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ASYMPTOTIC FORMULAE FOR RECURSIVELY DEFINED BASKAKOV-TYPE OPERATORS

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Abstract. We deal with Baskakov-type operators which are defined by replacing the binomial coefficients with general ones satisfying a recursively relation. We establish a Voronovskaja-type formula and also we give a global approximation property using some weighted norms.

Introduction. In [**7**] Campiti and Metafune introduced and studied a generalization of the classical Bernstein operators consisting in replacing the binomial coefficients with more general ones satisfying suitable recursive relations. Their work was motivated from the development of the study of connections between approximation processes and evolution problems through semigroup theory. During the last years, F. Altomare and his Bari school considered the study of elliptic-parabolic equations by means of positive operators. Among the former papers we quote [**2**], [**3**] and for a unified treatment of this subject we mention the monograph [**5**]. The class of evolution equations whose solutions can be approximated by constructive approximation processes has been enlarged and this fact has led to the new types of operators, see [**4**].

In the same manner, we considered [**1**] a generalization of the Baskakov operators which are related to functions defined on an unbounded interval. We obtained a decomposition of the Baskakov operator as a sum of elementary operators and our operators are expressed through a linear combination of these last ones. Also, we proved that these sequences of linear operators converge towards an operator multiplied by an analytic function.

In this paper we continue to study these Baskakov-type operators previously introduced pointing out new regularity properties. The main result is a

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Voronovskaja-type formula where the second order derivative is perturbed by a term depending on two fixed sequences. Also we establish global approximation properties of these operators in some spaces of functions of polynomial growth.

Background. Let two fixed sequences of real numbers be: $a = (a_n)_{n\geq 1}$, $b = (b_k)_{k\geq 0}$ with $a_1 = b_0$. We consider the numbers $c_{n,k}$ $(n \geq 1,\ k \geq 0)$ which satisfy the following recursive formulae:

$$\begin{gathered} c_{n,0} = a_n,\ n \geq 1; \quad c_{1,k} = b_k,\ k \geq 0; \\ c_{n+1,k} = c_{n,k} + c_{n+1,k-1},\ n \geq 1 \text{ and } k \geq 1. \end{gathered} \tag{1}$$

It is obvious that the sequences a and b determine uniquely these coefficients. Actually, the coefficients from (1) follow the same rule as the binomial ones. Now, we can define the linear operators $L_n^{\langle a,b\rangle}$ as follows:

$$(L_n^{\langle a,b\rangle} f)(x) = \sum_{k=0}^{\infty} c_{n,k} \frac{x^k}{(1+x)^{n+k}} f\Big(\frac{k}{n}\Big), \tag{2}$$

where $f \in C_B[0,\infty)$, the space of real functions continuous on $[0,\infty)$ such that the above series converges uniformly and $\frac{f(x)}{1+x^2}$ is convergent as $x \to \infty$. The space is endowed with the norm $\|f\|_* = \sup\limits_{x\geq 0} \frac{|f(x)|}{1+x^2}$. If $a_n = \lambda,\ n \geq 1$, and $b_k = \lambda,\ k \geq 0$, then $c_{n,k} = \lambda\binom{n+k-1}{k}$ and $L_n^{\langle \lambda,\lambda\rangle} = \lambda V_n$, where V_n represent the Baskakov operators.

We would like to mention some results from [**1**] which will be useful for the purposes of this paper. For the sake of simplification we set:

$$r_{k,n}(x) = \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0. \tag{3}$$

Theorem A. *For $f \in C_B[0,\infty)$ and $n \geq 1$ the following identity*

$$(L_n^{\langle a,b\rangle} f)(x) = \sum_{m=1}^{\infty} a_m (A_{m,n} f)(x) + \sum_{m=0}^{\infty} b_m (B_{m,n} f)(x) \tag{4}$$

holds, where:

$$(A_{1,n} f)(x) = (B_{0,n} f)(x) = \frac{\delta_{1,n}}{2(1+x)} f(0);$$

for any $m \geq 2$,

$$(A_{m,n} f)(x) = \begin{cases} 0, & m \geq n+1 \\ \sum\limits_{k=0}^{\infty} r_{k,n}(x) f(k/n), & m = n \\ \sum\limits_{k=1}^{\infty} \binom{n-m+k-1}{k-1} r_{k,n}(x) f(k/n), & m \leq n-1 \end{cases} \tag{5}$$

for any $m \geq 1$,

(6)
$$(B_{m,n}f)(x) = r_{m,n}(x)f(m/n) + (1-\delta_{1,n})\sum_{k=m+1}^{\infty}\binom{n+k-m-2}{k-m}r_{k,n}(x)f(k/n).$$

