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ON SETS OF PERIODIC AND OF RECURRENT POINTS

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Abstract. It is shown that if a continuum X contains the Gehman dendrite as its retract, then there exists a mapping f of X such that the closure of the set of periodic points of f is a proper subset of the closure of the set of recurrent points of f. Other continua with this property are presented, and a number of related questions are asked.

1. Introduction

All spaces considered in this paper are assumed to be metric and separable. By a *continuum* we mean a compact connected space. A locally connected continuum containing no simple closed curve is called a *dendrite.* A *tree* means a one-dimensional compact connected acyclic polyhedron. By an *end point* (in the classical sense) of an arcwise connected continuum X we mean a point p of X which is an end point of every arc A such that $p \in A \subset X$. Thus the concept of a dendrite having finitely many end points coincides with one of a tree. A *ramification point* is defined as a point being the vertex of a simple triod contained in the space; the number of arcs emanating from the point and pairwise disjoint out of it is taken as the *order of the point.*

We denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the set of reals, by $\mathbb{C}$ the set of complex numbers, by $\mathbb{I}$ the unit closed interval $[0,1]$ of reals, and by $\mathbb{S}$ the unit circle, i.e., $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.

Let X be a space, and let $f : X \to X$ be a mapping (i.e., a continuous function) of X to itself, and for any $n \in \mathbb{N}$ let $f^n : X \to X$ denote the n-th iteration of f. A point x of X is said to be:
— a *fixed* point of f, if $f(x) = x$;
— a *periodic* point of f, provided that there is $n \in \mathbb{N}$ such that $f^n(x) = x$;

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— a *recurrent* point of f, provided that for every neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(x) \in U$;
— a *nonwandering* point of f, provided that for every neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$.

The sets of fixed points, periodic points, recurrent points and nonwandering points of a mapping $f : X \to X$ will be denoted by $F(f)$, $P(f)$, $R(f)$ and $\Omega(f)$, respectively. Thus we have

$$F(f) \subset P(f) \subset R(f) \subset \Omega(f) \subset X. \tag{1.1}$$

1.2. *Remarks.* If $f : \mathbb{S} \to \mathbb{S}$ is a rotation of $\mathbb{S}$ by the angle $\pi/4$ (i.e., if f is defined by $f(z) = iz$), we have $F(f) = \emptyset$ and $P(f) = \mathbb{S}$. If $f : \mathbb{S} \to \mathbb{S}$ is an irrational rotation (i.e., a rotation by an angle α such that α/π is irrational), then $P(f) = \emptyset$, while $R(f) = \mathbb{S}$. A piecewise monotone mapping $f : \mathbb{I} \to \mathbb{I}$ with $\operatorname{cl} R(f) \subsetneq \Omega(f)$ is described in Section 3 of [**36**, pp. 183–184]. Finally, for $f : \mathbb{I} \to \mathbb{I}$ defined by $f(t) = t^2$ we have $R(f) = \{0, 1\} \subsetneq \mathbb{I}$. Thus we see that, in general, none of the inclusions in (1.1) can be replaced by the equality.

1.3. *Remarks.* Continua are known for which all the inclusions in (1.1) turn into the equalities. a) Cook has constructed ([**7**, Theorems 8 and 11, pp. 245 and 247]) hereditarily indecomposable and nonplanable continua X such that the identity is the only mapping of X into itself. b) Maćkowiak has constructed ([**23**, Section 4, Theorems 30 and 31, pp. 547 and 549]) hereditarily decomposable and planable continua X such that the identity is the only mapping of X onto itself. Thus for all these continua we have $F(f) = X$ for each $f : X \to X$.

The notions of periodic, recurrent and nonwandering points of a mapping $f : X \to X$ are ones of the most important notions in dynamical systems (see e.g. [**3**] and [**27**]). To formulate in a shorter way some properties of spaces related these notions let us accept the following definition.

1.4. *Definition.* A space X is said to have the *periodic-recurrent property* (shortly *PR-property*) provided that for every mapping $f : X \to X$ the equality holds

$$\operatorname{cl} P(f) = \operatorname{cl} R(f). \tag{1.5}$$

Coven and Hedlund proved in [**9**] the following result.

