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BETWEEN $\mathbf{TW}_{\to}$ AND $\mathbf{RW}_{\to}$

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To the memory of V. A. Smirnov (1931–1996)

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Abstract. We investigate some pure implicational systems placed between the implicational fragments $\mathbf{TW}_{\to}$ and $\mathbf{RW}_{\to}$ of the well-known relevance systems $\mathbf{TW}$ and $\mathbf{RW}$ For one them, $\mathbf{TRW}_{\to}$+RP, we prove (1) and (2):

(1) if both $\vdash A \to B$ and $\vdash B \to A$ in $\mathbf{TRW}_{\to}$+RP, then B can be obtained from A by substitution of occurrences of formulas of the form $D \to .C \to E$ for some occurrences of subformulas of A of the form $C \to .D \to E$ (CONGR);

(2) CONGR is equivalent to NOASS: for any A and B,

$$\nvdash A \to .A \to B \to B$$

in $\mathbf{TRW}_{\to}$+RP.

CONGR is a generalization of the solution to the P–W problem, solved for $\mathbf{TW}_{\to}$ in [**6**] (cf. also [**1**]–[**4**] for other solutions).

The equivalence of CONGR and NOASS is a generalization of the Dwyer-Powers theorem for $\mathbf{TW}_{\to}$ to the effect that the P–W problem is equivalent to NOID: there is no theorem of $\mathbf{TW}_{\to}$-ID of the form AA.

The proof of the equivalence of CONGR and NOASS is obtained by double induction applied jointly with a normal form theorem.

1. Introduction

The only connective in the propositional language investigated here is $\to$. We write (AB) for $(A \to B)$; furthermore, ABC and $A.BC$ stand for $(AB)C$ and $A(BC)$, respectively.

For any A the set $c(A)$ is the smallest set satisfying (1) and (2): (1) $A \in c(A)$; (2) let $B \in c(A)$ be such that $C.DE$ is a subformula of B, for some C, D and E,

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and let B^* be obtained by substitution of $D.CE$ for $C.DE$, at a single occurrence of $C.DE$ in B; then $B^* \in c(A)$.

A and B are *congruent* (in symbols $A \sim B$) iff $B \in c(A)$. $A \not\sim B$ means $B \notin c(A)$.

For any A by A^* we denote any formula B such that $A \sim B$.

In the sequel several sub-systems of $\mathbf{RW}_{\to}$ are investigated. If $\mathbf{S}$ and $\mathbf{T}$ are any two of them, we write $(\mathbf{S} \subset \mathbf{T})$, $\mathbf{S} \subseteq \mathbf{T}$, and $\mathbf{S} = \mathbf{T}$ if the set of theorems of $\mathbf{S}$ is a (proper) subset of or equal to the set of theorems of $\mathbf{T}$, respectively.

Let U be either an axiom-scheme or a rule; if U is adjoined (deleted) to (from) a system $\mathbf{S}$, the result is denoted by $\mathbf{S}$+U ($\mathbf{S}$–U).

$\mathbf{RW}_{\to}$ can be defined by modus ponens (MP) and the following axiom-schemata:

ID	AA
ASU	$AB.BC.AC$
APR	$BC.AB.AC$
AP	$A(BC).B.AC.$

An equivalent formulation of $\mathbf{RW}_{\to}$ is obtained by substitution of the axiom-scheme $A.ABB$ (axiom-scheme of *assertion*, ASS) for AP (axiom-scheme of *permutation*). Sometimes it is important to distinguish between these two formulations; on such occasions the first will be called $\mathbf{RW}_{\to \mathrm{AP}}$ and the second $\mathbf{RW}_{\to \mathrm{ASS}}$.

$\mathbf{RW}_{\to}$ is closed under substitution of equivalents and, hence, under the *rule of permutation*:

$$\text{P} \qquad \text{If } \vdash A, \text{ then } \vdash A^*$$

Also, $\mathbf{RW}_{\to}$ is closed under the rules

ASS1 If $\vdash A$, then $\vdash ABB$

ASS2 If $\vdash A$ and $\vdash B_1. \ \dots \ .B_k. \ \dots \ .B_nC$, then $\vdash B_1. \ \dots \ .AB_k. \ \dots \ .B_nC$.

The closure under ASS1 follows by ASS and MP, and the closure under ASS2 by P, ASS1 and TR. On the other hand, when we have ID, ASS1 is obtained by ASS2.

$\mathbf{TW}_{\to}$ is defined by MP and the axiom-schemata ID, ASU and APR. It has the Anderson-Belnap property (A-B): if both $\vdash AB$ and $\vdash BA$ in $\mathbf{TW}_{\to}$, then A and B denote the same formula.

A-B is equivalent to the Dwyer-Powers property (D-P): for any A, $\nvdash AA$ in $\mathbf{TW}_{\to}$–ID.

