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# A NEW RATIONAL FREE-FERMION SOLUTION OF THE YANG-BAXTER EQUATION

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**Abstract.** We give explicit formulae of a new family of rational $(4\times 4)$ $R$-matrices, satisfying the free-fermion condition. These solutions are of rank 2 and with rational irreducible spectral curves. That was the last geometrically possible situation in which solutions had not been known.

During the seventies and the eighties great efforts were made in finding and describing the solutions of the Yang-Baxter equation:

$$R^{12}(\theta_1)R^{13}(\theta_2)R^{23}(\theta_1-\theta_2)=R^{23}(\theta_1-\theta_2)R^{13}(\theta_2)R^{12}(\theta_1)$$

where $R(\theta)$ is a linear operator from $V\otimes V$ to $V\otimes V$, and $R^{ij}(\theta): V\otimes V\otimes V\to V\otimes V\otimes V$ is an operator acting on the $i$-th and the $j$-th components as $R(\theta)$ and as identity on the third component; $\theta$ is a complex parameter, called *spectral.* Almost always the "quasiclassical" property of the solutions was presumed: additionally they smoothly depend on parameter $\eta$ (Plank constant), $R(\theta,\eta)\big|_{\eta=0}=I$ and $r(\theta)=\frac{\partial}{\partial\eta}R(\theta,\eta)\big|_{\eta=0}$ is so-called classical $r$-matrix satisfying the classical Yang-Baxter equation. The other solutions, without quasiclassical property, were usually treated as strange and rarely investigated, and they were considered only in a few papers [**1**–**8**].

The best known among them is the Felderhof free-fermion solution [**1**–**8**]. It was found in the beginning of the seventies, almost in the same time as the celebrated Baxter solution. In this note we will use the parametrization of the Felderhof $R$-matrices given in [**6**–**8**]:

$$\Phi=\begin{pmatrix} b_1 & 0 & 0 & d\\ 0 & b_2 & c & 0\\ 0 & c & b_3 & 0\\ d & 0 & 0 & b_0\end{pmatrix}$$

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The key property of the matrix elements of the Felderhof $R$-matrix is *the free-fermion condition*

$$b_0 b_1 + b_2 b_3 = c^2 + d^2$$

The explicit formulae for the matrix elements will be given at the beginning of Section 2.

The classification problem for the solutions of the Yang-Baxter equation is far from being solved. The first serious steps were made by Krichever, in the basic $(4 \times 4)$ case, in 1981. He introduced and analyzed the vacuum vectors, the spectral curves and the rank of the solution (see below). He stated that in the general position all $(4 \times 4)$ rank 1 solutions are reduced to the Baxter's solution, and rank 2 solutions to the Felderhof's solutions. This solutions have elliptical spectral curves. In [**9, 10**] rank 1 solutions out of the general position were classified. They could have rational reducible spectral curves (represented by $R_{XXZ}$, [**9**]) and rational irreducible spectral curves (Cherednik's $R$-matrix [**10**]). Rank 2 solutions with rational reducible spectral curves are very well known free-fermion six-vertex $R$-matrices (see, for example [**6**–**8**]).

Here we give the missing solution – of rank 2 and with rational irreducible spectral curve. We use nontrivial gauge limit (see [**10**]) and the obtained solution is a free-fermion type analogue of Cherednik's $R$-matrix.

1. As it is well known, two families $R(\theta)$ and $R_1(\theta)$ of solutions of the Yang-Baxter equation, are gauge equivalent, if there exists a $(2 \times 2)$ matrix $T$ such that:

$$R_1(\theta) = T^{-1} \otimes T^{-1} R\, T \otimes T.$$

Let us also recall some of Krichever's definitions. An even-dimensional matrix $L$ can be understood as a function from $C^n \otimes C^2$ to $C^n \otimes C^2$. The vectors of the form $X \otimes U$ mapped by $L$ into the vectors of the same type $Y \otimes V$ are called the *vacuum vectors*. The vacuum vectors are parametrized by a curve $\Gamma$ defined by the equation

$$P(u,v) = \det(L^i_j(u,v)) = 0,$$

where $L^i_j(u,v) = \tilde{V}^\beta L^{\alpha i}_{\beta j} U_\alpha, \tilde{V} = (1,-v), U_2 = V_2 = X_n = Y_n = 1, U_1 = u, V_1 = v$. The curve $\Gamma$ is called the *spectral curve*, and $P(u,v)$ is the *spectral polynomial*.

