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CONSTRUCTING KRIPKE MODELS OF CERTAIN FRAGMENTS OF HEYTING'S ARITHMETIC

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Abstract. We present nontrivial methods of constructing Kripke models for the fragments of HA obtained by restricting the induction schema to instances with Π_1- and Π_2-induction formulae respectively. The model construction for Π_1-induction was applied in [W96a] and [W97] to investigate the provably recursive functions of this theory. The construction of Π_2-induction models is a modification of Smoryński's collection operation introduced in [S73].

1. Introduction

A Kripke structure K for a first-order language L is a pair

$$\mathsf{K} = ((K, \leq), (A_\alpha)_{\alpha \in K})$$

such that $(K, \leq)$ is a (nonempty) reflexive partial order (the frame of K) and for each $\alpha \in K$, A_α is a classical L-structure $A_\alpha = (A_\alpha, =_\alpha, (R_\alpha)_{R \in L}, (f_\alpha)_{f \in L})$ (not necessarily normal, i.e., $=_\alpha$ need not be true equality on A_α), with the proviso that the following 'monotonicity conditions' be fulfilled:

Whenever $\alpha \leq \beta$, then

1. A_α is a subset of A_β;
2. for every relation symbol R of L (including equality $=$): $R_\alpha \subseteq R_\beta$;
3. for every n-ary function symbol f of L: f_α is the function f_β restricted to ${A_\alpha}^n$.

The forcing relation $\Vdash_{\mathsf{K}}$, $\Vdash$ for short, is defined as usual. $\mathsf{K} \models \phi$ means that for all $\alpha \in K$, $\alpha \Vdash \phi$. We also use $\models$ for the classical satisfaction relation, sometimes

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writing $\alpha \models \phi$ instead of $A_\alpha \models \phi$. We consider here the usual arithmetical language L_{ar} given by $0, S, +, \cdot$ and $<$. $i\Delta_0$ is the intuitionistic L_{ar}-theory axiomatized by the usual axioms for PA^- (cf. [K91]) together with the axiom schema of induction for Δ_0-formulae. $I\Delta_0$ is the classical version of $i\Delta_0$, i.e., $i\Delta_0$ augmented by classical logic. Σ_1 is the class of L_{ar}-formulae of the form $\exists\bar{y}\,\varphi(\bar{x},\bar{y})$, Π_1 the class of L_{ar}-formulae $\forall\bar{y}\,\varphi(\bar{x},\bar{y})$ with, in both cases, φ in Δ_0. Given a formula class Γ, $i\Gamma$ is $i\Delta_0$ plus induction over all formulae in Γ ($I\Gamma$ being the corresponding classical theory). HA is $i\Delta_0$ together with the axiom schema of induction for arbitrary formulae of L_{ar}, its classical counterpart is PA. Note that in $i\Delta_0$, Δ_0-formulae are decidable.

As a consequence, we have the following

1.1 Lemma. *Let $\mathsf{K} = ((K,\leq),(A_\alpha)_{\alpha\in K})$ be any Kripke structure for L_{ar}. The following are equivalent:*

1. *$\mathsf{K} \models i\Delta_0$, i.e., for each $\alpha \in K$, $\alpha \Vdash i\Delta_0$.*
2. *For each $\alpha \in K$, A_α is a classical model of $I\Delta_0$ (i.e., $\alpha \models I\Delta_0$) and whenever $\alpha \leq \beta$ in K, $A_\alpha \prec_{\Delta_0} A_\beta$.*

Under these conditions, we have for each $\alpha \in K$, each Δ_0-formula $\phi(\bar{x})$ and each $\bar{a} \in A_\alpha$:

$$\alpha \Vdash \phi(\bar{a}) \iff \alpha \models \phi(\bar{a}). \quad \Box$$

For a proof, see [W96a]. If atomic formulae (and hence quantifier-free formulae, cf. [M84]) are decidable in K (as will be the case for $\mathsf{K} \models i\Delta_0$), we may assume without loss of generality that every A_α is a normal structure and that, whenever $\alpha \leq \beta$ in K, A_α is a substructure of A_β (such K will be called normal Kripke structures):

