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A NEW PROOF OF A THEOREM OF BELOV

Ljubica S. Velimirović

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Abstract. Belov in [**2**] gave necessary and sufficient condition for rotational surface generated by a special quadrangular meridian, to be rigid. Belov's theorem disproved the hypothesis of Boyarski that each toroid rotational surface with convex meridian is rigid. We give another proof of Belov's theorem. The field of infinitesimal bendings is determined, the rotational field is obtained too. The method, used here, can be applied in a case of every rotational surface generated by a simple polygon [**6**] .

0. Introduction

One of the basic problems of infinitesimal bendings theory in E_3 is to point out nonrigid surfaces, and, if possible, to determine the infinitesimal bending field and the field of rotations.

It is known [**3**] that a circular torus is rigid. Among surfaces topologically equivalent to the torus Belov [**2**] pointed out a class of nonrigid toroids with quadrangular meridian of a special form, which can be convex or nonconvex.

We give a new proof of Belov's theorem by using matrices. Such a proof of Belov's theorem makes possible to determine the infinitesimal bending field and the rotational field when the surface is nonrigid. The Belov's proof is only a proof of the existence of the bending field. Using this procedure we have proved the rigidity of toroid rotational surface with triangular meridian, not containing a side orthogonal to axis of rotation (see [**5**]). This procedure is also applied in the case of a toroid with a meridian in the form of a simple polygon [**6**].

We quote some well-known definitions and properties that we shall use afterwards. At first, we define infinitesimal deformation of a surface, and then infinitesimal bending, as a particular case ([**1**], [**3**], [**4**]).

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Definition 1. Let S be a regular surface of the class C^m $(m \geq 3)$, defined in a vector form

$$S : \bar{r} = \bar{r}(u, v),$$

included in a family of surfaces

$$S_\varepsilon : \bar{r}_\varepsilon(u, v) = \bar{r}(u, v) + \varepsilon \bar{z}(u, v), \tag{0.1}$$

where $\varepsilon(\varepsilon \to 0)$, $u, v \in R^1$ and $\bar{r}_0(u, v) = \bar{r}(u, v)$. It is said that the surfaces S_ε are **infinitesimal deformations** of the surface S.

Definition 2. The surfaces S_ε are **infinitesimal bendings** of a surface S if for line elements of these surfaces we have

$$ds_\varepsilon^2 - ds^2 = O(\varepsilon). \tag{0.2}$$

Infinitesimal bending can be defined in another manner, too, which follows from the following theorem:

THEOREM 1. [**1,3,4**] *The condition* (0.2) *is equivalent to each of the conditions*

$$ds_\varepsilon - ds = O(\varepsilon), \tag{0.3}$$

$$d\bar{r} d\bar{z} = 0, \tag{0.4}$$

$$\frac{\partial s_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = 0, \tag{0.5}$$

$$\bar{r}_u \bar{z}_u = \bar{r}_v \bar{z}_v = \bar{r}_u \bar{z}_v + \bar{r}_v \bar{z}_u = 0. \tag{0.6}$$

Definition 3. A vector field $\bar{z}(u, v)$, defined at points of the surface S satisfying the conditions of Definition 2, i.e., any of the conditions (0.2)–(0.6), is an *infinitesimal bending field* of the surface S.

The field

$$\bar{z} = \bar{a} \times \bar{r} + \bar{b}, \tag{0.7}$$

where $\bar{a}, \bar{b}$ are constant vectors and $\times$ denotes a vector product, satisfies identically (0.4). It is known [**3**] that the vector $\bar{z}$, in this case, induces only a motion of the surface as a solid body, without intrinsic deformations.

Definition 4. The field $\bar{z}$ of the form (0.7) is a *trivial* infinitesimal bending field or infinitesimal motion field.

If $\bar{z} = \overline{\text{const}}$, then we have a particular case of (0.7).

Definition 5. If a surface possesses only a trivial infinitesimal bending field, then it is **rigid**, otherwise it is nonrigid.

1. A new proof of Belov's theorem

THEOREM 2. (Belov [**2**]) *The quadrangle $\mathcal{B}$, with apexes $A(-1,b)$, $B(0,b+c_1)$, $C(1,b)$, $D(0,b-c_2)$, rotates around u-axis of the coordinate system $u0\rho$. A necessary and sufficient condition for nonrigidity of the toroid rotational surface generated by the meridian $\mathcal{B}$ is*

$$1/c_2 - 1/c_1 = k^2/b, \tag{1.1}$$

where $k \geq 2$ is an integer.