The $(4 \times 4)$ solution $R(\theta)$ of the Yang-Baxter equation is of rank 2 if the spectral polynomial $Q(u,v)$ of $(8 \times 8)$ matrix $\Lambda_1 = R^{12}(\theta_1)R^{23}(\theta_2)R^{13}(\theta_1 - \theta_2)$ can be represented as an exact square: $Q(u,v) = \tilde{Q}^2(u,v)$. Otherwise, the solution is of rank 1.

The matrices uniquely determine their spectral curves and the vacuum vectors. But also the spectral curves and the vacuum vectors, as functions on the curves with some analytical properties, uniquely define the matrices. That is a consequence of the Riemann-Roch theorem.

If the matrices $R_1$ and $R_2$ are gauge equivalent, i.e., if $R_1 = T \otimes T R_2 T^{-1} \otimes T^{-1}$ then their vacuum vectors are related by $(X_1, U_1, Y_1, V_1) = (TX_2, TU_2, TY_2, TV_2)$. So, we have

Proposition 1. *The spectral curve is a gauge invariant.*

2. The parametrization of Felderhof $R$-matrix given in [**6**–**8**] is:

$$\Phi(\varphi|q,p|k) = \begin{pmatrix} b_1 & 0 & 0 & d \\ 0 & b_2 & c & o \\ 0 & c & b_3 & 0 \\ d & 0 & 0 & b_0 \end{pmatrix}$$

where

$$b_0 = \rho(1 - pqe(\varphi)), b_1 = \rho(e(\varphi) - pq), b_2 = \rho(q - pe(\varphi)), b_3 = \rho(p - qe(\varphi)),$$

$$c = \frac{1}{2} i\rho\sqrt{(1-p^2)}\sqrt{(1-q^2)}(1 - e(\varphi))\Big/ \operatorname{sn}\frac{\varphi}{2},$$

$$d = \frac{1}{2} k\rho\sqrt{(1-p^2)}\sqrt{(1-q^2)}(1 + e(\varphi)) \operatorname{sn}\frac{\varphi}{2},$$

$$e(\varphi) = \operatorname{cn}\varphi + i \operatorname{sn}\varphi;$$

$\operatorname{sn}\varphi$ and $\operatorname{cn}\varphi$ are Jacobi elliptical functions of modules $k$, $\rho$ trivial common constant, $p$ and $q$ are arbitrary constants.

The free-fermion six-vertex model $R$-matrix $\Phi_{XXZ}$ is given by

$$\Phi_{XXZ} = \lim_{k\to 0} \Phi(\varphi|p,q|k)$$

Let

$$T(k) = \begin{pmatrix} (-1/k)^{1/4} & 0 \\ 0 & (-k)^{1/4} \end{pmatrix}$$

and

$$\Phi_1(\varphi|p,q) = \lim_{k\to 0} T^{-1}(k) \otimes T^{-1}(k)\ \Phi(\varphi|p,q|k)\ T(k) \otimes T(k)$$

Then we have explicit formulae of a new family of rational $(4 \times 4)$ $R$ matrices:

Proposition 2. *The explicit formula for $\Phi_1$ is*

$$\Phi_1(\varphi|p,q) = \begin{pmatrix} \tilde{b}_1 & 0 & 0 & 0 \\ 0 & \tilde{b}_2 & \tilde{c} & 0 \\ 0 & \tilde{c} & \tilde{b}_3 & 0 \\ \tilde{d} & 0 & 0 & \tilde{b}_0 \end{pmatrix}$$

*where*

$$\tilde{b}_0 = \rho(1 - pq\tilde{e}(\varphi)), \tilde{b}_1 = \rho(\tilde{e}(\varphi) - pq), \tilde{b}_2 = \rho(q - p\tilde{e}(\varphi)), \tilde{b}_3 = \rho(p - q\tilde{e}(\varphi)),$$

$$\tilde{c} = \frac{1}{2} i\rho\sqrt{(1-p^2)}\sqrt{(1-q^2)}(1 - \tilde{e}(\varphi))\Big/ \sin\left(\frac{\varphi}{2}\right),$$

$$\tilde{d} = \frac{1}{2} \rho\sqrt{(1-p^2)}\sqrt{(1-q^2)}(1 + \tilde{e}(\varphi)) \sin\left(\frac{\varphi}{2}\right),$$

$$\tilde{e}(\varphi) = \cos\varphi + i\sin\varphi.$$

The proof of the proposition is a consequence of the limit characteristics of the elliptical functions (see [**10, 11**]).