1.2 Lemma. *Let $\mathsf{K} = ((K,\leq),(A_\alpha)_{\alpha\in K})$ be a Kripke structure for a language L such that $\mathsf{K} \models \forall\bar{x}(P\bar{x} \vee \neg P\bar{x})$ for each atomic formula $P\bar{x}$ of L. Then there is a Kripke structure $\mathsf{K}^+ = ((K,\leq),(B_\alpha)_{\alpha\in K})$ for L such that every B_α is a normal structure and whenever $\alpha \leq \beta$ in K, B_α is a substructure of B_β, and such that $\mathsf{K} \models \phi \iff \mathsf{K}^+ \models \phi$ for each L-sentence ϕ.* $\Box$

This fact was already pointed out by Smoryński in [S73]; however, there are two pitfalls in proving it. Smoryński notes that one cannot just divide out $=_\alpha$ locally, since the equivalence classes may grow when passing from α to some $\beta \geq \alpha$, and so one will not obtain actual inclusion of domains. His proposal is to divide out globally as follows (note that he does not consider function symbols):

Let $D := \bigcup_{\alpha\in K} A_\alpha$. For $c, d \in D$ define $c \sim d :\iff \exists\alpha \in K\ c =_\alpha d$. Put $B_\alpha := \{[c] : c \in A_\alpha\}$, where $[c]$ is the equivalence class of c under $\sim$, let $R_{B_\alpha}([c_1],...,[c_n]) :\iff R_{A_\alpha}(d_1,...,d_n)$ for some elements $d_1,\ldots,d_n \in A_\alpha$ such that $d_i \in [c_i]$.

But there is a problem here: 'By accident', one and the same object may exist in worlds of incomparable nodes of the frame, and with incompatible properties in the respective worlds. Thus, let α, β and γ be nodes such that $\alpha < \beta$, $\alpha < \gamma$, but $\beta \not\leq \gamma$ and $\gamma \not\leq \beta$. Assume that A_α does not contain the object c, whereas $a \in A_\alpha$.

Then it is possible that $a =_\beta c$ and $a \neq_\gamma c$. Clearly, this makes it impossible to identify a and c.

Instead, it is necessary to proceed as follows: In a first step, divide out the equality relations $=_\alpha$ locally. In a second step, for $\alpha \leq \beta$, identify two equivalence classes $[a]_{=_\alpha}$ and $[b]_{=_\beta}$ if $[a]_{=_\alpha} \subseteq [b]_{=_\beta}$, then take the transitive closure. The easy but tedious verification is left to the reader.

Given some set of axioms T, a Kripke structure $\mathsf{K} = ((K, \leq), (A_\alpha)_{\alpha \in K})$ for the language of T is called T-normal or locally T if for each $\alpha \in K$, the structure A_α is a classical model of T, $A_\alpha \models T$. The interplay of sets of axioms being forced at α and those being classically true in A_α is intriguing, see e.g. [W96].

2. Smorynski's collection operation

In his [S73], Smoryński introduced a powerful *collection operation* on Kripke structures which he used to prove a large number of results on intuitionistic logic and arithmetic. The idea behind his operation $(\sum\ )'$ is this:

Let some family $\mathcal{F} = \{\mathsf{K}^i : i \in I\}$ of Kripke structures

$$\mathsf{K}^i = ((K^i, \leq^i), (A^i_\alpha)_{\alpha \in K^i})$$

for some version L of the arithmetical language (containing a closed term for each natural number n) be given. We may assume without loss of generality that for $i \neq j$ in I, $K^i \cap K^j = \emptyset$.

We obtain a new Kripke structure by forming the disjoint union $\Sigma\mathcal{F} = \mathsf{K} = ((K, \leq), (A_\alpha)_{\alpha \in K})$ of $\mathcal{F}$, putting

- $K = \bigcup_{i \in I} \mathsf{K}^i$;
- for $\alpha, \beta \in K$, $\alpha \leq \beta :\iff$ for some $i \in I$, $\alpha \leq^i \beta$;
- for $\alpha \in K$, $A_\alpha := A^i_\alpha$, where $i \in I$ is such that $\alpha \in K^i$.