1.6. THEOREM (Coven and Hedlund). *The closed unit interval* $\mathbb{I}$ *has the PR-property.*

The essential argument used in the proof of Theorem 1.6 was an equivalence saying that for each topological space X and for each mapping $f : X \to X$ the condition $x \in R(f)$ holds if and only if $x \in R(f^n)$ for each $n \in \mathbb{N}$ (Theorem I of [**11**, p. 126], which is a generalization of a similar result for homeomorphisms f due to Gottschalk, see [**14**, Theorem 1, p. 222]).

Special cases of this result have been proved by Block [**4**] (for mappings f with $\Omega(f)$ finite), by Coven and Hedlund [**8**] (for mappings f with $P(f)$ being closed), and by Young [**36**] (for piecewise monotone mappings). In Theorem 2.6 of [**34**, p. 349], Ye generalized the result to mappings on trees.

1.7. Theorem (Ye). *Every tree has the PR-property.*

Since each nondegenerate subcontinuum of a tree T is a tree, equality (1.5) holds not only for each mapping on T, but also for each mapping of a nondegenerate subcontinuum of T onto itself, i.e., every tree has the PR-property hereditarily.

1.8. *Question.* What continua have the PR-property hereditarily?

2. A property of Gehman's dendrite

Since dendrites have often appeared as Julia sets in complex dynamical systems (see [**28**], for example), the dynamical behavior of their automappings is both important and interesting in the study of dynamical systems (and in continuum theory, too). Therefore a question arises in a very natural way concerning a possibility of an extension of equality (1.5) to dendrites, which form the nearest (in a sense) class of curves containing trees. The question was discussed by Kato in [**20**], who has shown (exploiting a general method of constructing infinite telescopes $T(X)$ over an arbitrary compact metric space X as an application of inverse limits, due to Krasinkiewicz, see [**21**, Sections 3 and 4, pp. 98–105]) that equality (1.5) fails for the well-known Gehman dendrite G (see [**12**, the example on p. 42]; see also [**24**, pp. 422–423] for a detailed description, and [**26**, Fig. 1 on p. 203], for a picture; note that the infinite binary tree is another name of this dendrite, see [**10**, p. 12]). Recall that G can be characterized as the only dendrite whose set of end points is homeomorphic to the Cantor set, and whose ramification points are of order 3 only (see [**25**, p. 100]). Namely Kato defines a mapping $f : G \to G$ such that equality (1.5) does not hold, i.e., according to (1.1), that

$$\operatorname{cl} P(f) \subsetneq \operatorname{cl} R(f). \tag{2.1}$$

For further purposes we need some auxiliary definitions and results. A mapping $r : X \to Y$ from a continuum X onto a continuum $Y \subset X$ is called a *retraction* provided that $r(y) = y$ for each point $y \in Y$. Then Y is called a *retract* of X. A compact metric space Y is called an *absolute retract* provided that every homeomorphic image of Y lying in an arbitrary separable metric space X is a retract of X. Recall the following theorem (see [**22**, §53, III, Theorem 5, p. 341]).

(2.2) *A compact metric space Y is an absolute retract if and only if for each closed subset A of a separable metric space X each mapping from A to Y has a (continuous) extension over X.*

The following results concerning dendrites are known (see [**13**, Theorem, p. 157], and [**22**, §53, III, Theorem 16, p. 344]).

(2.3) *A continuum X is a dendrite*[2] *if and only if each subcontinuum of X is a monotone retract of X.*

(2.4) *Each dendrite is an absolute retract.*

To extend Kato's result to other continua, in particular to other dendrites, as well as to exhibit other important consequences of the existence of the mapping $f : G \to G$ satisfying (2.1) (whose proofs depend heavily on the definition and properties of the mapping) we have to recall both the construction of G and the description of the mapping. Since we need a geometric picture of G in the plane which helps to understand the mapping, we omit the general method of construction of infinite telescopes: this part of Kato's argument is not necessary to draw further consequences from the result.

