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# ON A CERTAIN EXTENSION OF THE CLASS OF SEMISYMMETRIC MANIFOLDS

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*Dedicated to Professor Witold Roter on his 65th birthday*

*Communicated by Mileva Prvanović*

**Abstract.** We study curvature properties of semi-Riemannian manifolds satisfying a new condition of pseudosymmetry type. Basing on obtained results we construct non-trivial examples of such manifolds.

## 1. Introduction

Let $(M, g)$ be a connected $n$-dimensional, $n \geq 3$, semi-Riemannian manifold of class $C^\infty$. We denote by $\nabla$, $\tilde{R}$, $R$, $C$, $S$ and $\kappa$ the Levi-Civita connection, the curvature operator, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of $(M, g)$, respectively.

A semi-Riemannian manifold $(M, g)$ is said to be semisymmetric [**18**] if

$$R \cdot R = 0$$

holds on $M$. As a proper generalization of locally symmetric spaces ($\nabla R = 0$) semisymmetric manifolds were studied by many authors. In the Riemannian case, Z. I. Szabó obtained in the early eighties a full intrinsic classification of semisymmetric Riemannian manifolds [**18**]. Very recently theory of Riemannian semisymmetric manifolds has been presented in the monograph [**1**]. The profound investigation of several properties of semisymmetric manifolds, gave rise to their next generalization: the pseudosymmetric manifolds.

A semi-Riemannian manifold $(M, g)$ is said to be *pseudosymmetric* [**10**] if at every point of $M$ the following condition is satisfied:

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$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.
This condition is equivalent to the relation

$$R \cdot R = L_R \, Q(g, R)$$

on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} \, G \neq 0 \text{ at } x\}$, where $L_R$ is some function on $\mathcal{U}_R$. The definitions of the tensors used will be given in Section 2. There exist various examples of pseudosymmetric manifolds which are non-semisymmetric and a review of results on pseudosymmetric manifolds is given in [**9**] (see also [**V**]).

It is easy to see that if $(*)_1$ holds on a semi-Riemannian manifold $(M, g)$, $n \geq 4$, then at every point of $M$ the following condition is satisfied:

$(*)_2$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.
The converse statement is not true [**8**] (cf. Example 3.1).

A semi-Riemannian manifold $(M, g)$, $n \geq 4$, is called *Weyl-pseudosymmetric* if at every point of $M$ the condition $(*)_2$ is fulfilled. If a manifold $(M, g)$ is Weyl-pseudosymmetric then the relation

$$R \cdot C = L_C \, Q(g, C)$$

holds on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where $L_C$ is some function on $\mathcal{U}_C$.

It is easy to see that at every point of pseudosymmetric Einstein manifold the following condition is fulfilled:

$(*)_3$ the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.
It is known that every hypersurface $M$,$\dim M \geq 4$, immersed isometrically in a semi-Riemannian space of constant curvature realizes $(*)_3$ ([**13**]). More precisely, the following relation $R \cdot R - Q(S, R) = -\frac{(n-2)\tilde{\kappa}}{n(n+1)} \, Q(g, C)$ holds on $M$, where $\tilde{\kappa}$ is the scalar curvature of the ambient space. Recently, pseudosymmetric manifolds satisfying $(*)_3$ were investigated in [**12**]. Semi-Riemannian manifolds realizing $(*)_1$–$(*)_3$ and other conditions of this kind, described in [**9**] or [**V**], are called *manifolds of pseudosymmetry type*.

The present paper concerns with semi-Riemannian manifolds satisfying the new condition of pseudosymmetry type:

$(*)$ the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent
at every point of $M$. This condition is equivalent to the relation

$$R \cdot C = L \, Q(S, C) \tag{1}$$

on the set $\mathcal{U} = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, for some function $L$ on $\mathcal{U}$, called the associated function of $M$. It is clear that every semisymmetric manifold satisfies $(*)$. The converse statement is not true (see Example 5.1).

In Section 2 of this paper we fix the notations and present auxiliary lemmas. In Section 3 we consider manifolds satisfying the equality $Q(S, C) = 0$

and we prove that such manifolds are pseudosymmetric. In Section 4 we investigate manifolds satisfying (1) and admitting a 1-form $a$ such that the cyclic sum $\sum_{X,Y,Z} a(X)\tilde{C}(Y,Z) = 0$. We prove that the associated function of such manifold must be equal to $1/(n-1)$ or $1/(n-2)$. Applying this result, we find in Section 5 the necessary and sufficient condition for a metric $\bar{g}$ with harmonic Weyl tensor $\bar{C}$ conformal to an essentially conformally symmetric metric $g$ to satisfy (1). As a consequence of these considerations, we give an example of a manifold realizing (1) with $L = 1/(n-2)$ which is not pseudosymmetric. Finally, Section 6 contains some results on concircular changes of metrics satisfying (1).

## 2. Preliminaries

Let $(M,g)$ be an $n$-dimensional, $n \geq 3$, semi-Riemannian manifold. A tensor $\tilde{B}$ of type $(1,3)$ on $M$ is said to be a generalized curvature tensor [**16**], if

$$\sum_{X_1,X_2,X_3} \tilde{B}(X_1,X_2)X_3 = 0,$$
$$\tilde{B}(X_1,X_2) + \tilde{B}(X_2,X_1) = 0,$$
$$B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2),$$

where $B(X_1,X_2,X_3,X_4) = g(\tilde{B}(X_1,X_2)X_3,X_4)$. The Ricci tensor $\operatorname{Ric}(\tilde{B})$ of $\tilde{B}$ is the trace of the linear mapping $X_1 \to \tilde{B}(X_1,X_2)X_3$. For a generalized curvature tensor $\tilde{B}$ we define the scalar curvature $\kappa(\tilde{B})$ by

$$\kappa(\tilde{B}) = \sum_{i=1}^{n} \epsilon_i \operatorname{Ric}(\tilde{B})(E_i,E_i), \quad \epsilon_i = g(E_i,E_i)\,,$$

where $E_1,\dots,E_n$ is an orthonormal basis. Let the tensor $G$ be defined by

$$G(X_1,X_2,X_3,X_4) = g((X_1 \wedge X_2)X_3,X_4)\,,$$
$$(X_1 \wedge X_2)X_3 = g(X_2,X_3)X_1 - g(X_1,X_3)X_2.$$

