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LITTLEWOOD-PALEY TYPE INEQUALITIES FOR $\mathcal{M}$-HARMONIC FUNCTIONS

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Abstract. We prove several Littlewood-Paley type inequalities for $\mathcal{M}$-harmonic and analytic functions on the unit ball B of C^n. Further, we give some characterizations of $\mathcal{M}$-harmonic and analytic Hardy spaces on B.

1. Introduction

Let B denote the open unit ball in C^n, $n > 1$, with boundary S. We denote by ν the normalized Lebesgue measure on B and by σ the rotation invariant probability measure on S.

The main purpose of this paper is to prove the following theorem.

THEOREM 1.1. *If f is a function in $L^p(S)$, $1 < p < \infty$, and u is a function on B defined via the invariant Poisson integral of f, then*

$$(1.1)\quad \left(\int_B |\widetilde{\nabla} u(z)|^p (1-|z|^2)^n d\tau(z) + |u(0)|^p\right) \succeq \int_S |f(\xi)|^p d\sigma(\xi), \text{ for } 1 < p \le 2,$$

and

$$(1.2)\quad \left(\int_B |\widetilde{\nabla} u(z)|^p (1-|z|^2)^n d\tau(z) + |u(0)|^p\right) \preceq \int_S |f(\xi)|^p d\sigma(\xi), \text{ for } 2 \le p < \infty,$$

where $\widetilde{\nabla}$ and τ denote the invariant gradient and invariant measure on B.

We also show that $|\widetilde{\nabla} u(z)|$ in (1.1) and (1.2) may be replaced by $(1-|z|^2)$ $\times|\nabla u(z)|$, where ∇ denotes the real gradient of u and by $(1-|z|^2)(|Ru(z)|+|\bar{R}u(z)|)$, where, as usual, $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ is the radial derivative.

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THEOREM 1.2. *If $f \in L^p(S)$, $1 < p < \infty$, and u is the invariant Poisson integral of f, then for $1 < p \le 2$ we have*

$$(1.3)\qquad \left(\int_B |\nabla u(z)|^p (1-|z|^2)^{p-1} d\nu(z) + |u(0)|^p\right) \succeq \int_S |f(\xi)|^p d\sigma(\xi)$$

$$(1.4)\quad \left(\int_B \big(|Ru(z)| + |\bar{R}u(z)|\big)^p (1-|z|^2)^{p-1} d\nu(z) + |u(0)|^p\right) \succeq \int_S |f(\xi)|^p d\sigma(\xi),$$

and for $2 \le p < \infty$ we have

$$(1.5)\quad \left(\int_B (|Ru(z)| + |\bar{R}u(z)|)^p (1-|z|^2)^{p-1} d\nu(z) + |u(0)|^p\right) \preceq \int_S |f(\xi)|^p d\sigma(\xi),$$

$$(1.6)\qquad \left(\int_B |\nabla u(z)|^p (1-|z|^2)^{p-1} d\nu(z) + |u(0)|^p\right) \preceq \int_S |f(\xi)|^p d\sigma(\xi).$$

For $n = 1$ (1.3) and (1.6) are well known inequalities of Littlewood and Paley [**11**]. Various generalizations of their result are referred to as a Littlawood-Paley type inequalities.

The method of proof of Theorem 1.1 we will present is based on local estimates for $\mathcal{M}$-harmonic functions (which will be defined in Section 2) and the following theorems that allow us to express the L^p norm of f in terms of some area integrals, and which are of interest on their own right.

THEOREM 1.3. *Let u be $\mathcal{M}$-harmonic function on B. If $1 < p < \infty$, then*

$$(1.7)\qquad \frac{d}{dr}\int_S |u(r\xi)|^p d\sigma(\xi) = \frac{c_p r^{1-2n}(1-r^2)^{n-1}}{2n} \int_{rB} |u(z)|^{p-2} |\widetilde{\nabla} u(z)|^2 d\tau(z),$$

where $c_p = p(p-1)$.