$\mathbf{TW}_{\to}$ and $\mathbf{TW}_{\to}$–ID have alternative formulations $\mathbf{TRW}_{\to}$ and $\mathbf{TRW}_{\to}$–ID, respectively, obtained by deleting MP and by adjoining the following rules instead:

SU	If $\vdash AB$, then $\vdash BC.AC$
PR	If $\vdash BC$, then $\vdash AB.AC$
TR	If $\vdash AB$ and $\vdash BC$, then $\vdash AC$

$\mathbf{TW}_{\rightarrow}$+P is an equivalent formulation of $\mathbf{RW}_{\rightarrow}$. On the other hand, adding P to $\mathbf{TRW}_{\rightarrow}$ does not suffice to produce $\mathbf{RW}_{\rightarrow}$ – we must add ASS1 as well.

THEOREM 1.1. $\mathbf{TW}_{\rightarrow}$+P = $\mathbf{TRW}_{\rightarrow}$+P+ASS1.

Proof. It was proved in [**5**, Theorem 5] that $\mathbf{TW}_{\rightarrow}$+P–ID = $\mathbf{TRW}_{\rightarrow}$+P+ASS1–ID. The inductive proof given there was to the effect that $\mathbf{TRW}_{\rightarrow}$+P+ASS1–ID is closed under the following Ackermann's rule δ (and hence under MP):

if

$$\text{(a)}\ \vdash A_i \qquad \text{and} \qquad \text{(b)}\ \vdash A_1.\ \ldots\ .A_{i-1}.A_i.A_{i+1}.\ \ldots\ .A_n p,$$

then

$$\text{(c)}\ \vdash A_1.\ \ldots\ .A_{i-1}.A_{i+1}.\ \ldots\ .A_n p.$$

The induction is on the weight of (b) and has to be extended here by considering ID. This is easy.

Can we substitute the rules SU, PR and TR for MP in $\mathbf{RW}_{\rightarrow \mathrm{AP}}$ and $\mathbf{RW}_{\rightarrow \mathrm{ASS}}$ such that the resulting systems are equivalent to the old ones and to each other?

The negative answer is a surprise. We shall show that these new systems are not closed under MP. Moreover, they are not equivalent to each other. This shows that between $\mathbf{TW}_{\rightarrow}$ and $\mathbf{RW}_{\rightarrow}$ there is more room for some interesting intermediate systems than we believed to be. One of them is $\mathbf{TRW}_{\rightarrow}$+RP; we shall show that it enjoys CONGR – a property analogous to A-B: if both $\vdash AB$ and $\vdash BA$ in $\mathbf{TRW}_{\rightarrow}$+RP, then $A \sim B$. This property is not shared by all systems between $\mathbf{TW}_{\rightarrow}$ and $\mathbf{RW}_{\rightarrow}$; for example, $\vdash AA.AAAA$ and $\vdash AAAA.AA$ in $\mathbf{TRW}_{\rightarrow}$+P; obviously, $AA \not\sim AAAA$.

2. The intermediate systems

Let us define the rule of *restricted permutation*:

$$\text{RP} \qquad \text{If } \vdash AB, \text{ then } \vdash A^*B^*$$

Since $\mathbf{TRW}_{\rightarrow}$+AP is closed under substitution of equivalents, it is easy to prove

THEOREM 2.1. $\mathbf{TRW}_{\rightarrow}$+RP = $\mathbf{TRW}_{\rightarrow}$+AP.

THEOREM 2.2. $\mathbf{TW}_{\rightarrow}$+RP = $\mathbf{RW}_{\rightarrow}$.

Proof. It is clear that $\mathbf{TW}_{\rightarrow}$+RP $\subseteq$ $\mathbf{RW}_{\rightarrow}$.

In $\mathbf{TW}_{\rightarrow}$+RP ID and RP yield AP. Also, $\mathbf{TW}_{\rightarrow}$+RP is closed under MP. Hence, $\mathbf{RW}_{\rightarrow}$ = $\mathbf{RW}_{\rightarrow \mathrm{AP}}$ $\subseteq$ $\mathbf{TW}_{\rightarrow}$+RP.

The main property of $\mathbf{TRW}_{\rightarrow}$+RP is given in the next theorem.

THEOREM 2.3. *If $A \not\sim B$, then*

$$\vdash AB \text{ in } \mathbf{TRW}_{\rightarrow}\text{+RP iff } \vdash AB \text{ in } \mathbf{TRW}_{\rightarrow}\text{+RP–ID}.$$

Proof. Suppose that $A \not\sim B$ and proceed by induction on theorems of $\mathbf{TRW}_{\to}$+RP.

If AB is an axiom of $\mathbf{TRW}_{\to}$+RP, then AB is an axiom of $\mathbf{TRW}_{\to}$+RP–ID.

Let $AB = CD.ED$, $\vdash CD.ED$ by $\vdash EC$ and SU; since $CD \not\sim ED$, it follows that $E \not\sim C$; by induction hypothesis, $\vdash EC$ in $\mathbf{TRW}_{\to}$+RP–ID, . Hence, by SU $\vdash CD.ED$ in $\mathbf{TRW}_{\to}$+RP–ID.

In the case of PR we proceed in a similar way.