Here $\delta_{n,m}$ represents Kronecker's symbol. Also we shall put $e_j(x) = x^j$, $j \in \mathbf{N}$.

THEOREM B. *For* $n \geq 3$, *the following identity holds:*

$$(L_n^{\langle a,b\rangle}e_0)(x) = \sum_{m\geq 1} b_m r_{m,1}(x) + \sum_{m=2}^{n-1} a_m r_{1,m-1}(x) + a_n/(1+x)^{n-1} \tag{7}$$

THEOREM C. *Assume that* f *belongs to* $C_B[0,\infty)$ *and the sequences* a, b *are bounded. Let the function* φ *be defined on* $(0,\infty)$ *as follows:*

$$\varphi = \sigma + \tau \quad \textit{where} \quad \sigma(x) = \sum_{m=2}^{\infty} a_m r_{1,m-1}(x), \quad \tau(x) = \sum_{m=1}^{\infty} b_m r_{m,1}(x). \tag{8}$$

Then $\lim_{n\to\infty} L_n^{\langle a,b\rangle} f = \varphi f$, *uniformly on any compact* $K \subset (0,\infty)$.

Results. The first comment concerns our notation (3) which implies:

$$\sum_{k=0}^{\infty} r_{k,0}(x) = 1+x, \quad \sum_{k=0}^{\infty} k r_{k,0}(x) = x(1+x), \tag{9}$$

$$\sum_{k=0}^{\infty} k^2 r_{k,0}(x) = x(1+x)(2x+1) \quad \text{and} \quad r_{k,n}(x) = (1+x)^{-n} r_{k,0}(x).$$

Also, we recall some useful relations which are fulfilled by V_n:

$$(V_n e_0)(x) = 1, \quad (V_n e_1)(x) = x, \quad (V_n e_2)(x) = x^2 + \frac{x(1+x)}{n}. \tag{10}$$

In what follows, we consider $n \geq 2$ and we calculate the expression of the elementary operators $A_{m,n}$, $B_{m,n}$ defined in Theorem A for the test function e_1. It is evident $(A_{1,n}e_1)(x) = (B_{0,n}e_1)(x) = 0$ and $(A_{m,n}e_1)(x) = 0$ too, for $m \geq n+1$. In accordance with (9) we can write:

$$(A_{n,n}e_1)(x) = \frac{1}{n(1+x)^n}\sum_{k=0}^{\infty} k r_{n,0}(x) = \frac{x}{n(1+x)^{n-1}}.$$

For $m \le n-1$, by using (10), we have:

$$\begin{aligned}(A_{m,n}e_1)(x) \\ &= \frac{x}{n(1+x)^m}\left\{\sum_{i=0}^{\infty}\binom{n-m+i}{i}r_{i,n-m+1}(x)i+\sum_{i=0}^{\infty}\binom{n-m+i}{i}r_{i,n-m+1}(x)\right\} \\ &= \frac{x}{n(1+x)^m}((n-m+1)x+1).\end{aligned}$$

Similarly, for any $m \ge 1$, we can write:

$$\begin{aligned}(B_{m,n}e_1)(x)-\frac{m}{n}r_{m,n}(x) &= \frac{x^m}{(1+x)^{m+1}}\sum_{i=1}^{\infty}\binom{n-2+i}{i}r_{i,n-1}(x)\frac{m+i}{n} \\ &= \frac{m}{n}\frac{x^m}{(1+x)^{m+1}}\left((V_{n-1}e_0)(x)-\frac{1}{(1+x)^{n-1}}\right)+\frac{n-1}{n}\frac{x^m}{(1+x)^{m+1}}(V_{n-1}e_1)(x),\end{aligned}$$

which implies $(B_{m,n}e_1)(x) = n^{-1}r_{m,1}(x)((n-1)x+m)$.