Proof. Designating ρ on sides $AB,\dots,DA$ with $\rho_{(1)}$, $\rho_{(2)}$, $\rho_{(3)}$, $\rho_{(4)}$, we get

$$\begin{aligned} \rho_{(1)} &= b + c_1\,(u+1)\,, \quad \rho_{(2)} = b + c_1\,(1-u)\,, \\ \rho_{(3)} &= b + c_2\,(u-1)\,, \quad \rho_{(4)} = b - c_2\,(u+1) \end{aligned} \tag{1.2}$$

If $\bar{e}$ is the ort of the u-axis, $\bar{a}(v)$ the ort of ρ-axis, v the angle between $\bar{a}$ and the plane of initial position of meridian $\rho = \rho\,(u)$, then [**1**, p. 90] the rotational surface generated by the meridian $\rho = \rho\,(u)$ at the coordinate system with orthonormal base $\bar{e}$, $\bar{a}\,(v)$, $\bar{a}'\,(v)$ is:

$$\bar{r}(u,v) = u\bar{e} + \rho(u)\bar{a}(v). \tag{1.3}$$

It is known, [**1**, p. 91], that for every $k \in \{2,3,\dots\}$ there is a field of infinitesimal bendings

$$\begin{aligned} \bar{z}(u,v) = \bar{z}_k(u,v) &= \left[\varphi_k(u)e^{ikv} + \tilde{\varphi}_k(u)e^{-ikv}\right]\bar{e} \\ &+ [\psi_k(u)e^{ikv} + \tilde{\psi}_k(u)e^{-ikv}]\bar{a}(v) + \left[\chi_k(u)e^{ikv} + \tilde{\chi}_k(u)e^{-ikv}\right]\bar{a}'(v) \end{aligned} \tag{1.4}$$

for the surface (1.3), where e.g., $\tilde{\varphi}_k\,(u)$ is the conjugate value for $\varphi_k(u)$. The functions $\psi_k\,(u)$ and $\chi_k\,(u)$ satisfy the same equation

$$\rho(u)\lambda''\,(u) + (k^2-1)\rho''\,(u)\,\lambda\,(u) = 0, \tag{1.5}$$

and also the equations

$$\begin{aligned} \varphi_k'(u) + \rho'(u)\psi_k'(u) = 0,\ \psi_k\,(u) + ik\chi_k\,(u) = 0, \\ ik\varphi_k\,(u) + \rho'\,(u)\,[ik\psi_k\,(u) - \chi_k\,(u)] + \rho\,(u)\,\chi_k'\,(u) = 0. \end{aligned} \tag{1.6}$$

are valid.

At turning points $u = \sigma$ of the meridian $\rho = \rho(u)$ we have [**1**, p. 112, eq. (15)]:

$$\rho(\sigma)[\psi_k'(\sigma+0) - \psi_k'(\sigma-0)] + (k^2-1)\psi_k(\sigma)[\rho'(\sigma+0) - \rho'(\sigma-0)] = 0, \tag{1.7}$$

supposing that the field $\bar{z}_k(u)$ is continuous at these points. We have analogous equation for $\chi_k(u)$.

Omitting the index k, we designate $\psi_k(u)$ by $\psi(u)$ and by $\psi_{(1)}(u), \dots, \psi_{(4)}(u)$ the corresponding values on the sides $AB, \dots, DA$ respectively. According to (1.5) and (1.2) we have the linearity of the functions $\psi_{(i)}(u)$, i.e.,

$$\psi_{(i)}(u) = M_i u + N_i, \quad (i = 1, \dots, 4). \tag{1.8}$$

From the continuity of the functions $\psi_{(i)}(u)$ at the apexes we have

$$\psi_{(1)}(-1) = \psi_{(4)}(-1),\ \psi_{(1)}(0) = \psi_{(2)}(0),\ \psi_{(2)}(1) = \psi_{(3)}(1),\ \psi_{(3)}(0) = \psi_{(4)}(0),$$

from which one gets

$$-M_1 + N_1 = -M_4 + N_4,\ \ N_1 = N_2,\ \ M_2 + N_2 = M_3 + N_3,\ \ N_3 = N_4. \tag{1.9}$$

At the apex A, according to (1.7), we will replace $\psi'_k(\sigma - 0)$ with $\psi'_{(4)}(-1)$ and $\psi'_k(\sigma + 0)$ with $\psi'_{(1)}(-1)$. Analogously at the apex C we will replace $\psi'_k(\sigma - 0)$ with $\psi'_{(2)}(1)$ and $\psi'_k(\sigma + 0)$ with $\psi'_{(3)}(1)$. In such a way we have at A:

$$\rho(-1)[\psi'_{(1)}(-1) - \psi'_{(4)}(-1)] + (k^2 - 1)\psi_{(1)}(-1)[\rho'_1(-1) - \rho'_4(-1)] = 0,$$

i.e.,

$$b(M_1 - M_4) + (k^2 - 1)(N_1 - M_1)(c_1 + c_2) = 0, \tag{1.10 a}$$

and for the apexes B, C, D:

$$(b + c_1)(M_2 - M_1) - 2(k^2 - 1)c_1 N_1 = 0, \tag{1.10 b}$$

$$b(M_3 - M_2) + (k^2 - 1)(c_1 + c_2)(M_2 + N_2) = 0, \tag{1.10 c}$$

$$(b - c_2)(M_4 - M_3) - 2(k^2 - 1)c_2 N_3 = 0. \tag{1.10 d}$$

We consider equations (1.9) and (1.10) as a system with respect to M_i, N_i $(i = 1, \dots, 4)$. At first, let us consider (1.9) as a system with respect to N_i. The extended matrix of this system is

$$P = \begin{array}{c} \begin{array}{rrrrcc} N_1 & N_2 & N_3 & N_4 & & \end{array} \\ \left[\begin{array}{rrrrcc} 1 & 0 & 0 & -1 & \vdots & M_1 - M_4 \\ 1 & -1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & -1 & 0 & \vdots & -M_2 + M_3 \\ 0 & 0 & 1 & -1 & \vdots & 0 \end{array}\right] \end{array} \sim \tag{1.11}$$

$N_1 \quad N_2 \quad N_3 \quad N_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & \vdots & M_1 - M_4 \\ 0 & -1 & 0 & 1 & \vdots & -M_1 + M_4 \\ 0 & 0 & -1 & 1 & \vdots & -M_1 - M_2 + M_3 + M_4 \\ 0 & 0 & 0 & 0 & \vdots & -M_1 - M_2 + M_3 + M_4 \end{bmatrix},$$

where M is the matrix of the system (1.9) (with respect to N_i formed by first four columns of P). The system (1.9) is compatible if and only if $\operatorname{rank} P = \operatorname{rank} M$, i.e.

$$-M_1 - M_2 + M_3 + M_4 = 0 \Leftrightarrow M_4 = M_1 + M_2 - M_3. \tag{1.12}$$

From (1.11,12) we get a reduced system,

$$N_1 - N_4 = M_1 - M_4, \quad -N_2 + N_4 = -M_1 + M_4, \quad -N_3 + N_4 = 0$$

and substituting M_4 from (1.12), we get

$$N_1 = N_2 = -M_2 + M_3 + N_4, \quad N_3 = N_4. \tag{1.13}$$

Substituting (1.12,13) at (1.10) we get a homogeneous system with unknowns M_1, M_2, M_3, N_4:

$$\begin{gathered} -(c_1 + c_2)(k^2 - 1)M_1 - [(c_1 + c_2)(k^2 - 1) + b]M_2 \\ + [(c_1 + c_2)(k^2 - 1) + b]M_3 + (c_1 + c_2)(k^2 - 1)N_4 = 0, \\ -(b + c_1)M_1 + [2c_1(k^2 - 1) + b + c_1]M_2 - 2c_1(k^2 - 1)M_3 - 2c_1(k^2 - 1)N_4 = 0, \\ -bM_2 + [(c_1 + c_2)(k^2 - 1) + b]M_3 + (c_1 + c_2)(k^2 - 1)N_4 = 0, \\ (b - c_2)M_1 + (b - c_2)M_2 + 2(c_2 - b)M_3 - 2c_2(k^2 - 1)N_4 = 0. \end{gathered} \tag{1.14}$$

The system (1.14) has nontrivial solutions if and only if the rank of the matrix N of this system is less then 4. We have to investigate the conditions under which that is valid. We have

$$N \sim \begin{array}{c} \begin{array}{cccc} N_4 & M_3 & M_2 & M_1 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & X & Y \end{bmatrix} \end{array}$$

where

$$\begin{aligned}
&a_{11} = (c_1 + c_2)(k^2 - 1), \quad a_{12} = (c_1 + c_2)(k^2 - 1) + b, \\
&a_{13} = (c_1 + c_2)(1 - k^2) - b, \quad a_{14} = (c_1 + c_2)(1 - k^2), \\
&a_{22} = \frac{2bc_1}{c_1 + c_2}, \quad a_{23} = b + c_1 - \frac{2bc_1}{c_1 + c_2}, \\
&a_{24} = 2c_1(1 - k^2) - b - c_1, \\
&X = -b + \frac{(c_1 + c_2)(b + c_1) - bc_2}{c_1} - \frac{(c_1 + c_2)(b + c_1)c_2k^2}{bc_1}, \\
&Y = \frac{(c_1 + c_2)b}{c_1} + \frac{(2c_1k^2 - c_1 + b)(c_1 + c_2)(c_2k^2 - b)}{bc_1}.
\end{aligned}$$

Let us consider three possible cases:

1. For $X = 0$ and $Y \neq 0$ or $X \neq 0$ and $Y = 0$ the system has only trivial solution and the surface is rigid.

2. $X = Y = 0$. From $X = 0$ we have

$$k^2c_2(b + c_1) = bc_1. \tag{1.15}$$

Substituting k^2 from this equation into the equation $Y = 0$, we have

$$b(c_2 - c_1) + c_1c_2 = 0. \tag{1.16}$$

For $c_1 = c_2$ we get $c_1c_2 = 0$, but then we have $c_1 = c_2 = 0$, which is impossible. If we find b from (1.16) and substitute in (1.15), we get $k = 1$, which is impossible too, because $k \in \{2, 3, \dots\}$. Therefore, for $X = Y = 0$ the surface is rigid.