If $\vdash AB$ by $\vdash AC$, $\vdash CB$ and TR, then either $A \not\sim C$ or $C \not\sim B$. If both $A \not\sim C$ and $C \not\sim B$, then by induction hypothesis, $\vdash AC$ and $\vdash CB$ in $\mathbf{TRW}_{\to}$+RP–ID; hence, $\vdash AB$ in $\mathbf{TRW}_{\to}$+RP–ID by TR.

If $A \sim C$, by induction hypothesis $\vdash CB$ in $\mathbf{TRW}_{\to}$+RP–ID; we obtain $\vdash AB$ by $\vdash CB$ and RP. If $C \sim B$, by induction hypothesis $\vdash AC$ in $\mathbf{TRW}_{\to}$+RP–ID; we obtain $\vdash AB$ by $\vdash AC$ and RP.

Let $AB = C^*D^*$ and $\vdash C^*D^*$ by $\vdash CD$ and RP; since $C^* \not\sim D^*$, we have $C \not\sim D$; by induction hypothesis, $\vdash CD$ in $\mathbf{TRW}_{\to}$+RP–ID, and $\vdash C^*D^*$ by RP.

Since $\mathbf{TRW}_{\to}$+RP–ID $\subset$ $\mathbf{TRW}_{\to}$+RP, the theorem is proved.

It was shown in [**4**] that $\mathbf{TRW}_{\to}$+P+ASS1–ID (called there **K**) has an equivalent Genzten-style formulation **J** that contains no theorem of any of the forms AA, $A.ABB$, $ABBA$ or $AABB$. We shall state this fact in the form of a theorem, for further reference.

Theorem 2.4. *In* $\mathbf{TRW}_{\to}$+P+ASS1–ID:

(a) (No identity, NOID) *There is no theorem of the form AA.*

(b) (No assertion, NOASS$_1$) *There is no theorem of the form $A.ABB$.*

(c) (NOASS$_2$) *There is no theorem of the form $ABBA$.*

(d) (NOE$_1$) $\vdash (A_1.A_2. \dots .A_nB)B$ *iff* $\vdash A_1, \dots, \vdash A_n$.

(e) (NOE$_2$) *There is no theorem of the form $AABB$.*

NOASS$_1$ and NOASS$_2$ are equivalent whenever we have SU. In the sequel we write NOASS both for NOASS$_1$ and NOASS$_2$.

Since $\mathbf{TRW}_{\to}$+RP–ID $\subseteq$ $\mathbf{TRW}_{\to}$+P–ID $\subseteq$ $\mathbf{TRW}_{\to}$+P+ASS1–ID, NOID, NOASS and NOE$_2$ hold for $\mathbf{TRW}_{\to}$+RP–ID as well.

Now we can prove CONGR.

Theorem 2.5. (CONGR) *If both* $\vdash AB$ *and* $\vdash BA$ *in* $\mathbf{TRW}_{\to}$+RP*, then* $A \sim B$.

Proof. Suppose that both $\vdash AB$ and $\vdash BA$ in $\mathbf{TRW}_{\to}$+RP, and that $A \not\sim B$. By Theorem 2.3, $\vdash AB$ and $\vdash BA$ in $\mathbf{TRW}_{\to}$+RP–ID. Hence, $\vdash AA$ in $\mathbf{TRW}_{\to}$+RP–ID by TR, contrary to NOID.

The Anderson–Belnap property A-B is a special case of CONGR.

Another surprise is that NOASS holds for some systems having the axiom-scheme ID.

Theorem 2.6. *For any A and any B, $\nvdash A.ABB$ in* $\mathbf{TRW}_{\rightarrow}$+RP.

Proof. Suppose that there are A and B such that $\vdash A.ABB$ in $\mathbf{TRW}_{\rightarrow}$+RP; by $A \not\sim ABB$ and Theorem 2.3, $\vdash A.ABB$ in $\mathbf{TRW}_{\rightarrow}$+RP –ID, contrary to NOASS.

Theorem 2.6 shows that $\mathbf{TRW}_{\rightarrow}$+RP is closed neither under MP nor under P; otherwise, from the axiom *pp.pq.pq* we would obtain *pp.p.pqq* by RP and then *p.pqq* by *pp* and MP.

Theorem 2.7. *In* $\mathbf{TRW}_{\rightarrow}$+RP CONGR *implies* NOASS.

Proof. Suppose that $\vdash A.ABB$ in $\mathbf{TRW}_{\rightarrow}$+RP, for some A and B. By SU we obtain (a) $\vdash ABB(CB).A.CB$ for any C. Also, (b) $\vdash C(AB).ABB.CB$ by ASU. Hence, (c) $\vdash A(CB).ABB.CB$ by RP. Now by (a), (c) and CONGR we have $A(CB) \sim ABB.CB$, which is impossible. Hence, in $\mathbf{TRW}_{\rightarrow}$+RP CONGR implies NOASS.

Let us compare $\mathbf{TRW}_{\rightarrow}$+AP and $\mathbf{TRW}_{\rightarrow}$+ASS.

Theorem 2.8. $\mathbf{TRW}_{\rightarrow}$+AP $\subset$ $\mathbf{TRW}_{\rightarrow}$+ASS.