Further, we define the Weyl curvature tensor $C(\tilde{B})$ associated with $\tilde{B}$ by

$$\begin{aligned} C(\tilde{B})(X_1,X_2,X_3,X_4) &= B(X_1,X_2,X_3,X_4) + \frac{\kappa(\tilde{B})}{(n-1)(n-2)} G(X_1,X_2,X_3,X_4) \\ &\quad - \frac{1}{n-2}(g((\widetilde{\operatorname{Ric}}(\tilde{B})X_1 \wedge X_2)X_3,X_4) - g((\widetilde{\operatorname{Ric}}(\tilde{B})X_1 \wedge X_2)X_4,X_3))\,, \end{aligned}$$

where the tensor field $\widetilde{\operatorname{Ric}}(\tilde{B})$ is defined by $\operatorname{Ric}(\tilde{B})(X,Y) = g(\widetilde{\operatorname{Ric}}(\tilde{B})X,Y)$. For an $(0,2)$-tensor field $A$ on $(M,g)$ we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y$, where $X,Y,Z \in \Xi(M)$. In particular we have $X \wedge_g Y = X \wedge Y$. For an $(0,k)$-tensor field $T$, $k \geq 1$, an $(0,2)$-tensor field $A$ and a generalized curvature tensor $\tilde{B}$ on $(M,g)$ we define the tensors $B \cdot T$ and $Q(A,T)$ by

$$\begin{aligned}(B \cdot T)(X_1, \ldots , X_k; X, Y) &= -T(\tilde{B}((X,Y)X_1, X_2, \ldots , X_k) - \cdots \\ &\quad - T(X_1, \ldots , X_{k-1}, \tilde{B}(X,Y)X_k)\,, \\ Q(A,T)(X_1, \ldots , X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, \ldots , X_k) - \cdots \\ &\quad - T(X_1, \ldots , X_{k-1}, (X \wedge_A Y)X_k)\,,\end{aligned}$$

where $X, Y, Z, X_1, X_2, \ldots \in \Xi(M)$. Putting in the above formulas

$$\tilde{B}(X,Y)Z = \tilde{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\ ,$$

$T = R$ or $T = C$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $Q(g,R)$, $Q(S,R)$, $R \cdot C$, $Q(g,C)$ and $Q(S,C)$, respectively.

Let $(M,g)$ be a semi-Riemannian manifold covered by a system of charts $\{W;\ x^k\}$. We denote by $g_{ij}$, $R_{hijk}$, $S_{ij}$, $S_i^{\ j} = g^{jk} S_{ik}$, $G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik}$ and

$$\begin{aligned}C_{hijk} = R_{hijk} &- \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\ &+ \frac{\kappa}{(n-1)(n-2)} G_{hijk}\end{aligned} \tag{2}$$

the local components of the metric tensor $g$, the Riemann–Christoffel curvature tensor $R$, the Ricci tensor $S$, the Ricci operator $\tilde{S}$, the tensor $G$ and the Weyl tensor $C$, respectively.

At the end of this section we present some results which will be used in the next sections. Let $g$ be a metric on a manifold $M$ and let $\bar{g}$ be another metric on $M$ conformally related to $g$, i.e., $\bar{g} = \exp(2p)g$, where $p$ is a nonconstant function on $M$. When $\Omega$ is a quantity formed with respect to $g$, we denote by $\overline{\Omega}$ the similar quantity formed with respect to $\bar{g}$. We shall use the following general formulas for conformally related metrics (cf. [**20**]):

$$\bar{g}_{ij} = \exp(2p) g_{ij}, \quad \bar{g}^{ij} = \exp(-2p) g^{ij}, \tag{3}$$

$$\bar{S}_{ij} = S_{ij} - (n-2)P_{ij} - (\Delta_2 p + (n-2)\Delta_1 p))g_{ij}, \tag{4}$$

$$\bar{\kappa} = \exp(-2p)(\kappa - (n-1)(2\Delta_2 p + (n-2)\Delta_1 p)), \tag{5}$$

$$\bar{R}_{hijk} = \exp(2p)(R_{hijk} - U_{hijk}), \tag{6}$$

$$\bar{C}^h_{\ ijk} = C^h_{\ ijk}, \quad \bar{C}_{hijk} = \exp(2p) C_{hijk}, \tag{7}$$

$$\bar{\nabla}_r \bar{C}^r_{\ ijk} = \nabla_r C^r_{\ ijk} + (n-3) p_r C^r_{\ ijk}, \tag{8}$$

where

$$\begin{gathered}\Delta_1 p = g^{ij} p_i p_j = \langle dp, dp \rangle, \quad \Delta_2 p = g^{ij} \nabla_j p_i, \\ U_{hijk} = g_{hk} P_{ij} - g_{hj} P_{ik} + g_{ij} P_{hk} - g_{ik} P_{hj} + \Delta_1 p\,(g_{hk} g_{ij} - g_{hj} g_{ik}),\end{gathered}$$