3. If $X \neq 0$ and $Y \neq 0$, we get

$$N \sim \begin{array}{c} \begin{array}{cccc} N_4 & M_3 & M_2 & M_1 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & Y - X \end{bmatrix} \end{array} \tag{1.17}$$

where

$$Y - X = \frac{2k^2}{bc_1}(c_1 + c_2)(c_1c_2k^2 + bc_2 - bc_1).$$

The system (1.14) has nontrivial solutions, i.e., the surface is nonrigid if and only if

$$Y - X = 0 \Leftrightarrow c_1c_2k^2 + bc_2 - bc_1 = 0 \Leftrightarrow 1/c_2 - 1/c_1 = k^2/b,$$

i.e., when (1.1) is valid. The theorem is proved.

In what follows the surface satisfying the conditions of the Theorem 2 will be named Belov's surface.

2. Determination of the infinitesimal bending field

In order to determine the bending field $\bar{z}(u,v)$ by virtue of (1.4), we have to find the functions $\varphi_{(i)}(u)$, $\psi_{(i)}(u)$, $\chi_{(i)}(u)$, $i=1,\dots,4$. The following lemma is valid:

Lemma 1. *The functions $\varphi_{(i)}(u)$, $\psi_{(i)}(u)$, $\chi_{(i)}(u), i=1,\dots,4$, where the index in the brackets is related respectively to the sides AB, BC, CD, DA of the meridian, have the values*

$$\begin{aligned}&\varphi_{(1)}(u)=-c_1M_1u,\quad \psi_{(1)}(u)=(u-P)M_1,\\&\chi_{(1)}(u)=\frac{i}{k}(u-P)M_1,\end{aligned}\tag{2.1.1}$$

$$\begin{aligned}&\varphi_{(2)}(u)=-c_1M_1u,\quad \psi_{(2)}(u)=-(u+P)M_1,\\&\chi_{(2)}(u)=-\frac{i}{k}(u+P)M_1,\end{aligned}\tag{2.1.2}$$

$$\begin{aligned}&\varphi_{(3)}(u)=-c_1M_1u,\quad \psi_{(3)}(u)=\left[\frac{c_1}{c_2}(u-1)-(P+1)\right]M_1,\\&\chi_{(3)}(u)=\frac{i}{k}\left[\frac{c_1}{c_2}(u-1)-(P+1)\right]M_1,\end{aligned}\tag{2.1.3}$$

$$\begin{aligned}&\varphi_{(4)}(u)=-c_1M_1u,\ \psi_{(4)}(u)=-\left[\frac{c_1}{c_2}(u+1)+P+1\right]M_1,\\&\chi_{(4)}(u)=-\frac{i}{k}\left[\frac{c_1}{c_2}(u+1)+P+1\right]M_1,\end{aligned}\tag{2.1.4}$$

where is $M_1\neq 0$ an arbitrary constant, and

$$P=\frac{b+c_1}{c_1(k^2-1)}=\frac{bc_2+c_1c_2}{bc_1-bc_2-c_1c_2},\tag{2.2}$$

Proof. According to (1.17) for Y-X=0 we can get reduced system

$$\begin{aligned}&(c_1+c_2)(k^2-1)N_4+[(c_1+c_2)(k^2-1)+b]M_3\\&\qquad+[(c_1+c_2)(1-k^2)-b]M_2+(c_1+c_2)(1-k^2)M_1=0,\\&\frac{2bc_1}{c_1+c_2}M_3+(b+c_1-\frac{2bc_1}{c_1+c_2})M_2+[2c_1(1-k^2)-b-c_1]M_1=0,\\&M_2+M_1=0.\end{aligned}\tag{2.3}$$

From (2.3) and (1.13) we have

$$\begin{aligned} &M_2 = -M_1, \quad M_3 = \Big(\frac{c_2}{c_1} + \frac{c_1+c_2}{b}k^2\Big)M_1,\\ &N_4 = -\left[\frac{c_2}{c_1} + \frac{c_1+c_2}{b}k^2 + \frac{k^2}{k^2-1} + \frac{b}{c_1(k^2-1)}\right] M_1 = N_3. \end{aligned} \tag{2.4}$$

Further, from (1.12,13), we get

$$M_4 = -\Big(\frac{c_2}{c_1} + \frac{c_1+c_2}{b}k^2\Big)M_1, \quad N_1 = N_2 = -\frac{b+c_1}{c_1(k^2-1)}M_1. \tag{2.5}$$