Observing that for each $\alpha \in K^i$, $\mathsf{K}^i_\alpha = \mathsf{K}_\alpha$, we see that if each $\mathsf{K}^i \models T$ for some theory T, then $\mathsf{K} = \Sigma\mathcal{F} \models T$. The operation of disjoint union becomes interesting only in conjunction with a second operation $\mathsf{K} \mapsto \mathsf{K}'$ which consists in attaching a (new) root to K:

Given a *normal* Kripke structure K, K' is obtained from K by adding a new node α_0 to K (i.e., $K' := K \uplus \{\alpha_0\}$), stipulating that α_0 be minimal in K' (i.e., $\leq' := \leq \cup \{(\alpha_0, \beta) : \beta \in K'\}$) and letting A_{α_0} be the standard model of arithmetic $\mathbb{N}$.

Note that the restriction to normal models, or at least an assumption that all worlds are normal in their 'standard part' given by the numerals, is important here: The interpretations f_α of function symbols f are actual functions, compatible with $=_\alpha$ but not multi-valued. Thus e.g. in the standard model A_{α_0}, we certainly have $S_\alpha(0) = 1$ (true equality!), whereas at some node β from K, only $S_\beta(0) =_\beta 1$ is required, and we may actually have $S_\beta(0) = c$ for some $c \in A_\beta$ such that $c \neq 1$ (but of course $c =_\beta 1$). This would collide with the requirement that S_{α_0} be the restriction of S_β to A_{α_0}. But this cannot happen if we exclude non-normal models from the start (which, by Lemma 1.2, is no real restriction).

It is now easy to see that $\mathcal{K}'$ is again a Kripke structure; more difficult is the question which theories T are preserved under this operation, i.e., for which theories T is $\mathcal{K}'$ a model of T, provided that $\mathcal{K} \models T$?

Smoryński has shown that HA and some of its extensions are preserved under the $'$ operation. We briefly go through his proof, pointing out the punch line and in fact showing that *every* fragment $i\Gamma$ of HA is preserved.

2.1 Theorem (Smoryński). *Let $\mathcal{K} \models i\Gamma$. Then $\mathcal{K}' \models i\Gamma$ too.*

Proof. Clearly it is sufficient to show that $\alpha_0 \Vdash' i\Gamma$ (writing $\Vdash'$ for $\Vdash_{\mathcal{K}'}$). The crucial axioms to check are the induction axioms. So let $\varphi(x, \bar{z}) \in \Gamma$; we must show that

$$\alpha_0 \Vdash' \forall \bar{z}[\varphi(0, \bar{z}) \wedge \forall x(\varphi(x, \bar{z}) \to \varphi(Sx, \bar{z})) \to \forall x\, \varphi(x, \bar{z})].$$

Assume the contrary. Then for some $\beta \geq' \alpha_0$ and $\bar{b} \in A_\beta$,

$$\beta \nVdash' \varphi(0, \bar{b}) \wedge \forall x(\varphi(x, \bar{b}) \to \varphi(Sx, \bar{b})) \to \forall x\, \varphi(x, \bar{b}).$$

The assumption that $\beta > \alpha_0$ leads to a contradiction, since at such β, $\Vdash$ and $\Vdash'$ coincide, and so $\beta \Vdash' i\Gamma$.

Hence $\beta = \alpha_0$ and $\bar{b} \in A_{\alpha_0} = \mathbb{N}$. We thus have (now suppressing parameters from A_{α_0}):

$$\alpha_0 \nVdash' \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Sx)) \to \forall x\, \varphi(x),$$

i.e., for some $\beta \geq' \alpha_0$,

$$\beta \Vdash' \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Sx))$$

but $\beta \nVdash' \forall x\, \varphi(x)$. Again $\beta > \alpha_0$ is impossible since then $\beta \Vdash' i\Gamma$. So we have

(i) $\alpha_0 \Vdash' \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Sx))$ and

(ii) $\alpha_0 \nVdash' \forall x\, \varphi(x)$.