$P_{ij}$ and $p_i$ are local components of the tensors $P = \nabla dp - dp \otimes dp$ and $dp$, respectively. Using (3), (6) and (7) we also have

$$\begin{aligned}\exp(-2p)(\bar{R}\cdot\bar{C})_{hijklm} &= (R\cdot C)_{hijklm} - \Delta_1 p Q(g,C)_{hijklm} - Q(P,C)_{hijklm}\\ &\quad - P_m{}^r(g_{hl}C_{rijk} + g_{il}C_{hrjk} + g_{jl}C_{hirk} + g_{kl}C_{hijr})\\ &\quad + P_l{}^r(g_{hm}C_{rijk} + g_{im}C_{hrjk} + g_{jm}C_{hirk} + g_{km}C_{hijr}).\end{aligned}$$

Lemma 2.1. [**5**, Lemma 1] *Let a tensor $A_{lmhs_1\dots s_N}$ of type $(0, N+3)$ be symmetric in $(l,m)$ and skew-symmetric in $(m,h)$. Then $A_{lmhs_1\dots s_N} = 0$.*

Lemma 2.2. [**17**] *We define the metric $g$ in $\mathbb{R}^n$ by the formula*

$$ds^2 = Q(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{10}$$

*where $\alpha,\beta = 2,\dots,n-1$, $[k_{\alpha\beta}]$ is a symmetric and nonsingular matrix consisting of constants, and $Q$ is independent of $x^n$. The only components of $\nabla$ and $C$, not identically zero are those related to:*

$$\Gamma^\alpha_{11} = -\frac{1}{2}k^{\alpha\omega}Q_{.\omega},\quad \Gamma^n_{11} = \frac{1}{2}Q_{.1},\quad \Gamma^n_{1\gamma} = \frac{1}{2}Q_{.\gamma}, \tag{11}$$

$$C_{1\lambda\mu 1} = \frac{1}{2}Q_{.\lambda\mu} - \frac{1}{2(n-2)}k_{\lambda\mu}(k^{\beta\omega}Q_{.\beta\omega}), \tag{12}$$

*where $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$ and the dot denotes partial differentiation with respect to coordinates.*

Lemma 2.3. [**11**, Theorem 1] *Let $\tilde{B}$ be a generalized curvature tensor at $x \in M$ such that the condition $\sum_{X,Y,Z}\omega(X)\tilde{B}(Y,Z) = 0$ is satisfied for $\tilde{B}$ and a covector $\omega$ at $x$, where $X,Y,Z \in T_x(M)$, $\Sigma$ denotes the cyclic sum. If $\omega \neq 0$ then $B\cdot B = Q(\mathrm{Ric}(\tilde{B}), B)$ at $x$*

Lemma 2.4. [**2**, Proposition 4.1] *Let $(M,g)$, $\dim M \geq 3$, be a semi-Riemannian manifold. Let $A$ be a nonzero symmetric $(0,2)$-tensor and $\tilde{B}$ a generalized curvature tensor at a point $x$ of $M$ satisfying the condition $Q(A,B) = 0$. Moreover, let $V$ be a vector at $x$ such that the scalar $\rho = a(V)$ is nonzero, where $a$ is a covector defined by $a(X) = A(X,V)$, $X \in T_x(M)$.*

- (*i*) *If the tensor $A - (1/\rho)\,a\otimes a$ vanishes, then the relation $\sum_{X,Y,Z} a(X)\tilde{B}(Y,Z) = 0$ holds at $x$, where $X,Y,Z \in T_x(M)$.*
- (*ii*) *If the tensor $A - (1/\rho)\,a\otimes a$ is nonzero, then the relation*

$$\rho B(X,Y,Z,W) = \lambda(A(X,W)A(Y,Z) - A(X,Z)A(Y,W))$$

*holds at $x$, where $\lambda \in \mathbb{R}$ and $X,Y,Z,W \in T_x(M)$.*

*Moreover, in both cases $B\cdot B = Q(\mathrm{Ric}(\tilde{B}), B)$ at $x$.*

Lemma 2.5. [**14**, Theorems 1 and 2] *Let (M,g) be a Weyl-pseudosymmetric semi-Riemannian manifold satisfying the condition* $\sum_{X,Y,Z} a(X)\tilde{C}(Y,Z) = 0$, *where a is a 1-form on M. If* $a \neq 0$ *and* $C \neq 0$ *at a point* $x \in M$, *then the following relations are satisfied at x:*

$$L_C = \frac{\kappa}{n(n-1)}, \quad S(W, \tilde{C}(X,Y)Z) = \frac{\kappa}{n} C(X,Y,Z,W),$$
$$Q\Big(S - \frac{\kappa}{n} g, C\Big) = 0, \quad R \cdot R = L_C Q(g,R).$$

Lemma 2.6. [**12**, Theorem 4.2] *Let* $(M,g)$ *be a semi-Riemannian manifold with the curvature tensor of the form*

$$R(X,Y,Z,W) = \phi(S(X,W)S(Y,Z) - S(X,Z)S(Y,W)) + \eta G(X,Y,Z,W)$$
$$+ \mu(S(X,W)g(Y,Z) + S(Y,Z)g(X,W) - S(X,Z)g(Y,W) - S(Y,W)g(X,Z))$$

*at* $x \in M$, *where* $X,Y,Z,W \in T_x(M)$ *and* $\phi, \mu, \eta \in \mathbb{R}$. *If* $C \neq 0$ *and* $S - (\kappa/n)\, g \neq 0$ *at x, then the following equalities hold at x:*

$$R \cdot R = L_R Q(g,R), \quad L_R = \frac{\mu}{\phi}((n-2)\mu - 1) - \eta(n-2),$$
$$R \cdot R = Q(S,R) + \Big(L_R + \frac{\mu}{\phi}\Big) Q(g,C).$$