Proof. As in the proof of Theorem 2.7, we have (a) and (b). Hence, we have $\vdash B(AC).A.BC$ in $\mathbf{TRW}_{\rightarrow}$+ASS, by TR; thus $\mathbf{TRW}_{\rightarrow}$+AP $\subseteq$ $\mathbf{TRW}_{\rightarrow}$+ASS. By NOASS, $\mathbf{TRW}_{\rightarrow}$+AP $\neq$ $\mathbf{TRW}_{\rightarrow}$+ASS.

Since $\mathbf{TRW}_{\rightarrow}$+ASS is closed under substitution of equivalents, it is closed under RP. Moreover, we have

Theorem 2.9. $\mathbf{TRW}_{\rightarrow}$+ASS *is closed under* P.

Proof. By induction on theorems of $\mathbf{TRW}_{\rightarrow}$+ASS.

Suppose that $\vdash D.EF$ in $\mathbf{TRW}_{\rightarrow}$+ASS. If it is an instance of ID, then $\vdash E.DF$ by ASS, and conversely, if $\vdash D.EF$ by ASS, then $\vdash E.DF$ by ID.

If $\vdash D.EF$ by ASU (APR), then $\vdash E.DF$ by APR (ASU).

Suppose that $\vdash D.EF$ by (a) $\vdash ED_1$ and SU, where $D = D_1F$. Now (b) $\vdash D_1.D_1FF$ is an axiom; hence, $\vdash E.DF$ by (a), (b) and TR.

Suppose that $\vdash D.EF$ by (a) $\vdash D_1F$ and PR, where $D = ED_1$. By (a) and PR we have (b) $\vdash ED_1D_1.ED_1F$. On the other hand, (c) $\vdash E.ED_1D_1$ by ASS. Therefore, $\vdash E.DF$ by (c), (b) and TR.

Suppose that $\vdash D.EF$ by (a) $\vdash DG$, (b) $\vdash G.EF$ and TR. By induction hypothesis, (c) $\vdash E.GF$. On the other hand, (d) $\vdash GF.DF$ by (a) and SU; hence, $\vdash E.DF$ by (c), (d) and TR.

Therefore, if $\vdash D.EF$ in $\mathbf{TRW}_{\rightarrow}$+ASS, then $\vdash E.DF$ in $\mathbf{TRW}_{\rightarrow}$+ASS. This, together with the closure under RP gives us the closure under P.

Corollary. $\mathbf{TRW}_{\rightarrow}$+ASS = $\mathbf{TRW}_{\rightarrow}$+P.

Proof. Obviously, $\mathbf{TRW}_{\rightarrow}$+ASS $\subseteq$ $\mathbf{TRW}_{\rightarrow}$+P, by ID and P. $\mathbf{TRW}_{\rightarrow}$+ASS is closed under P, by Theorem 2.9; hence, $\mathbf{TRW}_{\rightarrow}$+P $\subseteq$ $\mathbf{TRW}_{\rightarrow}$+ASS.

It is interesting to notice that $\mathbf{TRW}_{\rightarrow}$+ASS–ID $\neq$ $\mathbf{TRW}_{\rightarrow}$+P–ID. For, NOASS holds in $\mathbf{TRW}_{\rightarrow}$+P–ID, but in $\mathbf{TRW}_{\rightarrow}$+ASS–ID we have the axiom-scheme ASS. Hence, it is easy to prove AP in $\mathbf{TRW}_{\rightarrow}$+ASS–ID (look at (a), (b) and TR in the proof of Theorem 2.7). Thus, some instances of ID are theorems of $\mathbf{TRW}_{\rightarrow}$+ASS–ID. However, we can prove neither pp nor $pq.pq$ nor $pqr.pqr$ nor etc.

By induction on theorems one can prove that $\mathbf{TRW}_{\rightarrow}$+ASS contains no theorem of the form ABp; hence, it contains no theorem of the form App. This shows that $\mathbf{TRW}_{\rightarrow}$+ASS is not closed under ASS1. It follows that $\mathbf{TRW}_{\rightarrow}$+ASS is not closed under MP (otherwise, by MP and ASS, it is closed under ASS1).

It was shown in [**3**] that $\mathbf{TRW}_{\rightarrow}$+P–ID has an equivalent Genzten-style formulation **L** that enjoys the properties NOID and NOASS and contains no theorem of the form ABB. But we have more than that.

THEOREM 2.10. (NOABB). *For any A and any B, $\nvdash ABB$ in* $\mathbf{TRW}_{\rightarrow}$+RP.

Proof. Suppose that $\vdash ABB$ in $\mathbf{TRW}_{\rightarrow}$+RP, for some A and B. Since $B \not\sim AB$, by Theorem 2.3, $\vdash ABB$ in $\mathbf{TRW}_{\rightarrow}$+RP–ID $\subset$ $\mathbf{TRW}_{\rightarrow}$+P–ID, which is impossible.

What happens when either ASS1 or MP is added to any of these systems?

$\mathbf{TRW}_{\rightarrow}$+ASS1 and $\mathbf{TRW}_{\rightarrow}$+ASS1–ID are closed under MP. This is proved by induction on the weight of the major premiss of MP. In the first of these systems we have the axiom-scheme $AABB$ – the characteristic axiom-scheme of $\mathbf{EW}_{\rightarrow}$; in the latter we have NOE_2.