By (i) and the fact that for $\beta > \alpha_0$ we have $\beta \Vdash' i\Gamma$ we conclude that for all $\beta > \alpha_0$, $\beta \Vdash' \forall x\, \varphi(x)$. Hence we may infer from (ii) that for some $n \in A_{\alpha_0} = \mathbb{N}$, $\alpha_0 \nVdash \varphi(n)$. Here's the punch line: Let m be the least number n such that $\alpha_0 \nVdash'$ $\varphi(n)$. Such a minimal counterexample exists since we are in the standard model $\mathbb{N}$. (If A_{α_0} were nonstandard, we would have to know that the forcing relation at α_0 is suitably definable in A_{α_0}.) Now since $\alpha_0 \Vdash' \varphi(0)$, we have $m \neq 0$, so $m = Sk$ for some $k \in \mathbb{N}$. By minimality of m, $\alpha_0 \Vdash' \varphi(k)$, so by $\alpha_0 \Vdash' \forall x(\varphi(x) \to \varphi(Sx))$ we obtain $\alpha_0 \Vdash' \varphi(m)$, contradiction. $\Box$

2.2 Corollary. *Every theory of the form $i\Gamma$ has the explicit definability property (*ED*) and the disjunction property (*DP*), i.e., whenever $i\Gamma$ proves a sentence*

$\exists x\, \varphi(x)$, then for some $n \in \mathbb{N}$, $i\Gamma \vdash \varphi(n)$, and whenever $i\Gamma$ proves a sentence $\psi \vee \chi$, then either $i\Gamma \vdash \psi$ or $i\Gamma \vdash \chi$.

Proof. We consider only *(ED)*. Suppose that $i\Gamma \nvdash \varphi(n)$ for each $n \in \mathbb{N}$. By the completeness theorem, for each n there is a Kripke model $\mathsf{K}_n \models i\Gamma$ with $\mathsf{K}_n \not\models \varphi(n)$. Now consider $(\sum_{n\in\mathbb{N}} \mathsf{K}_n)' =: \mathsf{K}'$ which is a model of $i\Gamma$ by theorem 2.1. Obviously, $\alpha_0 \nVdash' \exists x\, \varphi(x)$, since this would imply $\alpha_0 \Vdash' \varphi(n)$ for some $n \in A_{\alpha_0} = \mathbb{N}$, which is impossible by $\mathsf{K}_n \not\models \varphi(n)$. By the soundness theorem, $i\Gamma \nvdash \exists x\, \varphi(x)$. $\square$

Smoryński's idea obviously yields a construction method for Kripke models that can be summarized as follows:

2.3 THEOREM. *Let $(K, \leq)$ be any conversely wellfounded tree. Attach arbitrary models of $I\Gamma$ to terminal nodes of $(K, \leq)$ and the standard model $\mathbb{N}$ of arithmetic to each internal node of $(K, \leq)$. The resulting Kripke structure is then a model of $i\Gamma$.*

3. Constructing models of Π_1-induction

Sam Buss has shown in [B93] that there is an easy way to construct Kripke models of $i\Sigma_1$ over arbitrary frames: *Every $I\Sigma_1$*-normal Kripke structure is a Kripke model of $i\Sigma_1$. But he also illustrates, by way of counterexample, that it is not as trivial to construct models of $i\Pi_1$. However, models of $i\Pi_1$ over conversely wellfounded frames are not too difficult to build:

3.1 LEMMA. *Let $\mathsf{K} = ((K, \leq), (A_\alpha)_{\alpha\in K})$ be a Kripke structure such that for each $\beta \in K$ there is a terminal node $\alpha \geq \beta$ in K (this is always the case when $(K, \leq)$ is conversely wellfounded). Suppose further that for all terminal nodes $\alpha \in K$, $A_\alpha \models I\Sigma_1$ and that for all internal nodes $\beta \in K$, $A_\beta \models I\Delta_0$. Assume that whenever $\alpha \leq \beta$ in K, A_β is a Δ_0-elementary extension of A_α. Then $\mathsf{K} \models i\Pi_1$.*