The above relations and Theorem A lead us to the following result:

Lemma 1. *If the operators $L_n^{\langle a,b\rangle}$ are defined by* (2) *then, for $n \ge 2$, we have:*

$$\begin{aligned}n(L_n^{\langle a,b\rangle}e_1)(x) \\ &= \sum_{m=2}^{n-1}a_m r_{1,m-1}(x)((n-m+1)x+1)+\frac{a_n x}{(1+x)^{n-1}}+\sum_{m\ge 1}b_m r_{m,1}(x)((n-1)x+m).\end{aligned}$$

This lemma allows us to state:

Theorem 1. *If the sequences a, b are bounded then the operators $L_n^{\langle a,b\rangle}$ satisfy:*

$$\lim_{n\to\infty} n(e_1(x)(L_n^{\langle a,b\rangle}e_0)(x)-(L_n^{\langle a,b\rangle}e_1)(x)) = -x(1+x)\varphi'(x),$$

where $x>0$ and φ is defined by (8).

Proof. Let $n \ge 3$. Taking lemma 1 and identity (7) into account we can write:

$$\begin{aligned}(11)\quad n(x(L_n^{\langle a,b\rangle}e_0)(x)-(L_n^{\langle a,b\rangle}e_1)(x)) &= \frac{(n-1)a_n x}{(1+x)^{n-1}} \\ &+\sum_{m\ge 1}b_m r_{m,1}(x)(x-m)+\sum_{m=2}^{n-1}a_m r_{1,m-1}(x)((m-1)x-1).\end{aligned}$$

Because of boundedness of a, the term $\dfrac{(n-1)a_n x}{(1+x)^{n-1}}$ converges to zero as $n\to\infty$.

On the other hand, from (8) we obtain:

$$\sigma'(x) = \sum_{m=2}^{\infty} a_m \frac{1-(m-1)x}{(1+x)^{m+1}}, \quad \tau'(x) = \sum_{m\geq 1} b_m \frac{x^{m-1}(m-x)}{(1+x)^{m+2}},$$

consequently:

$$\sum_{m\geq 2} a_m (m-1) x r_{1,m-1}(x) = \sigma(x) - x(1+x)\sigma'(x)$$

and

$$\sum_{m\geq 1} b_m (x-m) r_{m,1}(x) = -x(1+x)\tau'(x).$$

Using these relations in (11), for $n \to \infty$ we get to the desired result.

Now, we need the expression of operators $A_{m,n}$, $B_{m,n}$ $(n \geq 2)$ for the test function e_2. From (9) we deduce:

$$(A_{n,n}e_2)(x) = \frac{1}{n^2(1+x)^n} \sum_{k=0}^{\infty} k^2 r_{k,0}(x) = \frac{x(2x+1)}{n^2(1+x)^{n-1}}.$$

For $2 \leq m \leq n-1$, we take $p = n-m+1$ and in view of (10) we can write successively:

$$\begin{aligned}(A_{m,n}e_2)(x) &= \frac{x}{n^2(1+x)^m} \sum_{k=0}^{\infty} \binom{p+k-1}{k} r_{k,p}(x)(k+1)^2 \\ &= \frac{xp^2}{n^2(1+x)^m}((V_pe_2)(x) + \frac{2}{p}(V_pe_1)(x) + \frac{1}{p^2}(V_pe_0)(x)) \\ &= \frac{x}{n^2(1+x)^m}(p(p+1)x^2 + 3px + 1).\end{aligned}$$

For $m \geq 1$, we have:

$$\begin{aligned}(B_{m,n}e_2)(x) - \frac{m^2}{n^2} r_{m,n}(x) &= \frac{x^m}{n^2(1+x)^{m+1}} \sum_{i=1}^{\infty} \binom{n+i-2}{i} r_{i,n-1}(x)(m+i)^2 \\ &= \frac{x^m}{n^2(1+x)^{m+1}} \Big\{ m^2 \Big((V_{n-1}e_0)(x) - \frac{1}{(1+x)^{n-1}} \Big) \\ &\qquad + 2m(n-1)(V_{n-1}e_1)(x) + (n-1)^2 (V_{n-1}e_2)(x) \Big\}.\end{aligned}$$

Thus,

$$(B_{m,n}e_2)(x) = \frac{x^m}{n^2(1+x)^{m+1}}(n(n-1)x^2 + (n-1)(2m+1)x + m^2).$$

We can summarize these relations below.