It is easy to show that $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1 and $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1–ID are closed under MP. Since AP is a theorem of $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1–ID, the systems $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1 and $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1–ID are closed under P. Therefore, $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1 = $\mathbf{TRW}_{\rightarrow}$+P+ASS1 = $\mathbf{RW}_{\rightarrow}$.

However, in $\mathbf{TRW}_{\rightarrow}$+P+ASS1–ID we have both NOID and NOASS, but none of them in $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1–ID. Therefore, $\mathbf{TRW}_{\rightarrow}$+P+ASS1–ID $\subset$ $\mathbf{TRW}_{\rightarrow}$+ASS+ASS1–ID.

Adding ASS1 to $\mathbf{TRW}_{\rightarrow}$+RP destroys CONGR. Since (a) $\vdash AA(AA).AA$ and (b) $\vdash AA.AA.AA$ in $\mathbf{TRW}_{\rightarrow}$+RP+ASS1, we see that CONGR does not hold here. By (a) and SU we have (c) $\vdash AA(AA).(AA.AA).AA$; by (b), (c) and TR we obtain $\vdash AA.(AA.AA).AA$ – an instance of ASS. Therefore, NOASS does not hold in $\mathbf{TRW}_{\rightarrow}$+RP+ASS1. Since NOASS holds for $\mathbf{TRW}_{\rightarrow}$+RP+ASS1–ID, there are A and B such that $\vdash AB$ in $\mathbf{TRW}_{\rightarrow}$+RP+ASS1, $A \not\sim B$ and $\nvdash AB$ in $\mathbf{TRW}_{\rightarrow}$+RP+ASS1–ID.

Adding MP to $\mathbf{TRW}_{\rightarrow}$+RP would collapse it to $\mathbf{RW}_{\rightarrow}$. Adding MP to $\mathbf{TRW}_{\rightarrow}$+RP+ASS1–ID destroys NOABB. For example, let $A = pp.pp.pp$ and $B = (pp.pp)p.ppp$; then A and AB are instances of ASU. By RP and $AB.BC.AC$ we get (a) $\vdash AB.A.BCC$; hence, by (a) and MP, $\vdash BCC$.

Let us summarize the comparison of the systems investigated here.
$\mathbf{TW}_{\to} = \mathbf{TRW}_{\to} \subset \mathbf{TRW}_{\to}$+RP $= \mathbf{TRW}_{\to}$+AP $\subset \mathbf{TRW}_{\to}$+P $= \mathbf{TRW}_{\to}$+ASS $\subset \mathbf{TRW}_{\to}$+P+ASS1 $= \mathbf{TRW}_{\to}$+ASS+ASS1 $= \mathbf{TW}_{\to}$+RP $= \mathbf{TW}_{\to}$+P $= \mathbf{RW}_{\to}$ $= \mathbf{RW}_{\to \mathrm{AP}} = \mathbf{RW}_{\to \mathrm{ASS}}$.

Also, $\mathbf{TW}_{\to}$–ID $=$ $\mathbf{TRW}_{\to}$–ID $\subset$ $\mathbf{TRW}_{\to}$+RP –ID $\subset \mathbf{TRW}_{\to}$+P–ID $\subset$ $\mathbf{TRW}_{\to}$+P+ASS1–ID.

Only $\mathbf{TRW}_{\to}$+RP has the property stated in CONGR; it is not shared by stronger systems between $\mathbf{TRW}_{\to}$ and $\mathbf{RW}_{\to}$ investigated here.

The congruence relation $\sim$ defined for formulas is determined here by logical means only - by provability in $\mathbf{TRW}_{\to}$+RP.

3. The equivalence of congr and noass

The final surprise is the fact that in $\mathbf{TRW}_{\to}$+RP CONGR and NOASS are equivalent. To prove it we need some other facts.

In the sequel $\vdash A$ means $\vdash A$ in $\mathbf{TRW}_{\to}$+AP.

THEOREM 3.1. *If either* $\vdash Ap$ *or* $\vdash pA$, *then* $A = p$; *if* $\vdash B.pp$, *then* $B = pp$.

Proof. By induction on theorems.

Proofs in $\mathbf{TRW}_{\to}$+AP can be written in a normal form.

THEOREM 3.2. *For any proof of a theorem containing* n *applications of* TR *there is a proof of the same theorem containing* n *applications of* TR *such that no application of* TR *precedes an application of another rule.*

Proof. If $\vdash AC$ by (a) $\vdash AB$, (b) $\vdash BC$ and TR, and then $\vdash CD.AD$ by SU, then (c) $\vdash BD.AD$ by (a) and SU, as well as (d) $\vdash CD.BD$ by (b) and SU; hence, $\vdash CD.AD$ by (c), (d) and TR.

In a similar way we take care of PR.

If the old proof contains n applications of TR, so does the new one.

In the sequel we assume that in proofs of theorems no application of TR precedes an application of another rule.