## 3. Manifolds with vanishing tensor field Q(S,C)

Theotem 3.1. *Let (M,g),* $\dim M \geq 4$, *be a semi-Riemannian manifold satisfying at a point x of M the equality* $Q(S,C) = 0$. *If* $S \neq 0$ *and* $C \neq 0$ *at x, then the relation*

$$R \cdot R = \frac{\kappa}{n-1} Q(g,R) \tag{13}$$

*holds at x.*

*Proof.* It is easy to verify that the following identity is satisfied on $M$

$$(C \cdot C)_{hijklm} = (R \cdot C)_{hijklm} + \frac{1}{n-2}\left(\frac{\kappa}{n-1} Q(g,C)_{hijklm} - Q(S,C)_{hijklm}\right)$$
$$- \frac{1}{n-2}(g_{hl} S_{mr} C^r{}_{ijk} - g_{hm} S_{lr} C^r{}_{ijk} - g_{il} S_{mr} C^r{}_{hjk} + g_{im} S_{lr} C^r{}_{hjk}$$
$$+ g_{jl} S_{mr} C^r{}_{khi} - g_{jm} S_{lr} C^r{}_{khi} - g_{kl} S_{mr} C^r{}_{jhi} + g_{km} S_{lr} C^r{}_{jhi}).$$

According to Lemma 2.4, we may consider two cases (we will use notations of the mentioned lemma):

(i) $S = (1/\rho)\, a \otimes a$. In this case we have $a_l C_{hijk} + a_h C_{iljk} + a_i C_{lhjk} = 0$, which implies $a_r C^r{}_{ijk} = 0$ and consequently $S_{ir} C^r{}_{hjk} = 0$. Thus the equation $C \cdot C = 0$, which follows from Lemma 2.3, and our assumption turns (14) into

$$R \cdot C = -\frac{\kappa}{(n-1)(n-2)} Q(g, C).$$

Applying now Lemma 2.5 we obtain $\kappa = 0$ and next $R \cdot R = 0$.

(ii) $S - (1/\rho)\, a \otimes a \neq 0$. In this case we have $\rho C_{hijk} = \lambda(S_{hk}S_{ij} - S_{hj}S_{ik})$. This equation, in virtue of (2) leads to

$$R_{hijk} = \frac{\lambda}{\rho}(S_{hk}S_{ij} - S_{hj}S_{ik}) + \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) - \frac{\kappa}{(n-1)(n-2)} G_{hijk}.$$

Applying now Lemma 2.6 we obtain (13), which completes the proof.

From the above theorem it follows

COROLLARY 3.1. *Let $(M, g)$, $\dim M \geq 4$, be an analytic semi-Riemannian manifold with nonzero tensors $S$ and $C$. If the equality $Q(S, C) = 0$ is fulfilled on $M$, then $(M,g)$ is pseudosymmetric manifold satisfying* (13).

On the other hand, manifolds realizing $(*)$ for which $Q(S, C) \neq 0$, i.e., manifolds fulfilling (1), may be pseudosymmetric or not. This fact illustrates the following

*Example* 3.1. Let $(M, g)$ be the 4-dimensional manifold defined in [**4**, Lemme 1.1] As it was shown in [**4**] (see Lemme 1.1 and Remarqué 1.5), $(M, g)$ is a non-conformally flat and non-semisymmetric, Weyl-semisymmetric manifold, i.e., the tensors $C$ and $R \cdot R$ are nonzero and the condition $R \cdot C = 0$ holds on $M$. From these facts it follows that $(M, g)$ is a non-pseudosymmetric manifold.

(i) Let $V$ be a connected subset of the set $W = \{x \in M \mid u(x) \neq 0\}$, where $u$ is the function defined in [**4**, Lemme 1.1]. By formula (10) of [**4**] we have $W = U_C$. The scalar curvature $\kappa$ of $(M, g)$ satisfies the equality ([**4**, Lemme 1.1(iv)] $\kappa = u$, which implies that the Ricci tensor $S$ of $(M, g)$ is nonzero at every point of $V$. Using now Theorem 3.1 and the fact that the tensors $S$ and $C$ and the scalar curvature $\kappa$ are nonzero at every point of $V$ we can easily conclude that the tensor $Q(S, C)$ is nonzero at every point of $V$. Thus we have on $V$ the following equality:

$$R \cdot C = L\, Q(S, C) \quad \text{with } L = 0\,.$$

(ii) We consider now on $V$ the conformal deformation $g \to \bar{g} = (1/u^2)\, g$ of the metric $g$, where $u > 0$ or $u < 0$ on $V$. It is known that the manifold $(V, \bar{g})$ is an Einstein manifold [**4**, Lemme 1.1(viii)], i.e., $\bar{S} = (\bar{\kappa}/4)\, \bar{g}$ holds on $V$. Moreover, as it was shown in [**8**] (see Example 3) the relation

$$\bar{R} \cdot \bar{R} = -\frac{1}{12}(u^3 - pq) Q(\bar{g}, \bar{R}) \tag{15}$$

holds on $V$, where $\bar{R}$ is the Riemann-Christoffel curvature tensor of the metric $\bar{g}$ and $p, q$ are some constants. Evidently, if the Ricci tensor $\bar{S}$ vanishes at a point $x \in V$, then $Q(\bar{S}, \bar{C}) = 0$ holds at $x$ and, of course, the condition $(*)$ is fulfilled at $x$. If at a point $x \in M$ we have $\bar{S} \neq 0$, then (15) turns into

$$\bar{R} \cdot \bar{C} = -\frac{u^3 - pq}{3\bar{\kappa}}\, Q(\bar{S}, \bar{C})\,.$$

Thus the manifold $(V, \bar{g})$ realizes $(*)$.

Since the equality $Q(S, C) = 0$ at $x$ leads to the condition $(*)_1$ at $x$, we restrict our considerations in the remaining sections to the set $\mathcal{U}$.