If $q = (x, y) \in \mathbb{R}^2$, we define $\pi_x(q) = x$ and $\pi_y(q) = y$. Let $C \subset \mathbb{I}$ be the Cantor ternary set. Put $p = (1/2, 1)$ and join p with $(0, 0)$ and $(1, 0)$ by straight line segments. Let T_1 be the union of the two segments. Let $p(0)$, $p(1) \in T_1$ be such that

$$\pi_x(p(0)) < \pi_x(p(1)) \quad \text{and} \quad \pi_y(p(0)) = \pi_y(p(1)) = 1/3.$$

Join $p(0)$ with $(1/3, 0)$ and $p(1)$ with $(2/3, 0)$ by straight line segments, and let T_2 be the union of T_1 and the two segments. Next take four points

$$p(00),\ p(01),\ p(10),\ p(11) \in T_2$$

defined by

$$\pi_x(p(00)) < \pi_x(p(01)) < \pi_x(p(10)) < \pi_x(p(11))$$

and

$$\text{if} \quad \alpha_1, \alpha_2 \in \{0, 1\}, \quad \text{then} \quad \pi_y(p(\alpha_1\alpha_2)) = 1/3^2.$$

Joining them consecutively with the points $(1/3^2, 0)$, $(2/3^2, 0)$, $(7/3^2, 0)$, and $(8/3^2, 0)$ by straight line segments we define T_3 as the union of T_2 and the four recently constructed segments.

Continuing in this way we obtain an increasing sequence of trees

$$T_1 \subset T_2 \subset \cdots \subset T_n \subset \ldots$$

such that if $n > 1$ then T_n has $2^1 + 2^2 + \cdots + 2^{n-1}$ ramification points each of which is of order 3 and is of the form $p(\alpha_1 \ldots \alpha_{n-1})$, where $\alpha_i \in \{0, 1\}$. Further, each T_n has 2^n end points, each of which is of the form $(c, 0)$, where $c \in C$ is an end point of an open interval being a component of $\mathbb{I} \setminus C$. Then the Gehman dendrite G is defined as

$$G = \operatorname{cl}\left(\bigcup\{T_n : n \in \mathbb{N}\}\right),$$

and we see that if $\operatorname{End} G$ and $\operatorname{Ram} G$ stand for the sets of end points and of ramification points of G respectively, then

$$\operatorname{End} G = \{(c, 0) : c \in C\} \quad \text{and} \quad \operatorname{Ram} G = \{p(\alpha_1 \ldots \alpha_n) : \alpha_i \in \{0, 1\} \text{ and } n \in \mathbb{N}\}.$$

The key argument in constructing the above mentioned mapping $f : G \to G$ satisfying (2.1) is the existence of a homeomorphism $g : C \to C$ such that

$$P(g) = \emptyset \quad \text{and} \quad R(g) = C. \tag{2.5}$$

Such a homeomorphism g is defined as the binary adding machine, see [**20**, p. 461]. Since C is homeomorphic to $\operatorname{End} G \subset G$ under a homeomorphism $c \mapsto (c, 0)$ for $c \in C$, we can consider g as a mapping from $\operatorname{End} G$ to itself. Define $g_1 : \{p\} \cup \operatorname{End} G \to \{p\} \cup \operatorname{End} G$ by $g_1(p) = p$ and $g_1 | \operatorname{End} G = g$. Further, since G is an absolute retract (2.4), and since $\{p\} \cup \operatorname{End} G$ is a closed subset of G, there is by (2.2) an extension $f_1 : G \to G$ of g_1.

Note that for each $t \in [0, 1]$ the set $\pi_y^{-1}([t, 1])$ is a subdendrite of G. Thus by (2.3) there is a monotone retraction $r_t : X \to \pi_y^{-1}([t, 1])$ (observe that such a retraction is uniquely determined). Choose a homeomorphism $h : \mathbb{I} \to \mathbb{I}$ defined by $h(s) = \sqrt{s}$ and note that

$$h(0) = 0, \quad h(1) = 1, \quad \text{and} \quad h(s) > s \text{ for each } s \in (0, 1). \tag{2.6}$$