Proof. We proceed by brute force (see [W96a] for an alternative proof). Take any $\alpha \in K$ and let $\psi(x, \bar{y}, \bar{z})$ be a Δ_0-formula. We have to prove that α forces

$$\forall \bar{z}(\forall \bar{y}\, \psi(0, \bar{y}, \bar{z}) \wedge \forall x(\forall \bar{y}\, \psi(x, \bar{y}, \bar{z}) \to \forall \bar{y}\, \psi(Sx, \bar{y}, \bar{z})) \to \forall x \forall \bar{y}\, \psi(x, \bar{y}, \bar{z})).$$

So let $\beta \geq \alpha$, $\bar{b} \in A_\beta$ and suppose that

$$\beta \Vdash \forall \bar{y}\, \psi(0, \bar{y}, \bar{b}) \wedge \forall x(\forall \bar{y}\, \psi(x, \bar{y}, \bar{b}) \to \forall \bar{y}\, \psi(Sx, \bar{y}, \bar{b})).$$

Thus every terminal node above β classically satisfies this last sentence. Since all terminal nodes are classical models of $I\Sigma_1$ ($\equiv I\Pi_1$), every such node also satisfies $\forall x \forall \bar{y}\, \psi(x, \bar{y}, \bar{b}))$. This sentence is Π_1 and thus downwards preserved in Δ_0-elementary extensions. In particular, it is classically true in every $\gamma \geq \beta$. But we want to show that $\beta \Vdash \forall x \forall \bar{y}\, \psi(x, \bar{y}, \bar{b})$. However, this just means that for all $\gamma \geq \beta$ and all $c, \bar{d} \in A_\gamma$, $\gamma \Vdash \psi(c, \bar{d}, \bar{b})$, i.e., by 1.1 that $\gamma \models \forall x \forall \bar{y}\, \psi(x, \bar{y}, \bar{b})$, which, as we have just seen, is indeed the case. $\square$

3.2 *Remark.* This result shows that Buss' counterexample is not just accidentally not conversely wellfounded: Every PA-normal Kripke structure on conversely wellfounded frames validates $i\Pi_1$.

4. Some general Kripke model theory

In the next section, we will use a modification of Smoryński's $(\sum\ )'$ operation to construct models of $i\Pi_2$ from *arbitrary* classical models of $I\Sigma_2$. The present section develops a little theorem in general Kripke model theory as a preparation for that result.

4.1 *Definition.* A formula φ of some language L is positive if it is built up from atomic formulae using only $\wedge$, $\vee$ and $\exists$.

4.2 *Remark.* For all Kripke structures K, $\alpha \in K$, $\bar{a} \in A_\alpha$ and every positive formula $\varphi(\bar{x})$ we have: $\alpha \Vdash \varphi(\bar{a}) \iff \alpha \models \varphi(\bar{a})$. (In fact this property characterizes the positive formulae: If always $\alpha \Vdash \varphi(\bar{a}) \iff \alpha \models \varphi(\bar{a})$, then φ is intuitionistically equivalent to a positive formula; cf. [M83].) If in the respective Kripke model Δ_0-formulae are decidable, we may, by 1.1, relax the definition of positivity by replacing 'atomic' with 'Δ_0', and the property will continue to hold.

4.3 *Definition.* Let Γ be the smallest class of formulae containing the positive formulae, closed under application of $\forall$ and $\wedge$ and under the following rule: If φ is positive and $\psi \in \Gamma$, then $\varphi \to \psi$ is in Γ.

4.4 *Remark.* Under classical logic, Γ is the class of $\forall\exists$-formulae. This is also true in intuitionistic theories within which atomic formulae are decidable.