The sequence of theorems

$$\vdash AB.C_1D_1, \vdash C_1D_1.C_2D_2, \ldots, \vdash C_{n-1}D_{n-1}.C_nD_n, \vdash C_nD_n.EF$$

is called a *transitive chain* (TR-*chain*) from AB to EF iff TR is not applied in the proof of any member of the chain.

THEOREM 3.3. *If* (a) $\vdash AB$ *and* (b) $\vdash BC$ *such that* (a) *is obtained by an application of* SU (PR) *in the last step and* (b) *is obtained by an application of* PR (*of* SU) *in the last step, then there is a proof of* $\vdash AC$ *by* TR *such that the left premiss* (a') *in this application of* TR *is obtained by* PR (SU) *in the last step and the right premiss* (b') *is obtained by* SU (PR) *in the last step.*

Proof. Suppose that (a) $\vdash DE.FE$ by (a') $\vdash FD$ and SU, and that (b) $\vdash FE.FG$ by (b') $\vdash EG$ and PR. We have (b') $\vdash DE.DG$ by (b') and PR, and $\vdash DG.FG$ by (a') and SU.

Suppose that (a) $\vdash ED.EF$ by (a') $\vdash DF$ and PR, and (b) $\vdash EF.GF$ by (b') $\vdash GE$ and SU. We have $\vdash ED.GD$ by (b') and SU, and $\vdash GD.GF$ by (a') and PR.

THEOREM 3.4. *If* (a) $\vdash AB$ *and* (b) $\vdash BC$ *such that* (a) *is obtained by an application of* SU *in the last step and* (b) *is an instance of* AP, *then there is a proof of* $\vdash AC$ *by* TR *such that the left premiss* (a') *is an instance of* AP *and the right premiss* (b') *is obtained by* PR *in the last step.*

Proof. Suppose that (a) $\vdash D(EG).F.EG$ by (a") $\vdash FD$ and SU, and that (b) $\vdash F(EG).E.FG$ by AP. We have (a') $\vdash D(EG).E.DG$ by AP and (b") $\vdash DG.FG$ by (a") and SU, and then (b') $\vdash E(DG).E.FG$ by (b") and PR.

THEOREM 3.5. *If* (a) $\vdash AB$ *and* (b) $\vdash BC$ *such that* (a) *is obtained by* (a') *and* PR *in the last step*, (b) *is an instance of* AP, *and* (a') *is obtained from* (a") *either by* SU *or by* PR, *then there is a proof of* $\vdash AC$ *by* TR *such that the left premiss* (c) *is an instance of* AP *and the right premiss* (b') *is obtained either by* SU *or by* PR *in the last step, respectively.*

Proof. Suppose that (a") $\vdash ED$, (a') $\vdash DF.EF$, (a) $\vdash C(DF).C.EF$ and (b) $\vdash C(EF).E.CF$. We have (c) $\vdash C(DF).D.CF$ by AP, and $\vdash D(CF).E.CF$ by (a") and SU.

Suppose that (a") $\vdash ED$, (a') $\vdash FE.FD$, (a) $\vdash C(FE).C.FD$ and (b) $\vdash C(FD).F.CD$. We have (c) $\vdash C(FE).F.CE$ by AP, and $\vdash F(CE).F.CD$ by (a") and PR applied twice.

THEOREM 3.6. *Let*

$$\vdash AB.D_1E_1, \vdash D_1E_1.D_2E_2, \ldots, \vdash D_{n-1}E_{n-1}.D_nE_n, \vdash D_nE_n.CD$$

be a TR*-chain from AB to CD; if no member of the chain is an axiom, then either*

(a) $$\vdash CA \textit{ and } B = D,$$

if all members of the chain are obtained by SU *in the last step, or*

(b) $$\vdash BD \textit{ and } A = C,$$

if all members of the chain are obtained by PR *in the last step, or*

(c) $$\vdash CA \textit{ and } \vdash BD.$$

otherwise.

Proof. If all members of the TR-chain from AB to CD are obtained by SU (PR), then $\vdash CA$ and $B = D$ ($A = C$ and $\vdash BD$); the proof is by induction on the number of members of the chain.

Apply Theorem 3.3. Let every member in

$$\vdash AB.D_1E_1, \vdash D_1E_1.D_2E_2, \ldots, \vdash D_{k-1}E_{k-1}.D_kE_k,$$

be obtained by SU, and let every member in

$$\vdash D_kE_k.D_{k+1}E_{k+1}, \ldots, \vdash D_{n-1}E_{n-1}.D_nE_n, \vdash D_nE_n.CD$$

be obtained by PR; then $\vdash D_kA$, $B = E_k$, $C = D_k$, and $\vdash E_kD$.

Corollary. *Let*

$$\vdash AB.D_1E_1, \vdash D_1E_1.D_2E_2, \ldots, \vdash D_{n-1}E_{n-1}.D_nE_n, \vdash D_nE_n.CD$$

be a TR*-chain from AB to CD; if no member of the chain is an instance of either* ASU *or* APR*, then either $\vdash CA$ and $B \sim D$ (in case every member of the chain is either an instance of* AP *or obtained by* SU *in the last step) or else $\vdash BD$ and $A \sim C$ (in case every member of the chain is either an instance of* AP *or obtained by* PR *in the last step) or $\vdash CA$ and $\vdash BD$ otherwise.*

Proof. By Theorems 3.4 and 3.5, we may assume that all instances of AP in the TR-chain from AB to CD precede the members obtained by either SU or PR. Hence, apply Theorem 3.6.