For each point $q \in G$ put

$$t = h(\pi_y(q)) \tag{2.7}$$

and define the needed mapping $f : G \to G$ by

$$f(q) = r_t(f_1(q)), \tag{2.8}$$

where the index t depends on q according to (2.7). Note that f defined in this way is continuous and onto. Further, condition (2.6) and the definition of f imply that

$$f | (\{p\} \cup \operatorname{End} G) = g_1 \quad \text{and thus} \quad f | \operatorname{End} G = g, \tag{2.9}$$

and

$$\pi_y(q) < \pi_y(f(q)) \text{ for each point } q \in G \setminus (\{p\} \cup \operatorname{End} G). \tag{2.10}$$

Condition (2.10) means that each point $q \in G$ for which $0 < \pi_y(q) < 1$ moves "up" under f, whence it follows that no such a point is in $R(f)$. So, we conclude that $R(f) \subset \{p\} \cup \operatorname{End} G$. On the other hand, we see that $p \in F(f)$ and $R(g) \subset R(f)$ by the definition of f, whence

$$R(f) = \{p\} \cup \operatorname{End} G \tag{2.11}$$

by (1.1) and (2.5). Analogously, (2.10) and the definition of f imply

$$P(f) = \{p\}, \tag{2.12}$$

and thus (2.1) holds.

Therefore, Kato's result quoted above can be reformulated as follows.

2.13. Theorem (Kato)**.** *Gehman's dendrite does not have the PR-property.*

This result motivates asking the following question.

2.14. *Question***.** What continua have the PR-property?

A similar author's question for dendrites has recently been answered in full by A. Illanes (see Theorem 3.9 and Corollary 3.10 in the next section).

3. Generalizations

The lemma below is a consequence of definitions.

3.1. Lemma. *Let $r : X \to Y$ be a retraction of a continuum X onto a continuum $Y \subset X$, and let $f : Y \to Y$ be a mapping. Then*

$$(f \circ r)^n = f^n \circ r \quad \textit{for each} \quad n \in \mathbb{N}.$$

3.2. Proposition. *The PR-property is preserved under retractions, i.e., if a continuum X having the PR-property contains a subcontinuum Y which is a retract of X, then Y has the PR-property, too.*

Proof. Let $f : Y \to Y$ be a mapping satisfying (2.1), and let $r : X \to Y$ be a retraction. Define a mapping $g : X \to X$ by $g = f \circ r$. Then for each $n \in \mathbb{N}$ the equality $g^n = f^n \circ r$ holds by Lemma 3.1, and $r(x) = x$ if and only if $x \in Y$ by the definition of r. If $x \in Y$, then $g^n(x) = f^n(x)$. If $x \in X \setminus Y$, then putting $y = r(x)$ we have $g^n(x) = f^n(r(x)) = f^n(y)$. Thus in both cases we have $g^n = f^n$, whence the equalities hold

$$P(g) = P(f) \quad \text{and} \quad R(g) = R(f).$$

Thus $\operatorname{cl} P(g) \subsetneq \operatorname{cl} R(g)$ by (2.1), and the proof is finished.

Theorem 2.13 and Proposition 3.2 imply, by (2.4), the following assertion.

3.3. Theorem. *If a continuum contains a Gehman dendrite, then it does not have the PR-property.*

3.4. Corollary. *Each dendrite containing the Gehman dendrite does not have the PR-property.*

Let X be a continuum. We say that a point $p \in X$ is of order ω in X provided that it has arbitrarily small open neighborhoods U with finite boundaries $\operatorname{bd} U$ and $\operatorname{card}(\operatorname{bd} U)$ is not bounded by any natural number. It is well-known that a ramification point of a dendrite is either a point of a finite natural order $m \geq 3$ or of order ω (see [**22**, §51, VI, Theorem 4, p. 301]).

Let $m \in \{3, 4, \dots, \omega\}$. The *standard universal dendrite of order* m means a dendrite D_m such that each ramification point of D_m is of order m and for every arc A contained in D_m the set of all ramification points of D_m which belong to A is a dense subset of A. By results of WaDzewski (see [**30**, Chapter H, pp. 123–124, and Chapter K, p. 137]) it is known that any two such dendrites are homeomorphic and that they are universal in the class of dendrites whose ramification points are of order m at most (i.e., if all ramification points of a dendrite X are of order at most m, then D_m contains a homeomorphic copy of X). Hence we can write the following sequence of inclusions (to simplify notation, the homeomorphisms are omitted):

$$G \subset D_3 \subset D_4 \subset \cdots \subset D_m \subset \cdots \subset D_\omega.$$

Thus, according to Corollary 3.4, we have the next one.