LEMMA 2. *If the operators* $L_n^{\langle a,b\rangle}$ *are defined by* (2) *then, for* $n \geq 2$, *we have:*

$$n(L_n^{\langle a,b\rangle} e_2)(x) = \sum_{m=2}^{n-1} \frac{a_m}{n} r_{1,m-1}(x)(p(p+1)x^2 + 3px + 1) + \frac{a_n x(2x+1)}{n(1+x)^{n-1}}$$
$$+ \sum_{m\geq 1} \frac{b_m}{n} r_{m,1}(x)((n-1)nx^2 + (n-1)(2m+1)x + m^2),$$

where $p = n - m + 1$.

Set $\phi_{x,r}(t) := (t-x)^r \quad (t \geq 0,\ x \geq 0)$.

THEOREM 2. *If the sequences* a, b *are bounded then the operators* $L_n^{\langle a,b\rangle}$ *satisfy:*

$$\lim_{n\to\infty} n(L_n^{\langle a,b\rangle} \phi_{x,2})(x) = x(1+x)\varphi(x),$$

where $x > 0$ *and* φ *is defined by* (8).

Proof. Because $L_n^{\langle a,b\rangle}$ are linear operators, it is clear that:

$$(L_n^{\langle a,b\rangle} \phi_{x,2})(x) = (L_n^{\langle a,b\rangle} e_2)(x) - 2x(L_n^{\langle a,b\rangle} e_1)(x) + x^2(L_n^{\langle a,b\rangle} e_0)(x).$$

Using (7), Lemma 1 and Lemma 2 after a short calculation, we get:

$$n(L_n^{\langle a,b\rangle} \phi_{x,2})(x) = (x^2 + x) \sum_{m=2}^{n-1} a_m r_{1,m-1}(x) + (x^2 + x)\tau(x) + \frac{1}{n} R(n;x),$$

where

$$R(n;x) = \sum_{m=2}^{n-1} \alpha(m,x) a_m \frac{x}{(1+x)^m} + \sum_{m\geq 1} \beta(m,x) b_m \frac{x^m}{(1+x)^{m+1}} + \gamma(n,x)$$

and

$$\alpha(m,x) = (m^2 - 3m + 2)x^2 - 3(m-1)x + 1,$$

$$\beta(m,x) = -(2m+1)x + m^2,$$

$$\gamma(n,x) = a_n \frac{x}{(1+x)^{n-1}}((n^2 - 2n + 2)x + 1).$$

But $R(n;x)/n$ tends to 0 for $n \to \infty$ and according to (8) this completes the proof.

Now, we can state and prove the announced Voronovskaja-type formula.

Theorem 3. *Let $f \in C_B[0,\infty)$ be twice differentiable at some point $x > 0$ and assume that $f(t) = \mathcal{O}(t^2)$ as $t \to \infty$. If the operators $L_n^{\langle a,b\rangle}$ are defined by* (2) *and a, b are bounded then*

(12)
$$\lim_{n\to\infty} n((L_n^{\langle a,b\rangle} f)(x) - f(x)(L_n^{\langle a,b\rangle} e_0)(x)) = \frac{x(1+x)}{2}(\varphi(x)f''(x) + 2\varphi'(x)f'(x)).$$