From (1.1) we obtain

$$k^2 = \frac{(c_1-c_2)b}{c_1c_2}, \quad \frac{c_2}{c_1} + \frac{c_1+c_2}{b}k^2 = \frac{c_1}{c_2}, \tag{2.6}$$

$$\frac{c_2^2}{c_1} + \frac{c_1c_2+c_2^2}{b}k^2 = c_1, \quad \frac{k^2}{k^2-1} = \frac{b(c_1-c_2)}{bc_1-bc_2-c_1c_2}, \tag{2.7}$$

$$\frac{b}{c_1(k^2-1)} = \frac{bc_2}{bc_1-bc_2-c_1c_2}. \tag{2.8}$$

Using (2.6), we obtain

$$\frac{b+c_1}{c_1(k^2-1)} = \frac{bc_2+c_1c_2}{bc_1-bc_2-c_1c_2} \Rightarrow (2.2),$$

and from (2.4)–(2.8), we get

$$\begin{aligned} &M_2 = -M_1, \quad M_3 = \frac{c_1}{c_2}M_1,\\ &N_4 = N_3 = -\left(\frac{c_1}{c_2} + \frac{bc_1}{bc_1-bc_2-c_1c_2}\right) M_1 = -\left(\frac{c_1}{c_2} + P + 1\right) M_1. \end{aligned} \tag{2.9}$$

Further, by virtue of (1.12,13), (2.2) we have

$$M_4 = -\frac{c_1}{c_2}M_1, \quad N_1 = N_2 = -PM_1. \tag{2.10}$$

From (2.9,10) the magnitudes M_i N_i $(i = 1, \dots, 4)$ are expressed by M_1 (undetermined constant). Consequently, from (1.8) one gets the values of $\psi_{(i)}(u)$, given in (2.1).

The functions $\psi_{(i)}(u) = \psi_{(i)k}(u)$ are defined by (2.1, 2) for $k \in \{2, 3, \dots\}$. According to (1.6) we get the functions $\varphi_{(i)k}(u) = \varphi_{(i)}(u)$, $\chi_{(i)k}(u) = \chi_{(i)}(u)$, given at (2.1). The Lemma is proved.

As we mentioned, the problem of infinitesimal bendings can be considered solved only if the infinitesimal bendings field $\bar{z}(u,v)$ is determined, as in this case by virtue of (0.1) the deformed surface S_ϵ can be presented. In this case we have:

THEOREM 3. *The infinitesimal bendings field $\bar{z}(u,v)$ of Belov's toroid is defined by the equations*

$$\begin{aligned}\bar{z}_{(1)k}(u,v) = &-2c_1M_1u\cos kv\,\bar{e} + 2\,(u-P)\,M_1\cos kv\bar{a}(v)\\ &+2\,(P-u)\,\frac{M_1}{k}\sin kv\bar{a}'(v), \quad u\in[-1,0],\ v\in[0,2\pi],\end{aligned} \tag{2.11.1}$$

$$\begin{aligned}\bar{z}_{(2)k}(u,v) = &-2c_1M_1u\cos kv\,\bar{e} - 2\,(u+P)\,M_1\cos kv\,\bar{a}(v)\\ &+2\,(P+u)\,\frac{M_1}{k}\sin kv\,\bar{a}'(v), \quad u\in[0,1],\ v\in[0,2\pi],\end{aligned} \tag{2.11.2}$$

$$\begin{aligned}\bar{z}_{(3)k}(u,v) = &-2c_1M_1u\cos kv\,\bar{e} + 2\frac{c_1}{c_2}\,(u-Q)\,M_1\cos kv\,\bar{a}(v)\\ &+2\frac{c_1}{kc_2}\,(Q-u)\,M_1\sin kv\bar{a}'(v), \quad u\in[0,1],\ v\in[0,2\pi],\end{aligned} \tag{2.11.3}$$

$$\begin{aligned}\bar{z}_{(4)k}(u,v) = &-2c_1M_1u\cos kv\,\bar{e} - 2\frac{c_1}{c_2}\,(u+Q)\,M_1\cos kv\,\bar{a}(v)\\ &+2\frac{c_1}{kc_2}\,(Q+u)\,M_1\sin kv\,\bar{a}'(v), \quad u\in[-1,0],\ v\in[0,2\pi],\end{aligned} \tag{2.11.4}$$

where M_1 is an undetermined constant, P is given by (2.2)*, the parameters b, c_1, c_2, $k\in\{2,3,\dots\}$ satisfy* (1.1) *and*

$$Q = \frac{bc_1 - c_1c_2}{bc_1 - bc_2 - c_1c_2}.$$

The index in the brackets ($i=1,\dots,4$) is related to the parts of the surface obtained by the rotation of the sides $AB,\dots,DA$ of the Belov's quadrangle respectively.