3.5. Corollary. *For each $m \in \{3, 4, \dots, \omega\}$ the standard universal dendrite D_m of order m does not have the PR-property.*

3.6. *Remarks.* Observe that the inverse implications to one of Proposition 3.2 do not hold. Namely we have the following examples. a) Taking the Gehman dendrite G as X and any arc in G as Y we see that X does not have the PR-property, Y is a retract of X, and Y has the PR-property. b) Taking a disc as X and its boundary in the plane as Y we see that both X and Y do not have the PR-property (the disc by Theorem 3.3; its boundary, being homeomorphic to the unit circle, by the existence of an irrational rotation mentioned in Remark 1.2), while there is no retraction of X onto Y.

3.7. *Question.* Can the existence of a retraction $r : X \to Y \subset X$ in Proposition 3.2 be relaxed to the existence of any mapping from X onto Y?

Note that all the examples discussed above of dendrites which do not have the PR-property contain the Gehman dendrite G, according to Corollary 3.4. The converse implication to that of Corollary 3.4 is also true, as it recently was shown by Illanes, [**16**]. Namely the following result is both interesting and of a particular importance.

3.8. Theorem (Illanes). *Every dendrite which does not have the PR-property contains a homeomorphic image of the Gehman dendrite.*

Thus by Corollary 3.4 we have the following characterization of dendrites with the PR-property.

3.9. Theorem. *A dendrite has the PR-property if and only if it does not contain any homeomorphic copy of the Gehman dendrite.*

A continuum which is arcwise connected and hereditarily unicoherent is called a *dendroid.* It is well-known that each dendroid is hereditarily decomposable, thus one-dimensional, and that each locally connected dendroid is a dendrite. Therefore dendroids form the nearest (in a sense) class of curves containing the class of dendrites. An important example of a dendroid which is not a dendrite is the Cantor fan, i.e., the cone over the Cantor set. Let, as previously, $C \subset \mathbb{I}$ be the Cantor ternary set, and let $p = (1/2, 1) \in \mathbb{R}^2$. For each $c \in C$ let L_c denote the straight line segment joining p with $(c, 0)$. Then the Cantor fan F_C is defined as the union

$$F_C = \bigcup \{L_c : c \in C\}. \tag{3.9}$$

3.10. Theorem. *The Cantor fan does not have the PR-property.*

Proof. The proof for the Cantor fan runs in a similar way as Kato's proof of Theorem 2.13 for the Gehman dendrite. Namely since the set $\operatorname{End} F_C$ of end points of F_C is homeomorphic with C, we take the homeomorphism $g : \operatorname{End} F_C \to \operatorname{End} F_C$ such that

$$P(g) = \emptyset \quad \text{and} \quad R(g) = C, \tag{2.5}$$

we extend it to the needed mapping $f : F_C \to F_C$ defined so that $f(p) = p$, and if $q \in L_c \setminus \{p\}$ for some $c \in C$ (note that such a c is uniquely determined), then $f(q) \in L_d$, where $(d, 0) = g((c, 0))$ and $\pi_y(f(q)) = \pi_y(q)$. It can be observed that, even without using the homeomorphism h satisfying (2.6), as previously (see (2.11) and (2.12)) we have $P(f) = \{p\}$, and $R(f) = \{p\} \cup \operatorname{End} F_C$, whence the conclusion follows.

Theorem 3.10 and Proposition 3.2 imply a corollary.

3.12. COROLLARY. *If a continuum X contains the Cantor fan F_C as its retract, then X does not have the PR-property.*

3.13. *Remark.* Observe that the Cantor fan does not contain the Gehman dendrite, thus the assumption in Theorem 3.8 that the continuum X is a dendrite is indispensable.