Proof. In order to prove this identity we use Taylor's formula:

$$f\Big(\frac{k}{n}\Big) - f(x) = \Big(\frac{k}{n} - x\Big)f'(x) + \Big(\frac{k}{n} - x\Big)^2\Big(\frac{1}{2}f''(x) + \varepsilon\Big(\frac{k}{n} - x\Big)\Big),$$

where ε is bounded and $\lim\limits_{t\to 0}\varepsilon(t) = 0$. Then

$$(13)\quad (L_n^{\langle a,b\rangle} f)(x) - f(x)(L_n^{\langle a,b\rangle} e_0)(x) = \sum_{k=0}^{\infty} c_{n,k}\frac{x^k}{(1+x)^{n+k}}\Big(f\Big(\frac{k}{n}\Big) - f(x)\Big)$$
$$= f'(x)((L_n^{\langle a,b\rangle} e_1)(x) - x(L_n^{\langle a,b\rangle} e_0)(x)) + \frac{1}{2}f''(x)(L_n^{\langle a,b\rangle}\phi_{x,2})(x)$$
$$+ \sum_{k=0}^{\infty} c_{n,k}\frac{x^k}{(1+x)^{n+k}}\Big(\frac{k}{n} - x\Big)^2\varepsilon\Big(\frac{k}{n} - x\Big).$$

If λ is an upper bound for the sequences $|a|$, $|b|$, then (2) implies $|c_{n,k}| \le \lambda\binom{n+k-1}{k}$. It follows:

$$\lim_{n\to\infty} n\Big|\sum_{k=0}^{\infty} c_{n,k} r_{k,n}(x)\Big(\frac{k}{n} - x\Big)^2\varepsilon\Big(\frac{k}{n} - x\Big)\Big| \le \lambda\lim_{n\to\infty} n(V_n\psi_{x,2})(x) = 0,$$

where $\psi_x(t) = \phi_{x,2}(t)\varepsilon(t - x)$. We also used the Voronovskaja's formula for Baskakov operators [**8**], that is $\lim\limits_{n\to\infty} n((V_n f)(x) - f(x)) = (x^2 + x)f''(x)/2$. Therefore, from (13), Theorem 1 and Theorem 2, we obtain the desired result.

In what follows we assume $\varphi(x) \ne 0$ for every $x > 0$.

Remark. The right-hand side of (12) can be arranged under the form $(2\varphi(x))^{-1}x(1+x)\frac{d}{dx}(\varphi^2(x)f'(x))$. Therefore, theorem 3 shows that $(L_n^{\langle a,b\rangle} f)(x) - f(x)(L_n^{\langle a,b\rangle} e_0)(x)$ is of order not better than $1/n$ if $\frac{d}{dx}(\varphi^2(x)f'(x)) \ne 0$.

In order to prove a global approximation theorem for $L_n^{\langle a,b\rangle}$ in polynomial weight spaces, we define the weights w_N as follows $w_0(x) = 1$, $w_N(x) = (1 + x^N)^{-1}$, $x \ge 0$, $N \ge 1$. Also, we introduce the spaces $C_N = \{f \in C[0,\infty);\ w_N f$ uniformly continuous and bounded on $[0,\infty)\}$ endowed with the norm $\|f\|_N = \sup_{x\ge 0} w_N(x)|f(x)|$. Finally, we recall the notations

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x), \qquad \omega_N^2(f,\delta) = \sup_{0<h\le\delta}\|\Delta_h^2 f\|_N.$$

In the sequel, we consider that a, b are non negative bounded sequences and we shall denote by K_N a constant which may be different at each occurrence, it depends only on a, b and N.

Introducing $\mu_{n,r}$, $m_{n,r}$ the r-th order moment ($r \in \mathbf{N}$) of V_n respectively of $L_n^{\langle a,b\rangle}$, we need the following result due to Becker.

Lemma 3 [**6**, p. 131]. *Let $r \geq 2$ and* $\delta_r = \begin{cases} 1, & r \text{ odd} \\ 0, & r \text{ even.} \end{cases}$

Then, the r-th moment of the Baskakov operators is

$$\mu_{n,r}(x) = \sum_{j=1}^{[r/2]} b_{n,r,j} \Big(\frac{x(1+x)}{n}\Big)^j \Big(\frac{1+2x}{n}\Big)^{\delta_r},$$

where the $b_{n,r,j}$ coefficients are positive and bounded with respect to n. Moreover, $\mu_{n,r}(x)$ is a polynomial of degree r.