Proof. According to (1.4) with respect to (2.1.1–4), where, for example,

$$\begin{gathered}\varphi_k e^{ikv} + \tilde{\varphi}_k e^{-ikv} = 2\,\mathrm{Re}(\varphi_k)\cos kv - 2\,\mathrm{Im}(\varphi_k)\sin kv,\\ \varphi_k = \mathrm{Re}(\varphi_k) + i\,\mathrm{Im}(\varphi_k),\end{gathered}$$

we obtain (2.11.1–4).

3. A field of rotations

In the study of infinitesimal bendings of surfaces an important role plays the field of rotations $\bar{y}(u,v)$ (see [**1**], [**4**]), related to $\bar{z}(u,v)$, $\bar{r}(u,v)$ by the equation

$$d\bar{z} = \bar{y} \times d\bar{r}. \tag{3.1}$$

As given in [**3**, p. 331], the rotational field $\bar{y}$, in the coordinate system with base the $\bar{r}_1$, $\bar{r}_2$, $\bar{\nu}$, has coordinates

$$(3.1'a,b)\; y^1 = \frac{1}{\sqrt{a}}\Big(b_{12}z^1 + b_{22}z^2 + \frac{\partial z_0}{\partial u^2}\Big),\; y^2 = -\frac{1}{\sqrt{a}}\Big(b_{11}z^1 + b_{12}z^2 + \frac{\partial z_0}{\partial u^1}\Big),$$

$$y_0 = \frac{1}{2\sqrt{a}}\Big(\frac{\partial z_2}{\partial u^1} - \frac{\partial z_1}{\partial u^2}\Big), \tag{3.1$'$ c}$$

where $u^1 = u$, $u^2 = v$, $\bar{z} = z^i\bar{r}_i + z_0\bar{\nu}$, $\bar{y} = y^i\bar{r}_i + y_0\bar{\nu}$, $a = \det(a_{ij})$, a_{ij}, b_{ij} are the first and the second fundamental tensor of the surface, $\bar{r}_i = \frac{\partial \bar{r}}{\partial u^i}$, $\bar{\nu}$ is the unit normal vector of the surface.

By virtue of (2.11) we see that the field $\bar{z}$ is determined in the frame $\bar{e}$, $\bar{a}$, $\bar{a}'$, and from (3.1$'$) it follows that we need the coordinates of this field in the frame $\bar{r}_1$, $\bar{r}_2$, $\bar{\nu}$. We shall prove the needy lemma for the part $S_{(1)}$ of the surface. One can prove it analogously for the other parts, too.

LEMMA 2. *The infinitesimal bendings field $\bar{z}$ for the part $S_{(1)}$ of the Belov's surface in the frame $\bar{r}_1$, $\bar{r}_2$, $\bar{\nu}$ has the contravariant coordinates*

$$z^1_{(1)} = -\frac{2c_1PM_1}{p}\cos kv,\; z^2_{(1)} = (P-u)\,\frac{2M_1\sin kv}{k(c_1u+b+c_1)}, \tag{3.2 a,b}$$

$$z_{0(1)} = 2M_1\left[\frac{P}{\sqrt{p}} - \sqrt{p}u\right]\cos kv. \tag{3.2 c}$$

Proof. From (1.2,3) we get for $S_{(1)}$

$$\begin{gathered}\bar{r}_{(1)}(u,v) = u\bar{e} + \rho_{(1)}(u)\bar{a}(v) = u\bar{e} + (c_1u+b+c_1)\bar{a}(v),\\ \bar{r}_{(1)u} = \bar{e} + c_1\bar{a}(v),\;\; \bar{r}_{(1)v} = (c_1u+b+c_1)\bar{a}'(v),\end{gathered} \tag{3.3.1}$$

$$\bar{\nu}_{(1)} = \frac{(c_1u+b+c_1)(c_1\bar{e}-\bar{a})}{|c_1u+b+c_1|\,|c_1\bar{e}-\bar{a}|} = \frac{c_1\bar{e}-\bar{a}(v)}{\sqrt{p}},\; p = (c_1)^2+1, \tag{3.4.1}$$

as $c_1u+b+c_1 > 0$ ($c_1u+b+c_1 \le 0 \Leftrightarrow u \le -\frac{b}{c_1} - 1 < -1$, which is impossible, because in the considered case $|u| \le 1$, b, $c_1 > 0$).

According to (3.3,4) we have

$$\begin{aligned}\bar{z}_{(1)k} &= z^1_{(1)}\bar{r}_{(1)u} + z^2_{(1)}\bar{r}_{(1)v} + z_{0(1)}\bar{\nu}_{(1)}\\ &= \Big[z^1_{(1)} + \frac{c_1z_{0(1)}}{\sqrt{p}}\Big]\bar{e} + \Big[c_1z^1_{(1)} - \frac{z_{0(1)}}{\sqrt{p}}\Big]\bar{a}(v) + z^2_{(1)}(c_1u+b+c_1)\bar{a}'(v).\end{aligned}$$

Comparing this equation with (2.11.1), we get equations from which we get the values (3.2).