## 4. Manifolds satisfying some curvature conditions

THEOREM 4.1. *Let $(M, g)$, $\dim M \geq 4$, be a semi-Riemannian manifold satisfying $(*)$ and the following condition*

$$\sum_{X,Y,Z} a(X)\tilde{C}(Y, Z) = 0 \tag{16}$$

*for a 1-form $a$. If $a \neq 0$ and $Q(S, C) \neq 0$ at a point $x \in M$, then $L = 1/(n-2)$ or $L = 1/(n-1)$.*

*Proof.* First of all we note that (16), which in local coordinates takes the form

$$a_l C_{hijk} + a_j C_{hikl} + a_k C_{hilj} = 0, \tag{17}$$

leads to

$$a_r a^r = 0\ , \quad a_r C^r_{ijk} = 0 \tag{18}$$

and

$$C \cdot C = 0 \tag{19}$$

(cf. Lemma 2.3). In local coordinates the equation $R \cdot C = L\, Q(S, C)$ takes the form

$$\begin{aligned} R^r{}_{hlm} C_{rijk} + R^r{}_{ilm} C_{hrjk} + R^r{}_{jlm} C_{hirk} + R^r{}_{klm} C_{hijr} \\ = L(S_{hl} C_{mijk} - S_{hm} C_{lijk} + S_{il} C_{hmjk} - S_{im} C_{hljk} + S_{jl} C_{himk} \\ - S_{jm} C_{hilk} + S_{kl} C_{hijm} - S_{km} C_{hijl}). \end{aligned} \tag{20}$$

Transvecting (20) with $a^h$, in view of (18), we obtain

$$C_{rijk} R^r{}_{slm} a^s = L(d_l C_{mijk} - d_m C_{lijk}), \tag{21}$$

where $d_i = a^r S_{ri}$. Substituting (2) into (18) we have

$$R_{srlm}a^s = \frac{1}{n-2}(d_m g_{rl} - d_l g_{rm} + a_m S_{rl} - a_l S_{rm}) - \frac{\kappa}{(n-1)(n-2)}(a_m g_{rl} - a_l g_{rm}).$$

The substitution of the above equality into (21) and making use of $a_m C_{lijk} - a_l C_{mijk} = a_i C_{lmjk}$, which follows from (17), yields

$$(22)\quad ((n-2)L-1)(d_m C_{lijk} - d_l C_{mijk}) = a_m S_{lr} C^r{}_{ijk} - a_l S_{mr} C^r{}_{ijk} + \frac{\kappa}{n-1} a_i C_{mljk}\,.$$

Transvection of (22) with $a^m$, in virtue of (18), gives

$$((n-2)L-1)a^r d_r C_{lijk} = -a_l d_r C^r{}_{ijk}\,,$$

which immediately implies $d_r C^r{}_{ijk} = 0$.

Contracting now (22) with $g^{km}$ and using the above equality we have

$$(23)\qquad S^{rs} C_{rijs} = 0.$$

Transvecting (17) with $S_p{}^l$ we get $d_p C_{hijk} = a_k C_{hijr} S_p{}^r - a_j C_{hikr} S_p{}^r$. Substituting twice the above equality into (22) (taking suitable indices), we obtain

$$(24)\quad (n-2)L(a_l S_{mr} C^r{}_{ijk} - a_m S_{lr} C^r{}_{ijk}) = a_i \left(\frac{\kappa}{n-1} C_{mljk} + ((n-2)L-1)\right)(S_{mr} C^r{}_{ljk} - S_{lr} C^r{}_{mjk}).$$

Hence, by cyclic permutation in $m, j, k$, we get

$$(25)\qquad (n-2)L a_l T_{mijk} = ((n-2)L-1) a_i T_{mljk},$$

where $T_{mijk} = S_{mr} C^r{}_{ijk} + S_{jr} C^r{}_{ikm} + S_{kr} C^r{}_{imj}$. We assert that $T_{mijk} = 0$, i.e.,

$$(26)\qquad S_{mr} C^r{}_{ijk} + S_{jr} C^r{}_{ikm} + S_{kr} C^r{}_{imj} = 0.$$

In fact, if $L = 0$ then we immediatey have $T_{mijk} = 0$. Assume now that $L \neq 0$ at $x$. Using (25) we get

$$a_l T_{mijk} = \alpha a_i T_{mljk} = \alpha^2 a_l T_{mijk},$$

where $\alpha = \frac{(n-2)L-1}{(n-2)L}$. If $\alpha^2 \neq 1$ at $x$, then we get (26). On the other hand the equality $\alpha^2 = 1$ is equivalent to $(n-2)L = 1/2$. In this case (25) takes the form $a_l T_{mijk} + a_i T_{mljk} = 0$, which immediately leads to (26). The equalities (1), (14) and (19) imply

$$\begin{aligned}(L(n-2)-1)Q(S,C)_{hijklm} &+ \frac{\kappa}{n-1} Q(g,C)_{hijklm} \\ &= g_{hl} S_{mr} C^r{}_{ijk} - g_{hm} S_{lr} C^r{}_{ijk} - g_{il} S_{mr} C^r{}_{hjk} + g_{im} S_{lr} C^r{}_{hjk} \\ &\quad + g_{jl} S_{mr} C^r{}_{khi} - g_{jm} S_{lr} C^r{}_{khi} - g_{kl} S_{mr} C^r{}_{jhi} + g_{km} S_{lr} C^r{}_{jhi}.\end{aligned}\qquad(27)$$

Contracting (27) with $g^{hl}$, in virtue of (26) and (22), we obtain

$$L(n-2)\kappa C_{mijk} + (L(n-2)-1)S_{ir}C^r{}_{mjk} = (n-1)S_{mr}C^r{}_{ijk}. \tag{28}$$

