{|u_j|}{|z_i-\zeta_j||z_i-z_j|}\\ &< \frac{63}{55}|u_i|^2\sum_{j\neq i}\frac{|u_j|}{\frac{2n+1}{2n+3}d\cdot d} < \frac{3}{2d^2}|u_i|^2\sum_{j\neq i}|u_i|,\end{aligned}$$

wherefrom

$$|\hat{u}_i| < \gamma(n,d)|u_i|^2\sum_{j\neq i}|u_j|, \quad \gamma(n,d) = \frac{3}{2d^2}. \tag{21}$$

We have proved (20) under the assumption (8). Thus, the initial condition (8) implies the validity of the inequality $w^{(1)} < d^{(1)}/(2n+3)$. This inequality is of the same form as (8) (for $m=1$) so that, applying the same argumentation as for $m=0$, we derive all previous bounds for the next index, and so on. Therefore, we have the implication

$$w^{(m)} < \frac{d^{(m)}}{2n+3} \ \Rightarrow\ w^{(m+1)} < \frac{d^{(m+1)}}{2n+3}$$

and we conclude by induction that the initial condition (8) implies the inequality $w^{(m)} < d^{(m)}/(2n+3)$ for each $m = 1,2,\ldots$. Consequently, all previous relations and estimates are valid, too, for each $m = 1,2,\ldots$. Especially, regarding (18) and (21), we have

$$\frac{d^{(m)}}{d^{(m+1)}} < 2.1 \tag{22}$$

and
$$|u_i^{(m+1)}| < \gamma(n,d^{(m)})|u_i^{(m)}|^2 \sum_{j\neq i} |u_j^{(m)}|^2, \quad (i=1,\dots,n) \tag{23}$$
for each iteration index $m=0,1,\dots$ if (8) holds.

Substituting $t_i^{(m)} = \big[2.1(n-1)\gamma(n,d^{(m)})\big]^{1/2}|u_i^{(m)}|$, the inequalities (23) become

$$\begin{aligned} t_i^{(m+1)} &< \frac{[t_i^{(m)}]^2}{2.1(n-1)}\Big(\frac{\gamma(n,d^{(m+1)})}{\gamma(n,d^{(m)})}\Big)^{1/2} \sum_{j\neq i} t_j^{(m)} \\ &< \frac{[t_i^{(m)}]^2}{2.1(n-1)} \cdot \frac{d^{(m)}}{d^{(m+1)}} \sum_{j\neq i} t_j^{(m)}, \end{aligned}$$

wherefrom, by (22),
$$t_i^{(m+1)} < \frac{[t_i^{(m)}]^2}{n-1} \sum_{j\neq i} t_j^{(m)} \quad (i=1,\dots,n). \tag{24}$$

By virtue of (9) we find

$$\begin{aligned} t_i^{(0)} &= \Big[2.1(n-1)\gamma(n,d^{(0)})\Big]^{1/2}|u_i^{(0)}| < \frac{2d^{(0)}}{2n+3}\Big[2.1(n-1)\frac{3}{2[d^{(0)}]^2}\Big]^{1/2} \\ &\le \frac{2}{2n+3}\sqrt{3.15(n-1)} \le 0.558... < 1 \end{aligned}$$

for each $i=1,\dots,n$. Taking $t=\max_i t_i^{(0)}$ we observe that $t_i^{(0)} \le t < 1$ is valid for all $i=1,\dots,n$. According to this we conclude from (24) that the sequences $\{t_i^{(m)}\}$ (and, consequently, $\{|u_i^{(m)}|\}$) tend to 0 for all $i=1,\dots,n$. Therefore, the E-A method (1) is convergent.

Taking into account that the quantity $d^{(m)}$ which appears in (23) is bounded and tends to $\min_{i\neq j} |\zeta_i-\zeta_j|$, and setting $u^{(m)}=\max_i |u_i^{(m)}|$ from (23) we obtain
$$|u_i^{(m+1)}| \le u^{(m+1)} < (n-1)\gamma(n,d^{(m)})|u_i^{(m)}|^3,$$
which proves the second part of the theorem. $\square$

References

[1] O. Aberth, *Iteration methods for finding all zeros of a polynomial simultaneously.* Math. Comput. **27** (1973), 339–344.

[2] W. Börsch-Supan, *A posteriori error bounds for the zeros of polynomials.* Numer. Math. **5** (1963), 380–398.

[3] P. Chen, *Approximate zeros of quadratically convergent algorithms.* Math. Comp. **63** (1994), 247–270.

[4] J.H. Curry, *On zero finding methods of higher order from data at one point.* J. Complexity **5** (1989), 219–237.

[5] L.W. Ehrlich, *A modified Newton method for polynomials.* Comm. ACM **10** (1967), 107–108.

[6] M. Kim, *On approximate zeros and rootfinding algorithms for a complex polynomial.* Math. Comput. **51** (1988), 707–719.

[7] H.J. Maehly, *Zur iterativen Auflösung algebraischer Gleichungen.* Z. Angew. Math. Phys. **5** (1954), 260–263.

[8] M.S. Petković, *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros.* Springer-Verlag, Berlin-Heidelberg-New York 1989.

[9] M.S. Petković, *On initial conditions for the convergence of simultaneous root finding methods.* Computing **57** (1996), 163–178.

[10] M.S. Petković, C. Carstensen, M. Trajković, *Weierstrass' formula and zero-finding methods.* Numer. Math. **69** (1995), 353–372.

[11] M.S. Petković, Đ. Herceg, *Point estimation and safe convergence of root-finding simultaneous methods.* Scientific Review **21/22** (1996), 117–130.

[12] M.S. Petković, Đ. Herceg, S. Ilić, *Point estimation and some applications to iterative methods.* BIT (to appear).

[13] M.S. Petković, Đ. Herceg, S. Ilić, *Safe convergence of simultaneous methods for polynomial zeros* (submitted).

[14] S. Smale, *Newton's method estimates from data at one point.* in: *The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics,* Springer-Verlag, New York 1986, pp. 185–196.

[15] X. Wang, D. Han, *On dominating sequence method in the point estimate and Smale's theorem.* Sci. Sinica Ser. A (1989), 905–913.

[16] D. Wang and F. Zhao, *The theory of Smale's point estimation and its application.* J. Comput. Appl. Math. **60** (1995), 253–269.

[17] F. Zhao, D. Wang, *The theory of Smale's point estimation and the convergence of Durand-Kerner program* (in Chinese). Math. Numer. Sinica **2** (1993), 198–206.

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