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SOME REMARKS ON THE NONORIENTABLE SURFACES

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Abstract. It is a classical result of F. Klein that for any nonorientable (regular enough) surface $\mathbf{X}$ there is an orientable surface $\mathcal{O}_2$ and an involution without fixed point of $\mathcal{O}_2$ such that $\mathbf{X}$ is isomorphic to the quotient space of $\mathcal{O}_2$ with respect to the group generated by the respective involution.

In this note a reinforcement of the Klein's result is presented and the effect on the vector bundle of covariant tensors of second order on X produced by that involution is studied.

The projection $p : \mathcal{O}_2 \longrightarrow \mathbf{X}$ induces an isomorphism between the vector space of covariant tensors of order two on $\mathbf{X}$ and the space of covariant symmetric tensors of order two on $\mathcal{O}_2$ which are invariant with respect to the given involution.

By a *nonorientable surface* (NOS) $\mathbf{X}$ we understand in this paper a differentiable 2-dimensional connected manifold which does not admit a orientation. It is convenient to endow such a manifold with a so-called dianalytic structure, i.e., a structure defined by an atlas $\mathcal{A}$ such that every transition function is a conformal mapping or a mapping whose complex conjugate is conformal (called anticonformal mapping). It is evident that $\mathcal{A}$ does not admit an analtic subatlas $\mathcal{A}_1$, since then $(\mathbf{X}, \mathcal{A}_1)$ would be an orientable (Riemann) surface. Still Felix Klein has foreseen the role of these surfaces in the complex analysis. Despite the contradiction in terms, Teichmüller called them nonorientable Riemann surfaces. The name nonorientable Klein surface seems to become predominant (see [**7**]).

As in the orientable case, we can construct the universal covering

$$\pi : \widehat{\mathbf{X}} \longrightarrow \mathbf{X}$$

where $\widehat{\mathbf{X}}$ is a simply connected, hence orientable surface. This covering is a Galois covering, i.e., the automorphisms group $\mathbf{G}$ of $\widehat{\mathbf{X}}$ which conserve the fibers and acts

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transitively on each fiber. It contains necessarily conformal, but also anticonformal mappings.

If we denote by $\mathbf{G}_1$ the subgroup of $\mathbf{G}$ formed with the conformal transformations, then we can easily note that for $A \in \mathbf{G} \setminus \mathbf{G}_1$ we have $\mathbf{G} \setminus \mathbf{G}_1 = A\mathbf{G}_1$, i.e., $\mathbf{G}$ is the disjoint union

$$\mathbf{G} = \mathbf{G}_1 \cup A\mathbf{G}_1.$$

Moreover, $\mathbf{G}_1$ is itself transitive and therefore

$$\mathcal{O}_2 := \widehat{\mathbf{X}}/\mathbf{G}_1$$

is a Riemann surface whose universal covering is of the form $\pi_1 : \widehat{\mathbf{X}} \longrightarrow \mathcal{O}_2$. For a fixed $A \in \mathbf{G} \setminus \mathbf{G}_1$ we consider the mapping

$$h_A := h : \mathcal{O}_2 \longrightarrow \mathcal{O}_2$$

defined by $h(\hat{u}) = \widehat{Au}$, where $\hat{u}$ is the fiber of an $u \in \widehat{\mathbf{X}}$ by the action of $\mathbf{G}_1$. It is evident that h is correctly defined and it represents a fixed point free antianalytic involution of $\mathcal{O}_2$.

There is a canonical identification of $\mathbf{X}$ with $\mathcal{O}_2/\langle h\rangle$, where $\langle h\rangle$ is the two elements group generated by h.

Reciprocally, if there is a fixed point free antianalytic involution h of the Riemann surface $\mathcal{O}_2$, then the quotient space $\mathcal{O}_2/\langle h\rangle$ is a NOS endowed with a dianalytic structure.

On every NOS of class $\mathbf{C}^1$ we can define the Riemannian metric

$$ds = \lambda|dz + \mu d\bar{z}|,$$

where λ is a positive function [**6**]. If $\mu(z) \equiv 0$, the corresponding parameter and metric are called isothermal. It is easily seen that an isothermal metric on a NOS $\mathbf{X}$ induces a dianalytic structure on $\mathbf{X}$.