Symmetrizing this in $m, i$, we find $(L(n-2)-n)(S_{ir}C^r{}_{mjk} + S_{mr}C^r{}_{ijk}) = 0$. If $L(n-2) \neq n$, then we have

$$S_{ir}C^r{}_{mjk} = -S_{mr}C^r{}_{ijk}. \tag{29}$$

On the other hand contracting (1) with $g^{hk}$ we get the equality

$$L(S_{lr}C^r{}_{jim} + S_{mr}C^r{}_{jli} + S_{lr}C^r{}_{ijm} + S_{mr}C^r{}_{ilj}) = 0,$$

which, in virtue of (26), takes the form $L(S_{ir}C^r{}_{jlm} + S_{jr}C^r{}_{ilm}) = 0$. Thus in the case $L(n-2) = n$ we also have (29). Substituting (29) into (28) we obtain

$$L\kappa C_{mijk} = (L+1)S_{mr}C^r{}_{ijk}. \tag{30}$$

We shall show that $L \neq -1$. Suppose that $L = -1$. Thus from (30) it follows that $\kappa = 0$ and (27) and (22) take the forms

$$\begin{aligned}(1-n)Q(S,C)_{hijklm} &= g_{hl}S_{mr}C^r{}_{ijk} - g_{hm}S_{lr}C^r{}_{ijk} - g_{il}S_{mr}C^r{}_{hjk} + g_{im}S_{lr}C^r{}_{hjk} \\ &+ g_{jl}S_{mr}C^r{}_{khi} - g_{jm}S_{lr}C^r{}_{khi} - g_{kl}S_{mr}C^r{}_{jhi} + g_{km}S_{lr}C^r{}_{jhi}\end{aligned} \tag{31}$$

and

$$(1-n)(d_m C_{lijk} - d_l C_{mijk}) = a_m S_{lr}C^r{}_{ijk} - a_l S_{mr}C^r{}_{ijk}, \tag{32}$$

respectively. But using (29) we can rewrite the right hand side of the last equation as

$$\begin{aligned}-(a_m S_{ir}C^r{}_{ljk} - a_l S_{ir}C^r{}_{mjk}) &= -S_i{}^r(a_m C_{rljk} - a_l C_{rmjk}) \\ &= -S_i{}^r a_r C_{mljk} = -d_i C_{mljk}.\end{aligned}$$

Thus (32) takes the form

$$(n-1)(d_m C_{lijk} - d_l C_{mijk}) = d_i C_{mljk}\,.$$

Hence, by standard calculation, we can obtain $d_i = 0$. Applying this to (32) we have $a_m S_l{}^r C_{rijk} = a_l S_m^r C_{rijk}$ and, in virtue of (29),

$$a_m S_i{}^r C_{rljk} = -a_m S_l{}^r C_{rijk}\,.$$

We put $A_{mlijk} = a_m S_l{}^r C_{rijk}$. We see that the tensor $A$ is symmetric with respect to $m, l$ and antisymmetric with respect to $i, l$, which, in view of Lemma 2.1, implies

$A = 0$. Hence $S_l{}^r C_{rijk} = 0$ and (31) implies now $Q(S, C) = 0$, a contradiction. Thus we have $L \neq -1$ and we can rewrite (30) in the form

$$S_{mr} C^r{}_{ijk} = \phi C_{mijk}, \quad \text{where} \quad \phi = \frac{L\kappa}{L+1}. \tag{33}$$

Substituting (33) into (24) and using (17) we find

$$(n-2)L\phi a_i C_{mljk} = \left(2\phi((n-2)L - 1) + \frac{\kappa}{n-1}\right) a_i C_{mljk},$$

which implies

$$(n-2)L\phi = \left(2\phi((n-2)L - 1) + \frac{\kappa}{n-1}\right)$$

and next

$$\kappa\left(\frac{L}{L+1}((n-2)L - 2) + \frac{1}{n-1}\right) = 0.$$

We consider two cases:

(i) $\kappa = 0$. In this case from (33) we have $S_{mr} C^r{}_{ijk} = 0$ and taking into account (27), we obtain $L = 1/(n-2)$.

(ii) $\kappa \neq 0$. In this case we get the following equation

$$L(n-1)((n-2)L - 2) + L + 1 = 0,$$

which has two solutions: $L = 1/(n-2)$ or $L = 1/(n-1)$. This completes the proof.

Corollary 4.1. *Suppose that (M,g) satisfies the assumptions of the last theorem. If $L = 1/(n-1)$, then (M,g) is pseudosymmetric.*

*Proof.* For $L = 1/(n-1)$ (33) takes the form $S_{mr} C^r{}_{ijk} = (\kappa/n)\, C_{mijk}$. Substituting this into (27) we find

$$Q(S, C) = \frac{\kappa}{n} Q(g, C).$$

Now (1) implies

$$R \cdot C = \frac{\kappa}{n(n-1)} Q(g, C),$$

which denotes that $(M, g)$ is Weyl-pseudosymmetric at $x$. From Lemma 2.5 we conclude our assertion.

*Remark* 4.1. It will be shown in the next section that a manifold $(M, g)$ with the associated fundamental function $L = 1/(n-2)$ need not be pseudosymmetric.

## 5. Conformal deformations of e.c.s. manifolds

A semi-Riemannian manifold $(M, g)$ is said to be conformally symmetric if its Weyl conformal curvature tensor $C$ satisfies the condition $\nabla C = 0$. Conformally symmetric manifolds which are neither conformally flat nor locally symmetric are called essentially conformally symmetric (e.c.s. in short). It is known that every e.c.s. manifold is semisymmetric [**6**, Theorem 9].

Theorem 5.1. *Let (M,g) be an e.c.s. manifold. Assume that M admits a function p such that* $\bar{g} = \exp(2p)g$ *is a metric with harmonic Weyl conformal curvature tensor* $\bar{C}$. *Then:*

(*i*) *If* $(M, \bar{g})$ *satisfies the relation* (1) *and is not pseudosymmetric, then* $\Delta_2 p = 0$.