4.5 LEMMA. *Let $\varphi(\bar{z}) \in \Gamma$, let K be a Kripke structure, $\alpha \in K$, $\bar{a} \in A_\alpha$ and suppose that for all $\beta > \alpha$ we have $\beta \Vdash \varphi(\bar{a})$. Then:*

$$\alpha \Vdash \varphi(\bar{a}) \iff \alpha \models \varphi(\bar{a}).$$

Proof. We proceed by induction on the definition of $\varphi \in \Gamma$.

If $\varphi(\bar{z})$ is positive, then the claim follows from our remark above. Using the induction hypothesis, the case of conjunction is trivial.

So suppose that $\varphi(\bar{z})$ is of the form $\forall y\, \psi(y, \bar{z})$ with $\psi(y, \bar{z}) \in \Gamma$. By assumption for all $\beta > \alpha$, $\beta \Vdash \forall y\, \psi(y, \bar{a})$. (We will from now on suppress mention of the parameters $\bar{a}$ and just write $\forall y\, \psi(y)$.) First suppose that $\alpha \Vdash \forall y\, \psi(y)$. In particular, for all $c \in A_\alpha$, $\alpha \Vdash \psi(c)$, and so by persistence of forcing, for all $c \in A_\alpha$ and all $\beta \geq \alpha$ $\beta \Vdash \psi(c)$. By the induction hypothesis then for all $c \in A_\alpha$ $\alpha \models \psi(c)$, i.e., $\alpha \models \forall y\, \psi(y)$.

Now suppose that $\alpha \nVdash \forall y\, \psi(y)$. By definition, for some $\beta \geq \alpha$ and some $c \in A_\beta$, $\beta \nVdash \psi(c)$. But since all $\beta > \alpha$ force $\forall y\, \psi(y)$, we must have $\alpha \nVdash \psi(c)$ for some $c \in A_\alpha$. But for all $\beta > \alpha$ $\beta \Vdash \psi(c)$ and so by induction hypothesis $\alpha \not\models \psi(c)$ and thus $\alpha \not\models \forall y\, \psi(y)$.

For the case of implication, suppose φ is of the form $\psi \to \chi$, where ψ is positive and $\chi \in \Gamma$. (We are again suppressing parameters from A_α.) Assume that for all $\beta > \alpha$, $\beta \Vdash \psi \to \chi$.

First let $\alpha \Vdash \psi \to \chi$. We consider two cases. For the first case, suppose that $\alpha \Vdash \psi$. Then for all $\beta \geq \alpha$ $\beta \Vdash \psi$ and thus for all $\beta \geq \alpha$ $\beta \Vdash \chi$. By induction hypothesis then $\alpha \models \chi$ and of course $\alpha \models \psi \to \chi$. In the second case $\alpha \nVdash \psi$. Since ψ is positive, we then have $\alpha \not\models \psi$ and so vacuously $\alpha \models \psi \to \chi$.

For the other direction, assume that $\alpha \nVdash \psi \to \chi$. Then for some $\beta \geq \alpha$ $\beta \Vdash \psi$ and $\beta \nVdash \chi$. Since by assumption for all $\beta > \alpha$ $\beta \Vdash \psi \to \chi$, we must in fact have $\alpha \Vdash \psi$ and $\alpha \nVdash \chi$. Since $\alpha \Vdash \psi$, for all $\beta > \alpha$ $\beta \Vdash \chi$. Hence by induction hypothesis $\alpha \not\models \chi$. $\alpha \Vdash \psi$ and ψ is positive, so $\alpha \models \psi$. Together we get $\alpha \not\models \psi \to \chi$. $\square$

4.6 *Question.* Does the property exhibited in Lemma 4.5 characterize the class Γ?

4.7 *Remark.* If Δ_0-formulae are decidable in the model under consideration, the theorem remains true if we use the relaxed definition of positivity indicated in remark 4.2 in the definition of Γ (cf. lemma 1.1), so that we are then talking about Π_2-formulae.