We can prove now the theorem related to the determination of the field of rotations $\bar{y}$.

THEOREM 4. *The field $\bar{y}(u,v)$ of infinitesimal rotations of Belov's toroid is defined by equations*

$$\bar{y}_{(i)k} = y^1_{(i)}\bar{r}_{(i)u} + y^2_{(i)}\bar{r}_{(i)v} + y_{0(i)}\bar{\nu}_{(i)}, \quad i = 1,\dots,4$$

where

$$(3.5.1a)\quad y^1_{(1)} = \frac{2M_1 \sin kv}{k(c_1u+b+c_1)[(c_1)^2+1]}\Big\{uk^2[(c_1)^2+1] - u - \frac{b+c_1}{c_1}\Big\},$$

$$(3.5.1b)\quad y^2_{(1)} = \frac{2M_1 \cos kv}{c_1u+b+c_1},$$

$$(3.5.1c)\quad y_{0(1)} = \frac{-2M_1 \sin kv}{k\sqrt{(c_1)^2+1}}, \quad u\in[-1,0],\ v\in[0,2\pi],$$

$$(3.5.2a)\quad y^1_{(2)} = \frac{2M_1 \sin kv}{k(-c_1u+b+c_1)[(c_1)^2+1]}\Big\{u - uk^2[(c_1)^2+1] - \frac{b+c_1}{c_1}\Big\},$$

$$(3.5.2b)\quad y^2_{(2)} = \frac{2M_1 \cos kv}{c_1u-b-c_1},$$

$$(3.5.2c)\quad y_{0(2)} = \frac{2M_1 \sin kv}{k\sqrt{(c_1)^2+1}}, \quad u\in[0,1],\ v\in[0,2\pi],$$

$$(3.5.3a)\quad y^1_{(3)} = \frac{2M_1c_1 \sin kv}{kc_2(c_2u+b-c_2)[(c_2)^2+1]}\Big\{uk^2[(c_2)^2+1] - u - \frac{b-c_2}{c_2}\Big\},$$

$$(3.5.3b)\quad y^2_{(3)} = \frac{2M_1c_1 \cos kv}{c_2(c_2u+b-c_2)},$$

$$(3.5.3c)\quad y_{0(3)} = \frac{-2M_1c_1 \sin kv}{kc_2\sqrt{(c_1)^2+1}}, \quad u\in[0,1],\ v\in[0,2\pi],$$

$$(3.5.4a)\quad y^1_{(4)} = \frac{2M_1c_1 \sin kv}{kc_2(-c_2u+b-c_2)[(c_2)^2+1]}\Big\{u - uk^2[(c_1)^2+1] - \frac{b-c_2}{c_2}\Big\},$$

$$(3.5.4b)\quad y^2_{(4)} = \frac{2M_1c_1 \cos kv}{c_2(c_2u-b+c_2)},$$

$$(3.5.4c)\quad y_{0(4)} = \frac{2M_1c_1 \sin kv}{kc_2\sqrt{(c_2)^2+1}}, \quad u\in[-1,0],\ v\in[0,2\pi]$$

and the indices in the brackets are related to the parts of the surface obtained by rotation of the sides AB, BC, CD, DA of Belov's quadrangle, M_1 is an arbitrary constant.

Proof. From (3.3.1) we have

$$\bar{r}_{(1)uu} = 0,\ \bar{r}_{(1)uv} = c_1\bar{a}'(v),\ \bar{r}_{(1)vv} = (c_1u+b+c_1)\bar{a}''(v) = -(c_1u+b+c_1)\bar{a}(v),$$

and for the coefficients of the second fundamental form of the surface $b_{ij} = \bar{r}_{ij}\bar{\nu}$ we have

$$b_{(1)11} = b_{(1)12} = 0, \quad b_{(1)22} = \frac{1}{\sqrt{p}}(c_1 u + b + c_1).$$

According to $(3.1'a, b)$ and (3.2) one obtains

$$y^1_{(1)} = \frac{2M_1 \sin kv}{k\sqrt{a_{(1)}p}}\left[u\left(k^2 p - 1\right) - \frac{b + c_1}{c_1}\right], \quad y^2_{(1)} = \frac{2M_1 \cos kv}{c_1 u + b + c_1}$$

and because of

$$a_{(1)} = (c_1 u + b + c_1)^2 p, \quad p = (c_1)^2 + 1,$$

we obtain the first two equations in (3.5.1).