Now, we can state the following result

Lemma 4. *Let r be integer, $0 \leq r \leq N$. The quantities $m_{n,r}x^{N-r}(1+x^N)^{-1}$ are bounded with respect to x ($x \geq 0$) and to n ($n \in \mathbf{N}$).*

Proof. If the sequences a, b are non-negative and $\lambda = \max\Big\{\sup\limits_{n\geq 1} a_n, \sup\limits_{k \geq 0} b_k\Big\}$, then $L_n^{\langle a,b\rangle}|g| \leq L_n^{\langle \lambda,\lambda\rangle}|g| = \lambda V_n|g|$. We easily deduce

$$m_{n,2i} \leq \lambda \mu_{n,2i}, \quad i \geq 0. \tag{14}$$

If r is odd, according to Cauchy's inequality, we get

$$|m_{n,2i+1}| \leq m_{n,2}^{1/2} m_{n,4i}^{1/2} \leq \lambda \mu_{n,2}^{1/2}\mu_{n,4i}^{1/2} = \mathcal{O}(x^{2i+1}), \quad x \to \infty. \tag{15}$$

Also, the boundedness with respect to n follows from the coefficients of $\mu_{n,i}$ being bounded in n, see Lemma 3.

Lemma 5. *For $f \in C_N[0,\infty)$ the following inequality*

$$\|L_n^{\langle a,b\rangle} f\|_N \leq K_N \|f\|_N \tag{16}$$

holds.

Proof. We can write

$$\begin{aligned} w_N(x)L_n^{\langle a,b\rangle}(1/w_N;x) &= w_N(x)\sum_{k=0}^{\infty} c_{n,k} r_{k,n}(x)\Big(1 + \Big(\Big(\frac{k}{n} - x\Big) + x\Big)^N\Big) \\ &\leq \lambda (V_n e_0)(x)(1+x^N)^{-1} + \sum_{j=0}^{N} \binom{N}{j} m_{n,j}(x)x^{N-j}(1+x^N)^{-1} \leq K_N \end{aligned}$$

since the sum is bounded in x and in n - see lemma 4. For $f \in C_N$, we have

$$w_N(x)|(L_n^{\langle a,b\rangle} f)(x)| \le w_N(x) \sum_{k=0}^{\infty} w_N\left(\frac{k}{n}\right)\left|f\left(\frac{k}{n}\right)\right|\left(w_N\left(\frac{k}{n}\right)\right)^{-1} c_{n,k} r_{k,n}(x)$$
$$\le \|f\|_N w_N(x) L_n^{\langle a,b\rangle}(1/w_N; x) \le K_N \|f\|_N$$

and this implies (16).

LEMMA 6. *For $g \in C_N^2 = \{g: \ g, g', g'' \in C_N\}$ the inequality*

$$(17) \quad w_N(x)|(L_n^{\langle a,b\rangle} g)(x) - g(x)(L_n^{\langle a,b\rangle} e_0)(x)|$$
$$\le K_N\left(\sqrt{\frac{x(x+1)}{n}}\|g'\|_N + \frac{x(x+1)}{n}\|g''\|_N\right), \quad n \ge 1, \ x \ge 0,$$

holds.

Proof. We use the Taylor expansion

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du$$

where

$$\left|\int_x^t (t-u)g''(u)du\right| \le \|g''\|_N \int_x^t |t-u|\frac{1}{w_N(u)}du$$
$$\le \frac{1}{2}\|g''|\_N(t-x)^2\left(\frac{1}{w_N(t)} + \frac{1}{w_N(x)}\right).$$

This implies

$$(18) \quad w_N(x)|(L_n^{\langle a,b\rangle} g)(x) - g(x)(L_n^{\langle a,b\rangle} e_0)(x)|$$
$$\le (L_n^{\langle a,b\rangle}|\phi_{x,1}|)(x)\|g'\|_N + \frac{1}{2}\|g''\|_N\left(m_{n,2}(x) + w_N(x)L_n^{\langle a,b\rangle}\left(\frac{\phi_{x,2}}{w_N}; x\right)\right).$$

Taking into account (15), (14), (10) and Lemma 3 we can infer

$$(L_n^{\langle a,b\rangle}|\phi_{x,1}|)(x) \le \lambda \mu_{n,2}^{1/2} = \lambda\sqrt{\frac{x(x+1)}{n}},$$

and

$$\left(L_n^{\langle a,b\rangle}\frac{\phi_{x,2}}{w_N}\right)(x) = m_{n,2}(x) + \sum_{j=0}^{N}\binom{N}{j} m_{n,j+2} x^{N-j}$$
$$\le \frac{x(x+1)}{n} + P_{N+2,n}(x) = P_{N,n}(x)\frac{x(x+1)}{n},$$

$P_{k,n}$ being a polynomial of degree k with all coefficients bounded in n.