The result of Kato can be seen also from the inverse limits point of view. Namely the Gehman dendrite G is the inverse limit of an increasing sequence of trees $T_n \subset G$ with monotone retractions $r_n : T_{n+1} \to T_n$ as bonding mappings. Let $f : G \to G$ be the mapping satisfying (2.1), and take a sequence of mappings $f_n : T_n \to T_n$ such that $f = \operatorname{invlim} f_n$. Then for each $n \in \mathbb{N}$ the mapping f_n satisfies equality (1.5) by the result of Ye quoted above (Theorem 1.7), while the inverse limit mapping f does not. Thus we have the following statement.

3.14. STATEMENT. *Equality* (1.5) *is not preserved under the inverse limits of trees.*

This interpretation of the Kato's result directs our attention to inverse limits of arcs, and thus it motivates the following question.

3.15. *Question.* Is equality (1.5) preserved under the inverse limits of arcs? In other words: can the Coven and Hedlund's result (Theorem 1.6 above) be generalized to arc-like continua?

As a particular cases of the above question consider:
a) the $\sin(1/x)$-curve $S = \operatorname{invlim}\{X_n, \phi_n\}$, where for each $n \in \mathbb{N}$ we have $X_n = \mathbb{I}$, and $\phi_n \mathbb{I} :\to \mathbb{I}$ is defined by

$$\phi_n(t) = 2t \quad \text{for} \quad t \in [0, 1/2] \quad \text{and} \quad \phi_n(t) = 3/2 - t \quad \text{for} \quad t \in [1/2, 1];$$

it is well-known that S is homeomorphic to the following subcontinuum of the plane $\mathbb{R}^2$:

$$S = \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\} \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 : x \in (0, 1]\};$$

b) the simplest indecomposable continuum $B = \operatorname{invlim}\{X_n, \psi_n\}$, where for each $n \in \mathbb{N}$ we have $X_n = \mathbb{I}$, and $\psi_n : \mathbb{I} \to \mathbb{I}$ is defined by

$$\psi_n(t) = 2t \quad \text{for} \quad t \in [0, 1/2] \quad \text{and} \quad \psi_n(t) = 2 - 2t \quad \text{for} \quad t \in [1/2, 1];$$

it is well-known that B is homeomorphic to the plane continuum described as Example 1 of [**22**, §48, V, p. 204] (see e.g. [**29**]).

The following result has recently been obtained in [**6**, Corollary 5.10, p. 117].

3.16. Theorem. *The* $\sin(1/x)$*-curve* S *has the PR-property.*

3.17. *Question.* Has the simplest indecomposable continuum B the PR-property?

Recall that the inverse limits of continua, especially of arcs, were extensively studied during the last decade from the dynamical system point of view, see e.g. [**1**], [**2**], [**5**], [**17**], [**18**], and [**19**], where further references can be found. In particular, Ye have shown (see [**35**, Corollary 3.5, p. 92] that if an arc-like continuum X is hereditarily decomposable and $\mathrm{Order}(X)$ is finite (see [**35**, p. 87] for the definition; compare [**15**]), then the equality $P(f) = R(f)$ holds for every homeomorphism $f : X \to X$. Note that the $\sin(1/x)$-curve S is an example of such a continuum X.

4. Final remarks

In connection with the result of Coven and Hedlund (Theorem 1.6) and its generalization by Ye (Theorem 1.7) recall other concepts which are important in the study of dynamical behavior of mappings on continua. They are connected with another aspect of the result of Kato (Theorem 2.13). Repeat that for a mapping $f : X \to X$ on a space X the symbol $\Omega(f)$ means the set of all nonwandering points of X under f. Let $\Omega_1(f) = \Omega(f)$ and, for each positive integer n put $\Omega_{n+1}(f) = \Omega(f|\Omega_n(f))$. Then

$$\Omega_\infty(f) = \bigcap\{\Omega_n(f) : n \in \mathbb{N}\}$$

is called the (Birkhoff) *centre of* f. The minimal $n \in \mathbb{N} \cup \{\infty\}$ such that $\Omega_n(f) = \Omega_\infty(f)$ is called the *depth of the centre of* f.