Indeed, if the parameters z and z_1 correspond to two overlapping local charts, then in their common domain,

$$ds = \lambda|dz| = \lambda_1|dz_1|,$$

and therefore for the transition function $z \to z_1$ we have:

$$|dz_1| = \frac{\lambda}{\lambda_1}|dz|.$$

But $dz_1 = \dfrac{\partial z_1}{\partial z}dz + \dfrac{\partial z_1}{\partial \bar{z}}d\bar{z}$ and, by consequence, one of the two partial derivatives $\dfrac{\partial z_1}{\partial z}$ or $\dfrac{\partial z_1}{\partial \bar{z}}$ must be identically zero. It results that $z \to z_1$ is a conformal or an anticonformal mapping.

Reciprocally, if the NOS $\mathbf{X}$ is endowed with a dianalytic structure, i.e., for every transition function $z \to z_1$ we have $dz_1 = \dfrac{\partial z_1}{\partial z}dz$ or $dz_1 = \dfrac{\partial z_1}{\partial \bar{z}}d\bar{z}$, then there is a positive function ν such that $|dz_1| = \nu|dz|$. By consequence,

$$ds = \lambda_1|dz_1 + \mu_1 d\bar{z_1}| = \lambda_1|dz_1|\left|1+\mu_1\frac{d\bar{z_1}}{dz_1}\right| = \lambda_1\nu|dz|\left|1+\mu\frac{d\bar{z}}{dz}\right|,$$

which is possible if and only if $\mu = \mu_1 = 0$, i.e., if and only if ds is isothermal.

The following theorem represents a reinforcement of a classical result due to Klein.

THEOREM 1. *If $\mathbf{X}$ is a NOS endowed with a dianalytic structure and if $f : \mathbf{W} \longrightarrow \mathbf{X}$ is a covering of $\mathbf{X}$ with $\mathbf{W}$ orientable, then $\mathbf{W}$ admits a structure of Riemann surface with respect to which the projection f is dianalytic. For every symmetry k of $\mathbf{W}$ (if it exist any), there is a dianalytic atlas of the quotient space $\mathbf{W}/\langle k\rangle$ such that the canonical projection $\mathbf{W} \longrightarrow \mathbf{W}/\langle k\rangle$ is a dianalytic function. Moreover, $\mathbf{W}/\langle k\rangle$ is a nonorientable cover of $\mathbf{X}$. If the dianalytic structure of $\mathbf{X}$ has been induced by the isothermal metric $ds = \lambda|dz|$, then the analytic structure of $\mathbf{W}$ is induced by the isothermal metric $d\sigma = \lambda_1|dw|$, where $\lambda_1(w) = \lambda(z(w))\left|\dfrac{\partial z}{\partial w}+\dfrac{\partial z}{\partial \bar{w}}\right|$.*

Proof. Let $\mathcal{A}$ be a dianalytic atlas on $\mathbf{X}$. For every $q \in \mathbf{W}$ let us chose a neighborhood U_q of q such that $f|_{U_q}$ is a one to one mapping and $f(U_q)$ is included in the domain of a local parameter h_p on $\mathbf{X}$, where $p = f(q)$. It is easily seen that the atlas

$$\mathcal{B} = \{U_q, h_p \circ (f|_{U_q}) \mid q \in \mathbf{W}\}$$

defines a dianalytic structure on $\mathbf{W}$ and f is a dianalytic function with respect to this structure.

Indeed, given the fact that for two charts (V_1, g_1), (V_2, g_2) in $\mathcal{B}$ with $g_i = h_i \circ (f|V_i)$, $i = 1, 2$, we have $g_2 \circ {g_1}^{-1} = h_2 \circ {h_1}^{-1}$ in $g_1(V_1 \cap V_2)$ the dianalyticity of the atlas $\mathcal{B}$ and of the function f is granted. We can enlarge $\mathcal{B}$, if necessary, including for every chart, the chart defined by the complex conjugate parameter. Given the fact that $\mathbf{W}$ has been supposed orientable, there is a partial atlas of $\mathcal{B}$ previously enlarged which contains only charts compatible with a given orientation of $\mathbf{W}$. This atlas defines an analytic structure on $\mathbf{W}$.