Inserting the above relations into (18) we obtain (17).

THEOREM 4. *Let f belong to $C_N[0,\infty)$. If the sequences a, b are non-negative and bounded then*

$$w_N(x)|(L_n^{\langle a,b\rangle} f)(x) - f(x)(L_n^{\langle a,b\rangle} e_0)(x)| \leq K_N \omega_N^2\Big(f, \sqrt{\frac{x(x+1)}{n}}\Big). \tag{19}$$

Proof. For any $h > 0$ we introduce the modified Steklov means by

$$f_h(x) = (2/h)^2 \int_0^{h/2}\int_0^{h/2} (2f(x+s+t) - f(x+2s+2t))ds\,dt.$$

From above, we get

$$\begin{gathered} f(t) - f_h(x) = (2/h)^2 \int_0^{h/2}\int_0^{h/2} \Delta_{s+t}^2 f(x) ds\,dt, \\ f_h'(x) = (2/h)(\Delta_{t_0}^2 f(x) - \Delta_{t_0+h/2}^2 f(x)), \quad 0 < t_0 \leq h/2, \\ f_h''(x) = h^{-2}(8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)) \end{gathered}$$

and hence

$$\|f - f_h\|_N \leq \omega_N^2(f,h), \tag{20}$$

$$\|f_h'\|_N \leq 4h^{-1}\omega_N^2(f,h), \qquad \|f_h''\|_N \leq 9h^{-2}\omega_N^2(f,h). \tag{21}$$

Clearly, we have:

$$\begin{aligned} w_N(x)|(L_n^{\langle a,b\rangle} f)(x) - f(x)(L_n^{\langle a,b\rangle} e_0)(x)| &\leq w_N(x)|L_n^{\langle a,b\rangle}(f - f_h; x)| \\ + w_N(x)|(L_n f_h)(x) - f_h(x)(L_n e_0)(x)| &+ w_N(x)|f_h(x) - f(x)||(L_n e_0)(x)|. \end{aligned}$$

Below we estimate the terms on the right side separately. Using (16), (20) and (14) we can write

$$w_N(x)|L_n^{\langle a,b\rangle}(f - f_h; x)| \leq K_N \omega_N^2(f,h)$$

and

$$w_N(x)|f_h(x) - f(x)||(L_n^{\langle a,b\rangle} e_0)(x)| \leq \lambda \omega_N^2(f,h)$$

where λ is an upper bound for a and b.

Using (17) and (21) we get

$$\begin{aligned} &w_N(x)|(L_n^{\langle a,b\rangle} f_h)(x) - f_h(x)(L_n^{\langle a,b\rangle} e_0)(x)| \\ &\qquad \leq K_N\Big(\sqrt{\frac{x(x+1)}{n}}\frac{4}{h} + \frac{x(x+1)}{n}\frac{9}{h^2}\Big)\omega_N^2(f,h). \end{aligned}$$

Choosing $h = \sqrt{\dfrac{x(x+1)}{n}}$ and collecting the above inequalities we obtain (19).

Remark. In particular, if $0 < \alpha \le 2$ and $f \in Lip_N^2\alpha = \{f \in C_N[0,\infty) : \omega_N^2(f,\delta) = \mathcal{O}(\delta^\alpha),\ \delta \to 0^+\}$ then the conclusion of the theorem becomes

$$w_N(x)|(L_n^{\langle a,b\rangle}f)(x) - f(x)(L_n^{\langle a,b\rangle}e_0)(x)| \le K_N\Big(\frac{x(x+1)}{n}\Big)^{\alpha/2}.$$

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