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EXISTENCE RESULT FOR THE DISPLACEMENT FIELD OF ELASTIC BODY. THE CASE OF A LOCKING SUPPORT

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Abstract. The problem of existence result for displacement field of elastic body in contact with locking support is considered in this paper. Mathematically the problem is refered to, so called, hemivariational inequalities [**4**]. The existence result is obtained by making use of the theory of pseudo–monotone operators as in [**4**] or [**1**].

1. Introduction

Many inequality problems in mechanics are formulated not only as variational inequalities, but also in terms of, so called, hemivariational inequalities [**5**], [**4**]. Hemivariational inequalities are derived from nonconvex nondifferentiable superpotentials by making use of the generalized gradient introduced by Clarke [**2**].

This paper deals with the existence of solutions of problem related to hemivariational inequalities which correspond to superpotential on the boundary of the body having their nonconvex and nondifferentiable part with infinite branches on closed and convex subsets. The theory of pseudo–monotone set–valued mappings introduced by Browder and Hess [**1**] is the main tool of the problem under consideration.

For the reader's convenience let us recall some definitions (all of them can be found in [**4**]).

We denote by V a reflexive Banach space with dual V^*. The pairing over $V^* \times V$ and a norm on V will be denoted by $\langle \cdot, \cdot \rangle_V$ and $\| \cdot \|_V$ respectively.

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Let the convex function $f : V \to \bar{R} = R \cup \{+\infty\}$ be not everywhere differentiable. A vector $u^* \in V^*$ for which $f(v) - f(u) \geq \langle u^*, v - u\rangle_V$ for all $v \in V$ hold, where $f(v)$ is finite, is called a *subgradient* of f at u and the set of such vectors is denoted by $\partial f(u)$.

The directional derivative $f^0(u, v)$ of Clarke at u in the direction v is given for f Lipschitzian near u by the expression

$$f^0(u, v) = \limsup_{\substack{h \to 0 \\ \mu \to 0+}} \frac{1}{\mu} \left[f(u + h + \mu v) - f(u + h) \right]$$

and *the generalized gradient* $\bar{\partial} f(u)$ of f at $u \in V$, $f(u)$ finite, is defined as $\bar{\partial} f(u) = \left\{ u^* \in V^* \ : \ f^0(u^*, v - u) \geq \langle u, v - u\rangle_V \ \text{ for all } \ v \in V \right\}$. If f is convex, then $\bar{\partial} f(\,\cdot\,) = \partial f(\,\cdot\,)$.

Let T be a mapping from a real reflexive Banach space V into 2^{V^*}. Then T is said to be *pseudo–monotone* if the following conditions hold:

a) The set Tu is nonempty, bounded, closed and convex for all $u \in V$.

b) T is upper semicontinuous for each finite dimensional subset F of V to the weak topology of V^*.

c) If $\{u_i\}$ is a sequence in V converging weakly to u, and if $u_i^* \in Tu_i$ is such that $\limsup \langle u_i^*, u_i - u\rangle_V \leq 0$, then to each element $v \in V$ there exists $u^*(v) \in Tu$ with the property that $\liminf \langle u_i^*, u_i - u\rangle_V \geq \langle u^*(v), u - v\rangle_V$.

Let T be a mapping from V into V^*. Then T is said to be *quasi–bounded* if for each $M > 0$ there exists $K(M) > 0$ such that, whenever (u, u^*) lies in the graph $G(T) = \left\{ (v, v^*) \in V \times V^* \ : \ v^* \in Tv \right\}$ of T and $\langle u^*, u\rangle_V \leq M \, \|u\|_V$, $\|u\|_V \leq M$, then $\|u^*\|_{V^*} \leq K(M)$.

Let T be a mapping from V into V^*. Then T is said to be *strongly quasi–bounded* if for each $M > 0$ there exists $K(M) > 0$ such that for all $(u, u^*) \in G(T)$ with $\langle u^*, u\rangle_V \leq M$, $\|u\|_V \leq M$ we have $\|u^*\|_{V^*} \leq K(M)$.