For $1 < p < \infty$, let $\mathcal{H}^p = \mathcal{H}^p(B)$ denote the set of $\mathcal{M}$-harmonic functions u on B, $u \in \mathcal{M}$, for which $|u|^p$ has an $\mathcal{M}$-harmonic majorant on B. It is well known that $u \in \mathcal{H}^p(B)$ if and only if $\|u\|_{\mathcal{H}^p} = \sup_{0<r<1} M_p(r,u) < \infty$, where, as usual, $M_p^p(r,u) = \int_S |u(r\xi)|^p d\sigma(\xi)$.

THEOREM 1.4. *A function u $\mathcal{M}$-harmonic on B belongs to $\mathcal{H}^p$, $1 < p < \infty$, if and only if*

$$\int_B |u(z)|^{p-2} |\widetilde{\nabla} u(z)|^2 (1-|z|^2)^{-1} d\nu(z) < \infty$$

Furthermore, if $u \in \mathcal{H}^p$, $1 < p < \infty$, then

$$(1.8)\quad \|u\|^p_{\mathcal{H}^p} = \int_B |u(z)|^p d\nu(z) + \frac{p(p-1)}{4n^2} \int_B |u(z)|^{p-2} |\widetilde{\nabla} u(z)|^2 (1-|z|^2)^{-1} d\nu(z),$$

Let

$$G(\rho, r) = \frac{1}{2n}\int_\rho^r t^{1-2n}(1-t^2)^{n-1}dt, \quad 0 \le \rho \le r \le 1, \tag{1.9}$$

As a corollary of Theorem 1.3 we have another characterization of the Hardy space $\mathcal{H}^p$.

THEOREM 1.5. *Let $1 < p < \infty$. A function $u \in \mathcal{M}$ belongs to $\mathcal{H}^p$ if and only if*

$$\int_B |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 G(|z|, 1)d\tau(z) < \infty.$$

Moreover, if $u \in \mathcal{H}^p$, $1 < p < \infty$, then

$$\|u\|^p_{\mathcal{H}^p} = |u(0)|^p + p(p-1)\int_B |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 G(|z|, 1)d\tau(z). \tag{1.10}$$

The method of proof of Theorems concerning $\mathcal{M}$-harmonic functions we will present can also be applied to Hardy spaces H^p of holomorphic functions. Recall that a holomorphic function f on B, $f \in H(B)$, belongs to the Hardy space H^p, $0 < p < \infty$, if and only if $\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty$. An analogue of the identity (1.7) of the Hardy-Stein-Spencer type for analytic functions is as follows.

THEOREM 1.6. *Let $f \in H(B)$. If $0 < p < \infty$, then*

$$\frac{d}{dr}\int_S |f(r\xi)|^p d\sigma(\xi) = \frac{p^2}{4}\frac{r^{1-2n}(1-r^2)^{n-1}}{2n}\int_{rB} |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 d\tau(z), \tag{1.11}$$

An application of the identity (1.11) gives the following characterization of the Hardy spaces H^p.

THEOREM 1.7. *A function f holomorphic on B belongs to H^p, $0 < p < \infty$, if and only if*

$$\int_B |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 (1-|z|^2)^{-1} d\nu(z) < \infty.$$

Furthermore, if $f \in H^p$, $0 < p < \infty$, then

$$\|f\|^p_{H^p} = \int_B |f(z)|^p d\nu(z) + \frac{p^2}{8n}\int_B |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 (1-|z|^2)^{-1} d\nu(z). \tag{1.12}$$

THEOREM 1.8. *Let $0 < p < \infty$. A function $f \in H(B)$ belongs to H^p if and only if*

$$\int_B |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 G(|z|, 1)\, d\tau(z) < \infty$$

Furthermore, if $f \in H^p$, $0 < p < \infty$, *then*

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{4}\int_B |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 G(|z|,1)\,d\tau(z). \tag{1.13}$$

The characterizations of Hardy spaces $\mathcal{H}^p$, $1 < p < \infty$, and H^p, $0 < p < \infty$, given in Theorems 1.4 and 1.7 are known (see [**14**]). The proofs we will presente are based on the new identities (1.8) and (1.12) and they are simpler than the proofs given in [**14**].