In order to find $y_{0(1)}$ according to $(3.1'c)$, first of all we have to find $z_{1(1)}$, $z_{2(1)}$, where $z_i = a_{ip}z^p$. Based on (3.3.1), we have for $S_{(1)}$

$$a_{11} = (c_1)^2 + 1, \ a_{12} = 0, \ a_{22} = (c_1 u + b + c_1)^2,$$

and from (3.2) we have

$$z_{1(1)} = -2M_1 P c_1 \cos kv, \ z_{2(1)} = 2M_1(c_1 u + b + c_1)\,(P - u)\,\frac{\sin kv}{k},$$
$$\frac{\partial z_{1(1)}}{\partial u^2} = \frac{\partial z_{1(1)}}{\partial v} = 2M_1 k P c_1 \sin kv, \ \frac{\partial z_2}{\partial u^1} = (Pc_1 - 2c_1 u - b - c_1)\frac{\sin kv}{k}$$

and substituting into $(3.1'c)$ we get the third equation at (3.5.1).

The field of rotation for $S_{(1)}$ is

$$\bar{y}_{(1)k} = y^1_{(1)}\bar{r}_{(1)u} + y^2_{(1)}\bar{r}_{(1)v} + y_{0(1)}\bar{\nu}_{(1)}, \quad u \in [-1, 0], \quad v \in [0, 2\pi],$$

where $y^1_{(1)}, y^2_{(1)}, y_{0(1)}$ are given by $(3.5.1a{-}c)$. The same field $\bar{y}_{(1)k}$ in the coordinate system with the basis $\bar{e}$, $\bar{a}(v)$, $\bar{a}'(v)$ one gets according to (3.3.1), (3.4.1). In the same way we get components of the field of rotations for $S_{(2)}$ $S_{(3)}$ and $S_{(4)}$. For $S_{(2)}$ we have:

(3.3.2)
$$\bar{r}_{(2)}(u, v) = u\bar{e} + \rho_{(2)}(u)\bar{a}(v) = u\bar{e} + (-c_1 u + b + c_1)\bar{a}(v),$$
$$\bar{r}_{(2)u} = \bar{e} - c_1\bar{a}(v), \ \bar{r}_{(2)v} = (-c_1 u + b + c_1)\bar{a}'(v),$$

(3.4.2)
$$\bar{\nu}_{(2)} = -\frac{c_1\bar{e} + \bar{a}}{\sqrt{(c_1)^2 + 1}},$$

(3.5.2a)
$$y^1_{(2)} = \frac{2M_1 \sin kv}{k(-c_1 u + b + c_1)[(c_1)^2 + 1]}\{u - uk^2[(c_1)^2 + 1] - \frac{b + c_1}{c_1}\},$$

(3.5.2b)
$$y^2_{(2)} = \frac{2M_1 \cos kv}{c_1 u - b - c_1},$$

$$y_{0(2)}=\frac{2M_1\sin kv}{k\sqrt{(c_1)^2+1}}, \tag{3.5.2c}$$

$$\bar y_{(2)k}=y^1_{(2)}\bar r_{(2)u}+y^2_{(2)}\bar r_{(2)v}+y_{0(2)}\bar\nu_{(2)},\quad u\in[-1,0],\quad v\in[0,2\pi],$$

where $y^1_{(2)}, y^2_{(2)}, y_{0(2)}$ are given by (3.5.2). For $S_{(3)}$:

$$\bar r_{(3)}(u,v)=u\bar e+\rho_{(3)}(u)\bar a(v)=u\bar e+(c_2u+b-c_2)\bar a(v),$$
$$\bar r_{(3)u}=\bar e+c_2\bar a(v),\quad \bar r_{(3)v}=(c_2u+b-c_2)\bar a'(v),$$

$$\bar\nu_{(3)}=\frac{c_2\bar e-\bar a(v)}{\sqrt p},\quad q=(c_2)^2+1,$$

$$\bar y_{(3)k}=y^1_{(3)}\bar r_{(3)u}+y^2_{(3)}\bar r_{(3)v}+y_{0(3)}\bar\nu_{(3)},\quad u\in[0,1],\quad v\in[0,2\pi],$$

where $y^1_{(3)}$, $y^2_{(3)}$, $y_{0(3)}$ are given by (3.5.3).

For $S_{(4)}$:

$$\bar r_{(4)}(u,v)=u\bar e+\rho_{(4)}(u)\bar a(v)=u\bar e+(-c_2u+b-c_2)\bar a(v),$$
$$\bar r_{(4)u}=\bar e-c_2\bar a(v),\ \ \bar r_{(4)v}=(-c_2u+b-c_2)\bar a'(v),$$

$$\bar\nu_{(4)}=-\frac{c_2\bar e-\bar a(v)}{\sqrt q},\ q=(c_2)^2+1,$$

$$\bar y_{(4)k}=y^1_{(4)}\bar r_{(4)u}+y^2_{(4)}\bar r_{(4)v}+y_{0(4)}\bar\nu_{(4)},\quad u\in[-1,0],\quad v\in[0,2\pi],$$

where $y^1_{(4)}, y^2_{(4)}, y_{0(4)}$ are given by (3.5.4).

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Građevinski fakultet
18000 Niš
Yugoslavia

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