(*ii*) *If* $\Delta_2 p = 0$, *then* $\bar{R} \cdot \bar{C} = (1/(n-2))\, Q(\bar{S}, \bar{C})$.

*Proof.* We assert that all e.c.s. manifolds satisfy the condition (16). Every e.c.s. manifold satisfies the condition $\sum_{X,Y,Z} S(W,X)\tilde{C}(Y,Z) = 0$ [**7**, Theorem 7]. This implies (16) with $a \neq 0$ at any point at which $S \neq 0$ and, in virtue of parallelity of $C$, everywhere on $M$. Since $C$ is parallel and $\bar{C}$ is harmonic ($\bar{\nabla}_r \bar{C}^r{}_{ijk} = 0$), the equality (8) leads to $p_r C^r{}_{ijk} = 0$, whence

$$P_{lr} C^r{}_{ijk} = 0. \tag{34}$$

Now (9) takes the form

$$\exp(-2p)(\bar{R} \cdot \bar{C}) = -\Delta_1 p \colon Q(g, C) - Q(P, C). \tag{35}$$

Assume now that $(M, \bar{g})$ satisfies (1). Since $(M, \bar{g})$ also satisfies (16), so using Theorem 4.1 and Corollary 4.1 we can rewrite (35) in the form

$$Q\Big(\frac{1}{n-2}\bar{S}, C\Big) = -\Delta_1 p \colon Q(g, C) - Q(P, C).$$

Hence, in virtue of (4) and $Q(S, C) = 0$ [**6**, Lemma 7], we get $\Delta_2 p \colon Q(g, C) = 0$, which implies $\Delta_2 p = 0$ and ends the proof of (i).

Assume now that $\Delta_2 p = 0$. Substituting the equality

$$P = \frac{1}{n-2} S - \frac{1}{n-2}\bar{S} - \Delta_1 p g$$

into (35) and using $Q(S, C) = 0$, we easily obtain $\bar{R} \cdot \bar{C} = \dfrac{1}{n-2} Q(\bar{S}, \bar{C})$. This completes the proof.

*Example* 5.1. Let $M = \{x \in \mathbb{R}^5 \mid x^2 + x^3 > 0\}$ be endowed with the metric given by (10), where $Q = (A \colon k_{\lambda\mu} + a_{\lambda\mu}) x^\lambda x^\mu$. $A$ is nonconstant function of $x^1$ only and

$$[a_{\lambda\mu}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad [k_{\lambda\mu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is known that $(M,g)$ is essentially conformally symmetric and Ricci-recurrent manifold [**17**]. Further, it is easy to see, in view of (11) and (12), that the function $p(x) = x^2 + x^3$ satisfies equations: $p^r C_{\ rijk} = 0$, $\Delta_2 p = 0$ and $\Delta_1 p = 2$. Thus, according to Theorem 5.1, the metric $\bar{g} = \exp(2p)g$ satisfies the condition (1). We assert that this metric cannot be pseudosymmetric. Conversely, suppose that $\bar{g}$ is pseudosymmetric. Hence $\bar{g}$ is Weyl-pseudosymmetric. Applying now Theorem 3.1 of [**15**], we get $Q(P - (1/n)\ \mathrm{tr}(P)g, C) = 0$. But the only nonzero components of the tensor P are $P_{11}$ and $P_{22} = P_{23} = P_{33} = -1$. This, in virtue of (11) and (12), leads to $Q(P - (1/n)\ \mathrm{tr}(P)g, C)_{221441} \neq 0$, a contradiction. Thus the metric $\bar{g}$ is not pseudosymmetric and, consequently, it cannot be semisymmetric.

*Remark* 5.1. The 5-dimensional metric g, defined in the above example, can be easily extended on any dimension $n > 5$. Namely, we can enlarge matrices $[k_{\lambda\mu}]$ and $[a_{\lambda\mu}]$ such that the equality $a_{\lambda\mu}k^{\lambda\mu} = 0$ is still satisfied (this equality guaranties that the metric g is conformally symmetric).

## 6. Concircular changes of metrics satisfying (1)

Let $g$ be a metric on a manifold $M$ and let $\bar{g}$ be another metric conformally related to $g$, i.e., $\bar{g} = \exp(2p)g$, where $p$ is a non-constant function on $M$. If the tensor $P$ of conformal change of the metric, given by $P = \nabla(dp) - dp \otimes dp$, is proportional to $g$ at every point of $M$, then this conformal change is called concircular.

LEMMA 6.1. *Let (M,g) be a semi–Riemannian manifold and let on M be given a concircular change of metric* $g \longrightarrow \bar{g} = \exp(2p)g$. *Assume that the condition* (1) *is satisfied at a point x of M. Then:*

(*i*) *If* $L = 1/(n-1)$, *then* $\bar{R}\cdot\bar{C} = (1/(n-1))\, Q(\bar{S},\bar{C})$.

(*ii*) *If* $\bar{\kappa} = \exp(-2p)\kappa$, *then* $\bar{R}\cdot\bar{C} = LQ(\bar{S},\bar{C})$ *at x.*

*Proof.* For concircular change of metric we have $P = \dfrac{1}{n}\ \mathrm{tr}(P)g$, where $\mathrm{tr}(P) = \Delta_2 p - \Delta_1 p$. Hence, in virtue of (9), we get

$$\exp(-2p)\bar{R}\cdot\bar{C} = R\cdot C - \Delta_1 p\, Q(g,C) - 2\frac{\mathrm{tr}(P)}{n}Q(g,C) = R\cdot C - \frac{\alpha}{n}Q(g,C),$$

where $\alpha = (n-2)\Delta_1 p + 2\Delta_2 p = (\exp(2p)\bar{\kappa} - \kappa)/(n-1)$ (cf. (5)). Using now our assumption we obtain

$$\exp(-2p)\bar{R}\cdot\bar{C} = Q\Big(LS - \frac{\alpha}{n}g, C\Big). \tag{36}$$

But, in virtue of (4), we have $\bar{S} = S - \dfrac{(n-1)\alpha}{n}g$ and we can rewrite (36) in the form

$$\bar{R}\cdot\bar{C} = LQ(\bar{S},\bar{C}) + \frac{\alpha}{n}(L(n-1)-1)Q(g,\bar{C}).$$

Hence we easily get our assertions, which completes the proof.