5. Constructing models of Π_2-induction

In the variant indicated at the end of the previous section, we can use our result 4.5 to construct certain models of the intuitionistic version $i\Pi_2$ of $I\Pi_2$:

5.1 THEOREM. *Let K be an $I\Sigma_2$-normal Kripke structure over a conversely well-founded frame. Then K is a model of $i\Pi_2$, i.e., for each $\alpha \in K$, $\bar{a} \in A_\alpha$ and every Π_2-formula $\varphi(x, \bar{y})$ we have*

$$\alpha \Vdash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \to \varphi(Sx, \bar{a})) \to \forall x\, \varphi(x, \bar{a}).$$

Proof. Note that in $I\Sigma_2$-normal Kripke structures all extensions are Δ_0-elementary (since the MRDP theorem can be proved in $I\Sigma_2$, cf. [HP93]), and so $K \models i\Delta_0$ by 1.1.

We proceed by bar induction on α. For terminal α there is nothing to show since $I\Pi_2 \equiv I\Sigma_2$.

So let α be an internal node and suppose that

$$\alpha \nVdash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \to \varphi(Sx, \bar{a})) \to \forall x\, \varphi(x, \bar{a}).$$

Then for some $\beta \geq \alpha$ we have $\beta \Vdash \varphi(0, \bar{a})$, $\beta \Vdash \forall x(\varphi(x, \bar{a}) \to \varphi(Sx, \bar{a}))$ and $\beta \nVdash \forall x\, \varphi(x, \bar{a})$. But by induction hypothesis, we must have $\beta = \alpha$, so $\alpha \Vdash \varphi(0, \bar{a})$, $\alpha \Vdash \forall x(\varphi(x, \bar{a}) \to \varphi(Sx, \bar{a}))$ and $\alpha \nVdash \forall x\, \varphi(x, \bar{a})$.

By persistence of $\Vdash$ we obtain for each $\beta > \alpha$ that $\beta \Vdash \varphi(0, \bar{a})$, $\beta \Vdash \forall x(\varphi(x, \bar{a}) \to \varphi(Sx, \bar{a}))$ and hence by induction hypothesis that $\beta \Vdash \forall x\, \varphi(x, \bar{a})$. So it follows from $\alpha \nVdash \forall x\, \varphi(x, \bar{a})$ that already for some $b \in A_\alpha$, $\alpha \nVdash \varphi(b, \bar{a})$; however, for each $\beta > \alpha$ we have $\beta \Vdash \varphi(b, \bar{a})$ and φ is Π_2, so by 4.5 $\alpha \not\models \varphi(b, \bar{a})$. Now let c be the least element e of A_α such that $\alpha \not\models \varphi(e,\bar{a})$. (This is possible since $\alpha \models I\Pi_2$.)

Then the following obtain:

1. $\alpha \not\Vdash \varphi(c,\bar{a})$ (by 4.5, since $\alpha \not\models \varphi(c,\bar{a})$ and for all $\beta > \alpha$ we have $\beta \Vdash \varphi(c,\bar{a})$);
2. for each $d <_\alpha c$, $\alpha \Vdash \varphi(d,\bar{a})$ (again by 4.5, since for $d <_\alpha c$ $\alpha \models \varphi(d,\bar{a})$ and for each $\beta > \alpha$ $\beta \Vdash \varphi(d,\bar{a})$).

But now c is not 0, since $\alpha \Vdash \varphi(0,\bar{a})$ and $\alpha \not\Vdash \varphi(c,\bar{a})$. Hence $c = Sd$ for some $d \in A_\alpha$. But by 2. $\alpha \Vdash \varphi(d,\bar{a})$, so by $\alpha \Vdash \forall x(\varphi(x,\bar{a}) \to \varphi(Sx,\bar{a}))$ we get $\alpha \Vdash \varphi(Sd,\bar{a})$, i.e., $\alpha \Vdash \varphi(c,\bar{a})$, contradicting 1. $\square$

5.2 *Remark.* It is clear by Theorem 5.1 that every *PA*-normal Kripke structure over a wellfounded frame is a model of $i\Pi_2$. This result is in some sense optimal: Zambella and Visser show in a forthcoming paper that there are *PA*-normal Kripke structures over two-element frames, the respective *PA*-models an end-extension, which are not models of $i\Sigma_2$.

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