For another proof of (1.13) see [**4**]. See also [**14**].

The following identity due to Beatrous and Burbea [**2**] was first proved by Hardy, Stein and Spenser (see [**6**, p. 42]) for $n = 1$.

Theorem 1.9, *Let* $f \in H(B)$, $0 < p < \infty$ *and* $0 < r < 1$. *Then*

$$r\frac{d}{dr}M_p^p(r,f) = \frac{p^2}{2n}\int_{rB} |z|^{-2n}|f(z)|^{p-2}|Rf(z)|^2 d\nu(z). \tag{1.14}$$

An application of the identity (1.14) gives the criteria for f holomorphic in B to belong to the Hardy space H^p.

Theorem 1.10. [**2**] *Let* $0 < p < \infty$ *and* $f \in H(B)$. *Then the following statements are equivalent:*

(*i*) $f \in H^p$,

(*ii*) $\int_B |f(z)|^{p-2}|Rf(z)|^2(1-|z|^2)\,d\nu(z) < \infty$.

Since $|\widetilde{\nabla} f(z)| \geq (1-|z|)^2|\nabla f(z)| \geq (1-|z|^2)|Rf(z)|$, (see Section 2) the following theorem is an immediate consequence of Theorems 1.7 and 1.10.

Theorem 1.11. *A function* f *holomorphic on* B *belongs to* H^p, $0 < p < \infty$, *if and only if*

$$\int_B |\nabla f(z)|^2|f(z)|^{p-2}(1-|z|^2)d\nu(z) < \infty.$$

Remark 1. It is authors belief that the results of Theorems 1.10 and 1.11 should hold for the Hardy spaces $\mathcal{H}^p$, $1 < p < \infty$ of $\mathcal{M}$-harmonic functions

As a final result we have the inequalities of Littlewood-Paley type for analytic functions.

Theorem 1.12. *Let* $f \in H(B)$. *If* $0 < p \leq 2$, *then*

$$\|f\|_p^p \preceq \left(\int_B |\widetilde{\nabla} f(z)|^p(1-|z|^2)^{-1}d\nu(z) + |f(0)|^p\right), \tag{1.15}$$

$$\|f\|_p^p \preceq \left(\int_B |\nabla f(z)|^p(1-|z|^2)^{p-1}d\nu(z) + |f(0)|^p\right), \tag{1.16}$$

$$\|f\|_p^p \preceq \left(\int_B |Rf(z)|^p(1-|z|^2)^{p-1}d\nu(z) + |f(0)|^p\right), \tag{1.17}$$

and if $2 \le p < \infty$, *then*

$$\left(|f(0)|^p + \int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^{-1} d\nu(z)\right) \preceq \|f\|_p^p, \tag{1.18}$$

$$\left(|f(0)|^p + \int_B |\nabla f(z)|^p (1-|z|^2)^{p-1} d\nu(z)\right) \preceq \|f\|_p^p, \tag{1.19}$$

$$\left(|f(0)|^p + \int_B |Rf(z)|^p (1-|z|^2)^{p-1} d\nu(z)\right) \preceq \|f\|_p^p. \tag{1.20}$$

The usual method of proof of the inequality (1.20) ((1.19), resp.) is to apply the Riesz Convexity Theorem to the operator $f \to |Rf(z)|(1-|z|^2)\,(f \to |\nabla f(z)|(1-|z|^2))$ acting on functions f on the measure space (S, σ) and taking them to functions on $(B, (1-|z|^2)^{-1} d\nu(z))$. It is relatively easy to show that this operator is of type (2.2) as well as (∞, ∞) and the Riesz theorem produces (1.20) ((1.19), resp.). By duality we have (1.17) and (1.16) for $1 < p \le 2$. Additional considerations show that (1.17) holds also for $0 < p \le 1$. See [**3**]. Obviously (1.15) and (1.16) are consequences of (1.17). For the inequality (1.18) see [**5**].