Proposition 6.1. *Let (M,g) be a semi-Riemannian manifold satisfying the condition* (1) *and let on M be given a concircular change of metric* $g \longrightarrow \bar{g} = \exp(2p)g$. *Assume that* $\bar{g}$ *also satisfies* (1), *i.e.,*

$$\bar{R} \cdot \bar{C} = \bar{L} Q(\bar{S}, \bar{C}). \tag{37}$$

*If* $L = \bar{L}$ *at x, then* $L = 1/(n-1)$ *or* $\bar{\kappa} = \exp(-2p)\kappa$ *at x.*

*Proof.* Using (1), (9) and (37) we have

$$Q\Big(\bar{L}\bar{S} - LS + \frac{\alpha}{n} g, C\Big) = 0,$$

where $\alpha = (n-2)\Delta_1 p + 2\Delta_2 p = (\exp(2p)\bar{\kappa} - \kappa)/(n-1)$. Hence, in virtue of the relation

$$\bar{S} = S - \frac{(n-1)\alpha}{n} g, \tag{38}$$

which follows from (4), we get

$$Q(A, C) = 0, \quad \text{where} \quad A = S(\bar{L} - L) - \frac{\alpha}{n}(\bar{L}(n-1) - 1)g. \tag{39}$$

Because $\bar{L} = L$, the above equality implies $\bar{L} = 1/(n-1)$ or $\alpha = 0$ and we have the situation described in the previous lemma. This completes the proof.

Theorem 6.1. *Let (M,g) be a semi-Riemannian manifold satisfying the condition* (1) *and let on M be given a concircular change of metric* $g \longrightarrow \bar{g} = \exp(2p)g$. *Assume that* $\bar{g}$ *also satisfies* (1) *with the associated function* $\bar{L}$. *If* $L \neq \bar{L}$ *at x, then the following equation*

$$\kappa(\bar{L} + 1)(L(n-1) - 1) = \exp(2p)\, \bar{\kappa}\, (L + 1)(\bar{L}(n-1) - 1) \tag{40}$$

*holds at x. Moreover, metrics g and* $\bar{g}$ *are pseudosymmetric at x.*

*Proof.* In the same manner as in the proof of the previous proposition we get the equality (39). We shall consider two cases:

(I) $A = 0$. In this case we have

$$S = \frac{\alpha(\bar{L}(n-1) - 1)}{n(\bar{L} - L)} g, \quad R \cdot C = L \frac{\kappa}{n} Q(g, C).$$

So the metric $g$ is Einsteinian and Weyl-pseudosymmetric and consequently, pseudosymmetric. In virtue of (38) $\bar{g}$ is also Einsteinian. Pseudosymmetry of $\bar{g}$ follows immediately from Theorem 5.1 of [**3**].

(II) $A \neq 0$. According to Lemma 2.4 we have two possibilities:

(i) $A = (1/\rho)\, a \otimes a$. Since the covector $a$ satisfies the relation (17) we can apply Theorem 4.1. Thus we have $L = 1/(n-1)$ or $L = 1/(n-2)$. If $L = 1/(n-1)$, then, in virtue of Lemma 6.1, we have $\bar{L} = L$, a contradiction. If $L = 1/(n-2)$, then also $\bar{L} = 1/(n-2)$ (because $\bar{L} = 1/(n-1)$ implies $L = 1/(n-1)$), a contradiction.

(ii) $A - (1/\rho)\, a \otimes a \neq 0$. In this case we have

$$\rho C_{hijk} = \lambda(A_{hk}A_{ij} - A_{hj}A_{ik}). \tag{41}$$

Contracting (41) with $g^{hk}$ we get $A_{ir}A^{r}{}_{j} = \operatorname{tr}(A)A_{ij}$, where $\operatorname{tr}(A) = \kappa(\bar{L} - L) - \alpha(\bar{L}(n-1) - 1)$. Substituting (39) into the above equality we get

$$S_{ir}A^{r}{}_{j} = \phi A_{ij}, \quad \text{where} \quad \phi = \kappa - \frac{\alpha(n-1)}{n(\bar{L} - L)}(\bar{L}(n-1) - 1).$$

Transvecting (41) with $S_{l}{}^{r}$ we obtain $S_{l}{}^{r}C_{rijk} = \phi C_{lijk}$. Substitution of this equality into (14), in virtue of (19) and (1), leads to

$$(L(n-2) - 1)Q(S, C) = \Big(\phi - \frac{\kappa}{n-1}\Big)Q(g, C) = \left(\frac{(n-2)\kappa}{n-1} - \frac{(n-1)\beta}{n(\bar{L} - L)}\right) Q(g, C),$$

where $\beta = \alpha(\bar{L}(n-1) - 1)$.

On the other hand (39) implies $Q(S, C) = \dfrac{\beta}{n(\bar{L} - L)}\, Q(g, C)$. Substituting this relation into the previous one we get

$$\beta(L + 1) = \frac{n\kappa}{n-1}(\bar{L} - L), \tag{42}$$

which can be rewritten in the form (40).

In the same manner as in the proof of Theorem 3.1 we get that the metric $g$ is pseudosymmetric. Moreover, $L_R = \kappa/(n-1) - \beta/n(\bar{L} - L) = \beta L/n(\bar{L} - L)$ (in view of (42)). Pseudosymmetry of $\bar{g}$ we obtain as in the case (I). This completes the proof.

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