This construction is simple and well known (see [**12**]). Probably it was not applied previously to the Felderhof solutions, because it does not work with the most popular parametrizations (from [**1–3**], and [**5**]).

Since $\Phi_1$ is obtained as a limit of a family of rank 2 solutions, we have:

THEOREM 1. *$\Phi_1(\varphi|p,q)$ is a rank two solution of the Yang-Baxter equation.*

We will compare the spectral curves of the solutions $\Phi, \Phi_{XXZ}$ and $\Phi_1$.

PROPOSITION 3. $(a)$ *The spectral curve of the Felderhof R-matrix is elliptical.*

$(b)$ *The spectral curve of the six-vertex free-fermion model R-matrix is rational reducible.*

$(c)$ *The spectral curve of $\Phi_1$ is rational irreducible.*

*Proof.* Spectral curves in $(a), (b), (c)$ are given by $P_a(u,v) = u^2v^2 + \alpha u^2 + \beta v^2 + 1, P_b(u,v) = \alpha u^2 + \beta v^2$ and $P_c(u,v) = u^2v^2 + \alpha u^2 + \beta v^2$. The proposition follows by simple computation.

From Proposition 2, using the gauge invariance of the spectral curves, we get

THEOREM 2. *$\Phi_1$ is not gauge equivalent neither to Felderhof's matrix nor to R-matrix of the six-vertex model.*

On the first sight, it might look paradoxical that two families of equivalent solutions, in a limit, give solutions which are not equivalent. However, this is true since the family $T(k)$ does not have a limit when $k \to 0$.

3. Conclusion. Now we have examples of the rank 2 solutions of the Yang-Baxter equation in all geometrically possible cases: with elliptical, rational reducible and with rational irreducible spectral curves.

Krichever classified in [**5**], as it was mention above, the $(4 \times 4)$ solutions with genus 1 spectral curve of the Yang equation:

$$\mathcal{R}\ \mathcal{L}\ \mathcal{L}' = \mathcal{L}'\ \mathcal{L}\ \mathcal{R}$$

up to the gauge equivalence:

$$\begin{aligned}
\mathcal{R} &\mapsto T_X^{-1} \otimes T_{X'}^{-1}\ \mathcal{R}\ T_X \otimes T_{X'} \\
\mathcal{L} &\mapsto T_X^{-1} \otimes T_U^{-1}\ \mathcal{L}\ \ T_X \otimes T_U \\
\mathcal{L}' &\mapsto T_{X'}^{-1} \otimes T_U^{-1}\ \mathcal{L}'\ T_{X'} \otimes T_U
\end{aligned}$$

(This classification is, of course, weaker then up to the gauge equivalence defined in Section 1.)

One of the basic facts used by Krichever was the commuting property of the spectral polynomials $P_{\mathcal{L}}(u,v)$ and $P_{\mathcal{L}'}(u,v)$ of the solutions $\mathcal{L}, \mathcal{L}'$, in a sense of the 2–2 correspondences.

In our case the situation is practically the same:

LEMMA. *Polynomials* $P(u,v) = u^2v^2 + \alpha u^2 + \beta v^2$ *and* $P_1(u,v) = u^2v^2 + \alpha_1 u^2 + \beta_1 v^2$ *commute as relations if and only if* $\alpha + \beta = \alpha_1 + \beta_1$.

In order to get similar classification in our case, one can repeat other Krichever's arguments using functions on the rational curve with marked points $a, b$, satisfying conditions $f(a) = f(b)$, $\deg D_\infty f = 2$, instead of Jacobi elliptical functions and fractional–linear transformations $\psi : \psi(a) = a, \psi(b) = b$ instead of shift on the elliptical curves.

Let us note, at the end, that it would be very interesting to understand possible generalizations of the free-fermion condition in dimensions greater then 4. It seems like an important step toward the classification of the solutions of the Yang–Baxter equation in arbitrary dimension.

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