Let $\pi_1 : \mathbf{W} \longrightarrow \mathbf{W}/\langle k\rangle$ be the canonical projection. For every $p \in \mathbf{W}/\langle k\rangle$ we chose $q \in \mathbf{W}$ such that $p = \pi_1(q)$ and a neighborhood U_q of q with the property that $\pi_1|U_q$ is a one to one mapping and U_q is contained in the domain of a local parameter h_q on $\mathbf{W}$. A local chart at p is $(\pi_1(U_q), h_q \circ (\pi_1|_{U_q})^{-1})$. It is easily seen that the set of these charts forms a dianalytic atlas on $\mathbf{W}/\langle k\rangle$ and that π_1 is a dianalytic function with respect to this atlas. Taking into account that the partial order in the set of the coverings endows this set with a lattice structure and the fact that $\langle h\rangle$ contains only two elements, it results that we have the following chain of coverings

Let (V, g) be a local chart on $\mathbf{W}$. Then $V = U_{q_0}$ for a certain $q_0 \in \mathbf{W}$ and $g = h_{p_0} \circ (f|_V)$, for a local parameter h_{p_0} on $\mathbf{X}$, with $p_0 = f(q_0)$. If $z = h_{p_0}(p)$, $p \in f(V)$ and $w = g(q)$, $q \in V$, then $w \to z$ is a conformal mapping or anticonformal mapping. We define $\lambda_1(w) = \lambda(z(w))\left|\frac{\partial z}{\partial w}\right|$ if $w \to z$ is a conformal mapping and $\lambda_1(w) = \lambda(z(w))\left|\frac{\partial z}{\partial \bar{w}}\right|$ if $w \to z$ is an anticonformal mapping. In both cases $\lambda_1(w)|dw| = \lambda(z)|dz|$. It can be easily checked that $\lambda_1(w)|dw|$ is symmetric and it induces the previously mentioned conformal structure of $\mathbf{W}$.

In [**1**] and [**2**] a theory of integration on NOS endowed with dianalytic structures has been developed, in which the orientable surface $\mathcal{O}_2$ plays an *essential role.*

In the following we will define the coverings of certain geometric objects on $\mathbf{X}$ by $\mathcal{O}_2$ and we will study the effect on them produced by an antianalytic involution h of $\mathcal{O}_2$.

Let$S^2(T^*\mathbf{X}) = \bigcup_{x\in\mathbf{X}} S^2(T_x^*\mathbf{X})$ where $T_x^*\mathbf{X}$ is the cotangent space at the point x and $S^2(T_x^*\mathbf{X})$ is the space of covariant tensors of second order at x. The triplet

$$\mathbf{E} = (S^2(T^*\mathbf{X}), \pi, \mathbf{X})$$

where π is the canonical projection, is a vector bundle.

Analogously we define the vector bundle $\widetilde{\mathbf{E}} = (S^2(T^*\mathcal{O}_2), \widetilde{\pi}, \mathcal{O}_2)$.

Let us denote by $\mathcal{S}(\widetilde{\mathbf{E}})$ the space of differentiable sections of class C^∞ in the vector bundle $\widetilde{\mathbf{E}}$. A section $\theta \in \mathcal{S}(\widetilde{\mathbf{E}})$ is a covariant symmetric tensor field of order two on $\mathcal{O}_2$. We say that θ is *h-invariant* (respectively *h-antiinvariant*) if $h^*\theta = \theta$ (respectively $h^*\theta = -\theta$), where h^* is the transformation induced by h on $\mathcal{S}(\widetilde{\mathbf{E}})$.

Let us denote

$$\mathcal{S}_s(\widetilde{\mathbf{E}}) := \{\theta \in \mathcal{S}(\widetilde{\mathbf{E}}) | h^*\theta = \theta\} \text{ and } \mathcal{S}_a(\widetilde{\mathbf{E}}) := \{\theta \in \mathcal{S}(\widetilde{\mathbf{E}}) | h^*\theta = -\theta\}.$$

Obviously, $\mathcal{S}(\widetilde{\mathbf{E}})$ is a $C^\infty(\mathcal{O}_2, \mathbb{R})$-module.