For $X = \mathbb{I}$ the following result is known (see Nitecki's expository paper [**27**, Theorems (3.3a) and (3.3b), pp. 30–31]; for a simpler proof see [**33**]; Nitecki credits A. Block for the first equality in (4.2), and Coven and Hedlund [**9**] for the second one).

4.1. Theorem. *For every mapping* $f : \mathbb{I} \to \mathbb{I}$ *the equalities hold*

$$\Omega_2(f) = \mathrm{cl}\, P(f) = \Omega_\infty(f), \tag{4.2}$$

and the depth of the centre of f *is at most* 2.

This result has been extended by Wu in [**32**] to mappings of an n-od to itself (by an *n-od* is meant a set homeomorphic to $\{z \in \mathbb{C} : z^n \in \mathbb{I}\}$). A similar result, being a version of Theorem 4.1 and of the result of Wu, has been shown by Ye (see [**34**, Theorem 2.7, p. 349]) for mappings of trees. It runs as follows.

4.3. Theorem (Ye). *Let* T *be a tree equipped with a metric* ρ*, and let* $\mathrm{Ram}\, T$ *be the set of ramification points of* T*. Then for every mapping* $f : T \to T$ *there is a set* $A \subset \mathrm{Ram}\, T$ *such that* $\rho(A, \mathrm{cl}\, P(f)) > 0$*, and*

$$\Omega_2(f) = A \cup \mathrm{cl}\, P(f), \tag{4.4}$$

whence it follows that

$$\Omega_\infty(f) = \operatorname{cl} P(f), \tag{4.5}$$

and the depth of the centre of f is at most 3.

Consider once more the Gehman dendrite G and its mapping $f : G \to G$ defined by (2.8). Its property (2.10) assures us not only that (2.11) and (2.12) are true, but also that

$$\Omega(f) = R(f) = \{p\} \cup \operatorname{End} G, \tag{4.6}$$

whence it follows by the definition of f that

$$\Omega_n(f) = \{p\} \cup \operatorname{End} G \quad \text{for each} \quad n > 2, \tag{4.7}$$

and consequently

$$\Omega_\infty(f) = \{p\} \cup \operatorname{End} G, \tag{4.8}$$

and thus $\Omega_\infty(f) \neq \operatorname{cl} P(f)$ by (2.12) and (4.8). Therefore we see that neither (4.4) nor (4.5) of Theorem 4.3 is true if dendrites in place of trees are under consideration. So, we have the following observation.

4.9. OBSERVATION. *The mapping $f : G \to G$ defined by Kato on the Gehman dendrite G according to* (2.8) *shows not only that Theorem* 1.7 *of Ye is not true for dendrites, but also that Theorem* 4.3 *of Ye cannot be generalized from trees to arbitrary dendrites as well.*

Thus the result of Illanes (Theorem 3.8 above) motivates the following question.

4.10. *Question.* Can Theorem 4.3 of Ye be generalized from trees to dendrites which do not contain any homeomorphic copy of the Gehman dendrite?

The main result of Worth's paper [**31**, Theorem 1.3, p. 623] states that every mapping $f : T \to T$ of a tree T into itself such that the intersection $\bigcap\{f^n(T) : n \in \mathbb{N}\}$ is not a singleton must have at least two periodic points. Below it is shown that this result cannot be extended to mappings of dendrites.

4.11. THEOREM. *For every dendrite D which contains a Gehman dendrite G there exists a mapping $m : D \to G \subset D$ such that the intersection $\bigcap\{m^n(D) : n \in \mathbb{N}\}$ is G and that $P(m)$ is a singleton.*

Proof. Let $r : D \to G$ be a retraction (see (2.3)), and let $f : G \to G$ be a surjective mapping defined by (2.8). Put $m = f \circ r$. Then, according to Lemma 3.1, we get $m^n(D) = f^n(r(D)) = f^n(G) = G$ for each $n \in \mathbb{N}$, whence the first part of the conclusion follows. Further, again by Lemma 3.1, we have $P(m) = P(f)$, so the second part holds by (2.12). The argument is complete.

4.12. *Question.* Can the above mentioned result of Worth be extended to mappings of dendrites which do not contain any homeomorphic copy of the Gehman dendrite?

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