Now we can prove that NOASS implies CONGR.

Theorem 3.7. *If $\vdash AB$, $\vdash BA$, then $A \sim B$.*

Proof. Assume NOASS in $\mathbf{TRW}_{\rightarrow}$+AP (forgetting that it is already proved) and $\vdash AB$ and $\vdash BA$. If either $A = p$ or $B = p$ or $A = pp$ or $B = pp$, then $A = B$, by Theorem 3.1.

By Theorem 3.2, we have the TR-chain $(*)$

$$\vdash AB_1, \vdash B_1B_2, \ldots, \vdash B_{k-1}B_k, \vdash B_kB_{k+1}, \vdash B_{k+1}B_{k+2}, \ldots \vdash B_{m-1}B_m, \vdash B_mA$$

Lemma 3.8 *No member of $(*)$ is an instance of either* ASU *or* APR.

Proof of the lemma. Suppose that $\vdash DE.EF.DF$ is a member of $(*)$; then we have (a) $\vdash (EF.DF).DE$ as well. But $\vdash EF.DE.DF$ and (b) $\vdash (DE.DF)(DF).EF.DF$ by APR and SU; hence, $\vdash (DE.DF)(DF).DE$ by (a), (b) and TR, contrary to NOASS.

Let $\vdash EF.DE.DF$ be a member of $(*)$; then (a) $\vdash (DE.DF).EF$ as well. Now $\vdash DE.EF.DF$ and (b) $\vdash (EF.DF)(DF).DE.DF$ by ASU and SU; hence, $\vdash (EF.DF)(DF).EF$ by (a), (b) and TR, contrary to NOASS.

Lemma 3.9 *No member of the chain* $(*)$ *is obtained by* PR *in the last step from either* ASU *or* APR.

Proof of the lemma. Let (a') $\vdash DE.EF.DF$, (a) $\vdash G(DE).G.EF.DF$ by (a') and PR, and let (a) be a member of $(*)$; then (b) $\vdash G(EF.DF).G.DE$, for there is a member $\vdash HI$ of $(*)$ such that $H^*I^* \sim G(EF.DF).G.DE$. Now (c) $\vdash EF.G(DE).G.DF$ by $\vdash EF.DE.DF$, $\vdash DE(DF).G(DE).G.DF$ and TR. Hence, (d) $\vdash (G(DE).G.DF)(DF).EF.DF$ by (c) and SU, (e)

$$\vdash G((G(DE).G.DF)(DF)).G.EF.DF$$

by (d) and PR, (f) $\vdash ((G(DE).G.DF).G.DF).G.EF.DF$ by (e) and RP, and, eventually, $\vdash ((G(DE).G.DF).G.DF).G.DE$ by (f), (b) and TR, contrary to NOASS.

Suppose that (a') $\vdash EF.DE.DF$, (a) $\vdash G(EF).G.DE.DF$ by (a') and PR, and that (a) is a member of $(*)$; then (b) $\vdash G(DE.DF).G.EF$, for there is a member $\vdash HI$ of $(*)$ such that $H^*I^* \sim G(DE.DF).G.EF$. Now (c) $\vdash DE.G(EF).G.DF$ by $\vdash DE.EF.DF$, $\vdash EF(DF).G(EF).G.DF$ and TR, (d) $\vdash G(EF)(G.DF)(DF).DE.DF$ by (c) and SU, (e)

$$\vdash G((G(EF).G.DF)(DF)).G.DE.DF$$

by (d) and PR, (f) $\vdash ((G(EF).G.DF).G.DF).G.DE.DF$ by (e) and RP, and, eventually, $\vdash ((G(EF).G.DF).G.DF).G.EF$ by (f), (b) and TR, contrary to NOASS.

Returning to the proof of the theorem, proceed by double induction: on the degree of A and on the length l of the TR-chain $(*)$. Suppose that the theorem is true for any formula of degree smaller than the degree of A and any TR-chain (Hyp 1), and for A and any TR-chain of length smaller than l (Hyp 2).

Let us analyze the TR-chain $(*)$; by theorems 3.2-6 and lemmas 3.8-9, there is a TR-chain $(*)$ from A to A such that no member of $(*)$ is an instance of either ASU or APR. Moreover, we may assume that all instances of AP precede all instances obtained by either SU or PR in the last step.

If the member $\vdash B_mA$ of $(*)$ is an instance of AP, so are all members of $(*)$ and the theorem is proved.

Suppose that

$$\vdash AB_1, \vdash B_1B_2, \ldots, \vdash B_{k-1}B_k,$$

are instances of AP, and that

$$\vdash B_kB_{k+1}, \vdash B_{k+1}B_{k+2}, \ldots, \vdash B_{m-1}B_m, \vdash B_mA$$

are obtained either by SU or by PR in the last step. By Theorem 3.3 all applications of SU in the last steps precede all applications of PR in the last step.