Let us define the *symmetrisation* and *antisymmetrisation* operators S and A on $\mathcal{S}(\widetilde{\mathbf{E}})$ respectively by:

$$S\theta := \frac{1}{2}(\theta + h^*\theta); \quad A\theta := \frac{1}{2}(\theta - h^*\theta).$$

It can be easily checked that S and A are orthogonal projectors on $\mathcal{S}(\widetilde{\mathbf{E}})$, i.e., they fulfill the following equalities:

$$S \circ S = S; A \circ A = A; S \circ A = A \circ S = 0; S + A = Id_{\mathcal{S}(\widetilde{\mathbf{E}})} \text{ and } S - A = h^*.$$

(h^* is an involution).

Let us notice that $\mathcal{S}_s(\widetilde{\mathbf{E}})$ and $\mathcal{S}_a(\widetilde{\mathbf{E}})$ are invariant spaces with respect to the operators S and A. Moreover, we have the following result:

THEOREM 2. *a)* $\mathcal{S}_s(\widetilde{\mathbf{E}})$ *and* $\mathcal{S}_a(\widetilde{\mathbf{E}})$ *are vector subspaces of* $\mathcal{S}(\widetilde{\mathbf{E}})$*;*

b) $\mathcal{S}(\widetilde{\mathbf{E}}) = \mathcal{S}_s(\widetilde{\mathbf{E}}) \bigoplus \mathcal{S}_a(\widetilde{\mathbf{E}})$*;*

c) The projection $p : \mathcal{O}_2 \longrightarrow \mathbf{X}$ *induces an isomorphism between the vector space of covariant tensors of order two on* $\mathbf{X}$*, on the space* $\mathcal{S}_s(\widetilde{\mathbf{E}})$.

Proof. The points a) and b) are obvious. To prove c), let us denote

$$\alpha : \mathcal{S}(\mathbf{E}) \longrightarrow \mathcal{S}_s(\widetilde{\mathbf{E}})$$

an application defined punctually by $\alpha(\theta) = \widetilde{\theta}$, where $\widetilde{\theta}$ is chosen such that for every $u \in \mathcal{O}_2$,

$$\widetilde{\theta}_u(\widetilde{A}, \widetilde{B}) = \theta_x(p_*\widetilde{A}, p_*\widetilde{B}), \tag{1}$$

where $x = p(u)$ and p is the dual application of p^*, p being a local diffeomorphism. Then

$$p_* : T_u\mathcal{O}_2 \longrightarrow T_x(\mathbf{X})$$

is an isomorphism, therefore α is correctly defined and it is a one-to-one application.

Let us show now that $\widetilde{\theta}$ is h-invariant, i.e., it belongs to $\mathcal{S}_s(\widetilde{\mathbf{E}})$. We have $p \circ h = p$ and this relation implies

$$p_* \circ h_* = p_* \tag{2}$$

where p_* is the differential.

Let p^* be the dual of p_*. Then, for every $u \in \mathcal{O}_2$ and arbitrary $\widetilde{A}, \widetilde{B}$ in $T_u\mathcal{O}_2$, we have

$$\begin{aligned}(h^*\widetilde{\theta})_u(\widetilde{A}, \widetilde{B}) &= \widetilde{\theta}_{hu}(h_*\widetilde{A}, h_*\widetilde{B}) = \widetilde{\theta}_u(h_*\widetilde{A}, h_*\widetilde{B}) \\ &= \theta_x(p_*h_*\widetilde{A}, p_*h_*\widetilde{B}) = \theta_x(p_*\widetilde{A}, p_*\widetilde{B}),\end{aligned}$$

where $x = p(u) = p(hu)$. Hence, indeed $h^*\widetilde{\theta} = \widetilde{\theta}$.