2. Classical Formulation

Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$, $n = 2$ or 3, occupied by a linear elastic body in its undeformed state. The body is refered to an orthogonal Cartesian coordinate system. The boundary Γ of Ω is assumed to be Lipschitzian.

In the framework of linear elasticity and small deformations, the following relations hold

$$\div \sigma + f = 0 \qquad \text{in } \Omega \tag{1}$$

$$2\, \varepsilon(u) = \nabla u + (\nabla u)^T \qquad \text{in } \Omega \tag{2}$$

$$\sigma = \mathbb{C}\, \varepsilon(u) \qquad \text{in } \Omega \tag{3}$$

where $\sigma = \{\sigma_{ij}\}$ (resp. $\varepsilon = \{\varepsilon_{ij}\}$), $i, j = 1, \ldots, n$, is the stress (resp. strain) tensor and $\mathbb{C}$ is Hooke's elasticity tensor fulfilling the well known ellipticity and symmetry properties [**4**]. Moreover, let $u = \{u_i\}$ and $f = \{f_i\}$, $i = 1, \ldots, n$ be the displacement and volume force respectively. Further, let $\Gamma = \bar{\Gamma}_U \cup \bar{\Gamma}_F \cup \bar{\Gamma}_S$ with properties: $\Gamma_U \cap \Gamma_F \cap \Gamma_S = \emptyset$ and $\operatorname{meas}(\Gamma_U) > 0$.

We assume that

$$u = 0 \qquad \text{on } \Gamma_U\,, \tag{4}$$

$$\sigma\, n = F \qquad \text{on } \Gamma_F\,, \tag{5}$$

where $n = \{n_i\}$, $i = 1, \ldots, n$, is the outward unit normal vector to Γ_F.

Let K be a given convex and closed subset of displacement vector field of points on Γ_S, and let $I_K(\,\cdot\,)$ be indicator function for the displacement vectors on Γ_S, i.e., $I_K(v) = 0$ if $v \in K$ and $I_K(v) = +\infty$ if $v \notin K$.

Here the nonmonotone multivalued reaction–displacement law with infinite branches on K will be defined by a nonmonotone superpotential $j(x, u)$; then the boundary conditions on Γ_S is given by [**4**]:

$$-S \in \bar{\partial} j(\,\cdot\,, u) + \partial I_K(u) \qquad \text{on } \Gamma_S\,. \tag{6}$$

The relation (6) describes the adhesive contact with a locking support, e.g. a rubber support with limited compressibility [**4**].

Now we can formulate the classical **problem (P)** for displacement field as: For given f, $\mathbb{C}$, F, j and K find the displacement field $u(x)$, $x \in \bar{\Omega}$, such that the relations (1)–(6) will be satisfied.

3. Variational Formulation

To give the variational, i.e. hemivariational formulation of the classical problem (P) we introduce the kinematically admisible space

$$V = \left\{ v \in \left(H^1(\Omega)\right)^n \;:\; v = 0 \text{ on } \Gamma_U \right\}. \tag{7}$$

From (1) and (2) we obtain the variational equality (by application of the Green–Gauss theorem):

$$\begin{aligned} \int_\Omega \sigma \cdot \big(\varepsilon(v) - \varepsilon(u)\big)\, d\Omega &= \int_\Omega f \cdot (v - u)\, d\Omega + \int_{\Gamma_F} F \cdot (v - u)\, d\Gamma \\ &+ \int_{\Gamma_S} S \cdot (v - u)\, d\Gamma \qquad \text{for all } v \in V\,. \end{aligned} \tag{8}$$

The relation (6) is equivalent to the following one:

(9)
$$\xi \in \mathbb{R}^n\,, \qquad j^0(x,\,\xi;\,\eta-\xi) + I_K(\eta) - I_K(\xi) \geq (-S)\,(\eta-\xi) \qquad \text{for all } \eta \in \mathbb{R}^n\,.$$

Then, by (9), the equality (8) becomes an hemivariational inequality of the form:

(10)
$$a(u,\,v-u) + I_K(v) - I_K(u) + \int_{\Gamma_S} j^0(x,\,u(x);\,v(x)-u(x))\,d\Gamma \geq l(v-u) \qquad \text{for all } v \in V\,,$$

where, by definition

(11)
$$a(u,\,v) = \int_\Omega \mathbb{C}\varepsilon(u)\cdot\varepsilon(v)\,d\Omega\,,$$

(12)
$$l(v) = \int_\Omega f\cdot v\,d\Omega + \int_{\Gamma_F} F\cdot v\,d\Gamma\,.$$

If we introduce the linear operator $A : V \to V^*$ as $\langle Au,\,v\rangle_V = a(u,\,v)$, and the linear functional $g : V \to \mathbb{R}$ by $\langle g,\,v\rangle_V = l(v)$ then we can formulate the following hemivariational **problem (V)**: For given $f \in \big(L_2(\Omega)\big)^n$, $\mathbb{C} \in \big(L_\infty(\Omega)\big)^{n\times n\times n\times n}$, $F \in \big(L_2(\Gamma_F)\big)^n$, j and $K \subset \big(L_2(\Gamma_S)\big)^N$, $N \geq 1$, find the displacement field $u(x)$, $x \in \bar{\Omega}$, such that

(13)
$$\langle Au-g,\,v-u\rangle_V + I_K(v) - I_K(u) + \int_{\Gamma_S} j^0(x,\,u(x);\,v(x)-u(x))\,d\Gamma \geq 0 \qquad \text{for all } v \in V\,.$$

The functional $J_S : L_2(\Gamma_S;\,\mathbb{R}^N) \to \mathbb{R}$, $N \geq 1$, indicated in (13), defined by

(14)
$$J_S(v) = \int_{\Gamma_S} j(x,\,v(x))\,d\Gamma\,,$$

to be locally Lipschitz on $L_2(\Gamma_S;\,\mathbb{R}^N)$, for the function $j : \Gamma_S \times \mathbb{R}^N \to \mathbb{R}$, the following conditions are introduced [**2**]:

(i) for all $\xi \in \mathbb{R}^N$ the function $x \to j(x,\,\xi)$ is measurable on Γ_S;

(ii) for almost all $x \in \Gamma_S$ the function $\xi \to j(x,\,\xi)$ is locally Lipschitz on $\mathbb{R}^N$;

(iii) the function $j(\,\cdot\,,0)$ is finitely integrable on Γ_S, i.e., $j(\,\cdot\,,0) \in L_1(\Gamma_S)$;

(iv) for almost all $x \in \Gamma_S$ and each $\xi \in \mathbb{R}^N$: $-S \in \bar{\partial}j(x,\,\xi) \Rightarrow |S| \leq c\,(1+|\xi|)^{p-1}$ for some constant $c > 0$ not depending on $x \in \Gamma_S$. (In our example $p = 2$).

Moreover, we suppose that

(v) for almost all $x \in \Gamma_S$ and each $\xi \in \mathbb{R}^N$: $j^0(x,\, \xi;\, -\xi) \leq \beta(x)\,(1 + |\xi|^s)$, where $0 \leq s < 2$ and $\beta(\,\cdot\,)$ is a nonnegative function from $L_q(\Gamma_S)$ with $q = p/(p-s)$. (In our example $p = 2$).

Then the hemivariational inequality (13) can be presented in the form:

$$(15)\quad \langle Au - g,\, v - u\rangle_V + I_K(v) - I_K(u) + J_S^0(i\,u;\, i\,v - i\,u) \geq 0 \qquad \text{for all } v \in V\,,$$

where i is the compact injection from $Y(\Gamma_S) = \big(H^{1/2}(\Gamma_S)\big)^n$ into $L_2(\Gamma_S;\, \mathbb{R}^N)$.

The dual of $Y(\Gamma_S)$ is denoted by $Y^*(\Gamma_S)$, i.e. $Y^*(\Gamma_S) = \big(H^{-1/2}(\Gamma_S)\big)^n$.