Let $A = A_1.A_2A_3$ and let $B_k = B_k^1.B_k^2B_k^3, \ldots, B_m = B_m^1.B_m^2B_m^3$.

Case I $A = B_k$.

I.1 B_mA is obtained by SU; hence, $\vdash A_1B_m^1$ and $A_2A_3 = B_m^2B_m^3$. Also, all members of $\vdash B_{k+1}B_{k+2}, \dots, \vdash B_{m-1}B_m$ are obtained by SU in the last step. Hence, $\vdash B_{k+1}^1A_1$ and $A_2A_3 = B_{k+1}^2B_{k+1}^3$. By Theorem 3.6 $\vdash B_m^1B_{k+1}^1$. We have $\vdash A_1B_m^1$ and $\vdash B_m^1A_1$; by Hyp 1 $A_1 \sim B_m^1$, $A \sim B_m$ and there is a TR-chain from A to A of length $l-1$. By Hyp 2, $A \sim B_1 \sim \dots \sim B_m \sim A$.

I.2 B_mA is obtained by PR; hence $A_1 = B_m^1$ and $\vdash B_m^2B_m^3.A_2A_3$.

I.2.1 $\vdash AB_{k+1}$ by PR; hence, $A_1 = B_{k+1}^1$, $\vdash A_2A_3.B_{k+1}^2B_{k+1}^3$ and all members of $\vdash B_{k+1}B_{k+2}, \dots, \vdash B_{m-1}B_m$ are obtained by PR in the last step. Hence, $\vdash A_2A_3.B_{k+1}^2B_{k+1}^3, \dots \vdash B_m^2B_m^3.A_2A_3$. By Hyp 1, $A_2A_3 \sim B_{k+1}^2B_{k+1}^3 \sim \cdots \sim B_m^2B_m^3$ and $A \sim B_1 \sim \dots \sim B_m \sim A$.

I.2.2 $\vdash AB_{k+1}$ by SU; hence, $\vdash B_{k+1}^1A_1$ and $A_2A_3 = B_{k+1}^2B_{k+1}^3$. By Theorem 3.6, $\vdash B_m^1B_{k+1}^1$ and $\vdash B_{k+1}^2B_{k+1}^3.B_m^2B_m^3$. We have $\vdash A_1B_{k+1}^1$ and $\vdash B_{k+1}^1A_1$. By Hyp 1, $A_1 \sim B_{k+1}^1$, $A \sim B_{k+1}$ and there is a TR-chain from A to A of length $l-1$. By Hyp 2, $A \sim B_1 \sim \dots \sim B_m \sim A$.

Case II $A_2.A_1A_3 = B_k^1.B_k^2B_k^3$.

II.1 B_mA is obtained by SU and so are all members of $\vdash B_{k+1}B_{k+2}, \dots, \vdash B_{m-1}B_m$; hence, $A_1A_3 = B_{k+1}^2B_{k+1}^3 = \dots = B_m^2B_m^3 = A_2A_3$. Therefore, $A_1 = A_2$ and we may proceed as in Case I.

II.2 B_mA is obtained by PR; hence $A_1 = B_m^1$ and $\vdash B_m^2B_m^3.A_2A_3$.

II.2.1 $\vdash B_kB_{k+1}$ by PR; hence, $A_2 = B_{k+1}^1$, $\vdash A_1A_3.B_{k+1}^2B_{k+1}^3$ and all members of $\vdash B_{k+1}B_{k+2}, \dots, \vdash B_{m-1}B_m$ are obtained by PR in the last step. Hence, $\vdash A_1A_3.B_{k+1}^2B_{k+1}^3, \dots, \vdash B_m^2B_m^3.A_2A_3$. By Theorem 3.6, $A_1 = A_2$ and we may proceed as in Case I.

II.2.2 $\vdash B_kB_{k+1}$ by SU; hence, $\vdash B_{k+1}^1A_2$ and $A_1A_3 = B_{k+1}^2B_{k+1}^3$. By Theorem 3.6, $\vdash B_m^1B_{k+1}^1$ and $\vdash B_{k+1}^2B_{k+1}^3.B_m^2B_m^3$. Hence, $\vdash A_1B_{k+1}^1$ and $\vdash B_{k+1}^1A_2$. We get $\vdash A_1A_2$ and hence $\vdash A_2A_3.A_1A_3$ by SU. But we also have $\vdash A_1A_3.A_2A_3$. By Hyp 1, $A_1A_3 \sim A_2A_3$. Hence, $A_1 \sim A_2$ and we may proceed as in Case I.

This completes the proof of the theorem.

4. A concluding remark

It is both logically and philosophicaly interesting that substitution of formulas of the form $B.CD$ for subformulas of the form $C.BD$ in a formula A can be identified with CONGR - with the derivability of certain formulas in the weak logical system $\mathbf{TRW}_{\rightarrow}$+RP.

Also, it is interesting that the same substitution can be identified with NOASS - with the non-derivability of certain formulas.

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