It remains to prove that α is onto. Let $\widetilde{\theta} \in \mathcal{S}_s(\widetilde{\mathbf{E}})$ be arbitrary. We chose θ in $\mathcal{S}(\mathbf{E})$ such that for every couple $A, B \in T_x\mathbf{X}$,

$$\theta_x(A, B) = \widetilde{\theta}_u(\widetilde{A}, \widetilde{B}), \tag{3}$$

where $x = p(u)$ and where $\widetilde{A}, \widetilde{B} \in T_u\mathcal{O}_2$ are uniquely defined by the isomorphism p_* such that $p_*\widetilde{A} = A$ and $p_*\widetilde{B} = B$.

Let us show that θ_x is correctly defined, i.e., it does not depend on the choice of u for which $x = p(u)$. Indeed, θ being h-invariant, we have

$$\widetilde{\theta}_u(\widetilde{A}, \widetilde{B}) = \widetilde{\theta}_{hu}(h_*\widetilde{A}, h_*\widetilde{B}). \tag{4}$$

On the other hand, taking into account (2), we have

$$p_*h_*\widetilde{A} = p_*\widetilde{A} = A \text{ and } p_*h_*\widetilde{B} = p_*\widetilde{B} = B. \tag{5}$$

The relations (3), (4) and (5) imply

$$\theta_x(A, B) = \widetilde{\theta}_{hu}(h_*\widetilde{A}, h_*\widetilde{B}), \tag{6}$$

which shows that θ_x is indeed correctly defined. The $\mathbb{R}$-linearity of α implies that it is an isomorphism.

We recall that a Riemannian metric g on $\mathbf{X}$ is a symmetric, doubly covariant, positively defined tensor field at every point of $\mathbf{X}$. The set of Riemannian metrics on $\mathbf{X}$ is denoted by $\mathcal{M}(\mathbf{X})$. Every metric $g \in \mathcal{M}(\mathbf{X})$ is a section of the vector bundle E such that at every point $x \in \mathbf{X}$,

$$g(x) \in \mathcal{J}_x(\mathbf{X}) \subset S^2(T^*\mathbf{X}),$$

where $\mathcal{J}_x(\mathbf{X})$ is the convex cone of bilinear symmetric and positively defined forms on $T_x(\mathbf{X})$.

The previous theorem implies that there is an isomorphism between the cone of h-invariant Riemannian metrics on $\mathcal{O}_2$ and the cone $\mathcal{M}(\mathbf{X})$. From the same theorem it results that every Riemannian metric $\widetilde{g}$ on $\mathcal{O}_2$ has a *canonical decomposition*

$$\widetilde{g} = \widetilde{g}_s + \widetilde{g}_a$$

where $\widetilde{g}_s$ is a h-invariant Riemannian metric and $\widetilde{g}_a \in \mathcal{S}_a(\widetilde{\mathbf{E}})$.

The involution h is an isometry with respect to the component $\widetilde{g}_s$ of g and the projection p is a *local isometry* between the Riemannian manifolds $(\mathcal{O}_2, \widetilde{g}_s)$ and $(\mathbf{X}, g)$, where g is the element of $\mathcal{M}(\mathbf{X})$ corresponding to g_s by the isomorphism mentioned in the previous theorem.

Example 1. Let $\mathbf{M} = \{z = x + iy \mid y > 0, -1/2 < x \leq 1/2,\ |z| > 1\} \cup \{e^{i\phi} \mid \pi/3 \leq \phi \leq \pi/2\}$. For every $\tau \in \mathbf{M}$, let us denote by Σ_τ the lattice $\{m + n\tau \mid m, n \in \mathbb{Z}\}$ identified with the group of translations of the complex plane $\mathbb{C}$, $z \longrightarrow z + m + n\tau$.

Let us denote by $\mathbf{G}$ the group generated by the mappings $U, V : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$U(z) = z + \tau; \quad V(z) = \bar{z} + \frac{1}{2}.$$

Let $\mathbf{G}_1$ be the subgroup of $\mathbf{G}$ formed with the conformal mappings. It can be easily checked that $\mathbf{G}$ consists of all translations with elements from Σ_τ and by consequence the orbit space $\mathbb{C}/\mathbf{G}_1 := T(\tau)$ is a torus.

It is known (see [**4**]) that every torus is conformally equivalent to a torus $T(\tau)$ and that $T(\tau)$ is symmetric if and only if $\operatorname{Re}(\tau) = 0$ (see [**3**] and [**7**]).