4. Existence

The hemivariational inequality (15) is equivalent to the inclusion [**4**]:

$$(16)\qquad g \in Au + \partial I_K(u) + \bar{\partial}^i J_S(u)\,,$$

where

$$(17)\quad \bar{\partial}^i J_S(u) = \big\{\chi \in Y^*(\Gamma_S)\ :\ J_S^0(i\,u,\, i\,v) \geq \langle \chi,\, v\rangle_{Y(\Gamma_S)} \text{ for all } v \in Y(\Gamma_S)\big\}\,.$$

The theorem below provides conditions which guarantee the existence of solution to the problem (V). This theorem is a slight modification of theorems 4.28 and 4.26 in [**4**].

Theorem 1. *Let A be a pseudo–monotone operator from the reflexive Banach space V into V^*. Let us suppose that the injection $Y(\Gamma_Q) \subset L_p(\Gamma_Q;\, \mathbb{R}^N)$, $N \geq 1$, $\Gamma_Q \subset \Gamma$, is compact for some $2 \leq p < \infty$, and that $j : \Gamma_Q \times \mathbb{R}^N \to \mathbb{R}$ fulfills the requirements (i)–(v). Further, assume that the functional $\varphi_Q : Y(\Gamma_Q) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and proper. Suppose that the following hypotheses hold:*

(H1) $u_0 \in Dom\,(\partial\varphi_Q)$;

(H2) either A_{u_0} ($A_{u_0} v = A(v + u_0)$) is quasibounded or $\partial\varphi_{Q\,u_0}$ is strongly quasibounded;

(H3) there exists a function $c : \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \approx r$ as $r \to \infty$, such that for all $v \in V$ and $\chi \in \bar{\partial}^i F_Q(v)$:

$$\langle Av,\, v - u_0\rangle_V + \langle \chi,\, v - u_0\rangle_{Y(\Gamma_Q)} \geq c(\|v\|_V)\, \|v\|_V\,,$$

where the functional $F_Q : L_p(\Gamma_Q;\, \mathbb{R}^N) \to \mathbb{R}$, $N \geq 1$, is defined by $F_Q(v) = \int_{\Gamma_Q} j(x,\, v)\, d\Gamma$, and $\bar{\partial}^i F_Q(v)$ as $\bar{\partial}^i J_S(u)$ by (17).

Then the hemivariational inequality

(18) $\langle Au - g,\, v - u\rangle_V + \varphi_Q(v) - \varphi_Q(u) + F_Q^0(i\,u;\, i\,v - i\,u) \geq 0 \qquad$ *for all* $v \in V$

has at least one solution.

Proof: Hemivariational inequality (18) can be written equivalently as $g \in Au + \partial\varphi_Q(u) + \bar{\partial}^i F_Q(u)$. Thus the problem is reduced to the question whether g belongs to the range of $A + \partial\varphi_Q + \bar{\partial}^i F_Q$. The operator $\partial\varphi_Q$ is maximal monotone by properties of the functional φ_Q [**3**]. Similarly as in theorem 4.23 in [**4**] we can prove that the operator $\bar{\partial}^i F_Q$ is pseudo–monotone. Since $A + \bar{\partial}^i F_Q$ is coercive, $A + \bar{\partial}^i F_Q + \partial\varphi_Q$ is coercive too, because of the existence of an affine minorant of φ_Q. Then theorem 2.12 in [**4**] implies that the range of $A + \bar{\partial}^i F_Q + \partial\varphi_Q$ coincides with the whole V^*. This establishes the existence of solution of (18). $\square$

All conditions of Theorem 1 for the problem (V) are fulfilled. Obviously $\Gamma_Q = \Gamma_S$, $\varphi_Q = I_K$, $p = 2$ and $F_Q = J_S$, and it remains to verify the hypotheses (H2) and (H3). The operator $A_0 = A$ is quasi–bounded ($u_0 = 0$, for example) because the bilinear form $a(u, v)$ is bounded. By the ellipticity, Korn's inequality and the estimate

$$\langle \chi,\, v\rangle_{Y(\Gamma_S)} \leq \|\chi\|_{Y^*(\Gamma_S)}\, \|v\|_{Y(\Gamma_S)} \leq c\, \|v\|_V\,,$$

the hypothesis (H3) is fulfiled.

Then the problem (V) has at least one solution.

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