If $T(\tau)$ is symmetric then it represents the orientable double cover of the Klein bottle $\mathbb{C}/\mathbf{G}$.

The Euclidian metric $g = |dz|^2$ in the complex plane $\mathbb{C}$ being invariant with respect to both U and V, is invariant with respect to $\mathbf{G}_1$, so it induces a Riemannian metric on the torus $\mathbb{C}/\mathbf{G}_1$. Such a metric can be defined by $\tilde{g}(\dot{z}) = |dz|^2$, where $\dot{z}$ is the orbit of z with respect to $\mathbf{G}_1$:

$$\dot{z} = z + \Sigma_\tau = \{z + m + n\tau \mid m, n \in \mathbb{Z}\}.$$

It is obvious that $h : T(\tau) \longrightarrow T(\tau)$ defined by

$$h(\dot{z}) := \bar{z} + \frac{1}{2} + \Sigma_\tau$$

is an antianalytic involution. The h-invariant component of $\tilde{g}$ is given by

$$\begin{aligned}\tilde{g}_s(\dot{z}) &= \frac{1}{2}\left[\tilde{g}(\dot{z}) + \tilde{g}(h(\dot{z}))\right] = \frac{1}{2}\left[|dz|^2 + |d(\bar{z} + m + n\tau)|^2\right] \\ &= \frac{1}{2}(|dz|^2 + |d\bar{z}|^2) = |dz|^2 = \tilde{g}(\dot{z}).\end{aligned}$$

By consequence $\tilde{g}$ is a symmetric metric. We should have expected this result since the Riemannian metric induced by g on the Klein bottle $\mathbb{C}/\mathbf{G}$ should be the same, given its obvious invariance with respect to $\mathbf{G}$.

Example 2. Let $R > 1$ and let $\mathbf{A}_R$ be the annulus $\{z \in \mathbb{C} \mid 1/R < |z| < R\}$. Let $h : \mathbf{A}_R \longrightarrow \mathbf{A}_R$ defined by $h(z) = -1/\bar{z}$. $(\mathbf{A}_R, h)$ is a symmetric Riemann surface and the quotient space $\mathbf{A}_R/\langle h \rangle$ is a Möbius strip. The Euclidian metric

$$\widetilde{g} = ds^2 = dx^2 + dy^2$$

is *not* h-invariant. Its h-invariant component is

$$\widetilde{g_s} = d\sigma^2 = \frac{1}{4}\Big(1 + \frac{1}{x^2 + y^2}\Big)^2 (dx^2 + dy^2).$$

Indeed, with the notations $z = x + iy$ and $w = u + iv = -1/\bar{z}$ we have:

$$d\sigma = \frac{1}{2}\,(ds + ds \circ h) \quad \text{that is} \quad d\sigma(z) = \frac{1}{2}\,(|dz| + |dw|) = \frac{1}{2}\Big(1 + \frac{1}{|z|^2}\Big)|dz|,$$

equality which means exactly the previous formula for $\widetilde{g_s}$.

Example 3. One denotes, as usual, by $\hat{\mathbb{C}}$ the extended complex plane, identified with the Riemann sphere. Let $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ defined by $h(z) = -1/\bar{z}$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$). The pair $(\hat{\mathbb{C}}, h)$ is a symmetric Riemann surface and the quotient space $\mathbb{P}^2 := \widehat{\mathbb{C}}/\langle h \rangle$ is the real projective plane.

The *natural* (Riemannian) metric on $\hat{\mathbb{C}}$ is the spheric metric $\tilde{g} = da^2$ defined by

$$da(z) = \frac{|dz|}{1+|z|^2} \,.$$

The h-invariant component of $\tilde{g}$ is $\tilde{g_s} = d\sigma^2$, where

$$d\sigma(z) = \frac{1}{2}\left(\frac{|dz|}{1+|z|^2} + \frac{|dh(z)|}{1+|h(z)|^2}\right) = da(z).$$

This last equality means that the spheric metric on $\hat{\mathbb{C}}$ is h-invariant and therefore

$$\tilde{g} = \tilde{g_s} = \frac{dx^2 + dy^2}{(1+x^2+y^2)^2}.$$

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