Our argument is different. We show that

$$\int_B |\widetilde{\nabla} f(z)|^p (1-|z|^2)^{-1} d\nu(z) \preceq \int_B |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 (1-|z|^2)^{-1} d\nu(z),$$

for $2 \le p < \infty$, and that the reverse inequality holds for $0 < p \le 2$, and then we apply the identity (1.12) established in Theorem 1.7.

2. Notations and preliminary results

Let $\widetilde{\Delta}$ be the invariant Laplacian on B. That is, $\widetilde{\Delta} u(z) = \Delta(u \circ \varphi_z)(0)$, where Δ is the ordinary Laplacian and φ_z the standard authomorphism of B taking 0 to z [**13**]. (It can be shown that $\widetilde{\Delta}$ is equal $(n+1)$ times the Laplacian with respect to the Bergman metric). As in [**13**],

$$\widetilde{\Delta} = 4(1-|z|^2) \sum_{i,j} (\delta_{i,j} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

The real valued functions annihilated by $\widetilde{\Delta}$ are called invariantly harmonic or $\mathcal{M}$-harmonic.

The operator $\widetilde{\Delta}$ is $\mathcal{M}$-invariant in the sense that $\widetilde{\Delta}(f \circ \varphi) = (\widetilde{\Delta} f) \circ \varphi$, for $\varphi \in \mathrm{Aut}(B)$ [**13**, Theorem 4.1.2]. This implies that the class $\mathcal{M}$ of $\mathcal{M}$- harmonic functions on B is $\mathcal{M}$-invariant.

For a function $u \in C^1(B)$ let $\widetilde{D} u(a) = D(u \circ \varphi_a)(0)$ and $\widetilde{\nabla} u(a) = \nabla(u \circ \varphi_a)(0)$ be the invariant complex and invariant real gradient respectively, where

$Du = \left(\frac{\partial u}{\partial z_1}, \ldots, \frac{\partial u}{\partial z_n}\right)$ denotes the complex gradiant and $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{2n}}\right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, \ldots, n$, is the real gradient of u.

For $u \in C^1(B)$ and $\phi \in \mathrm{Aut}(B)$ we have $|\widetilde{D}(u \circ \phi)| = |(\widetilde{D}u) \circ \phi|$ and $|\widetilde{\nabla}(u \circ \phi)| = |(\widetilde{\nabla}u) \circ \phi|$. In other words, $|\widetilde{D}|$ and $|\widetilde{\nabla}|$ are $\mathcal{M}$-invariant. See [**12**].

The length of the invariant gradient of a function $u \in C^1(B)$ is given in coordinates by

$$|\widetilde{\nabla}u(z)|^2 = 2(1 - |z|^2)(|Du(z)|^2 - |Ru(z)|^2 + |D\bar{u}(z)|^2 - |R\bar{u}(z)|^2). \tag{2.1}$$

It is easy to check that

$$|z|^2 |Du(z)|^2 = |Ru(z)|^2 + \sum_{i<j} |T_{i,j}u(z)|^2, \quad u \in C^2(B),$$

where $T_{i,j}u = \bar{z}_i \frac{\partial u}{\partial z_j} - \bar{z}_j \frac{\partial u}{\partial z_i}$ are tangential derivatives of u. Using this and (2.1) we find that

$$\begin{aligned} |z|^2 |\widetilde{\nabla}u(z)|^2 = 2(1 - |z|)^2 \big[&(1 - |z|^2)(|Ru(z)|^2 + |R\bar{u}(z)|^2) \\ &+ \sum_{i<j} |T_{i,j}u(z)|^2 + \sum_{i<j} |T_{i,j}\bar{u}(z)|^2 \big], \end{aligned} \tag{2.2}$$

For $z \in B$ and r between 0 and 1, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$.

In this note we follow the custom of using the letter C to stand for a positive constant which changes its value from one appearence to another while remaining independent of the important variables.

We write $A \preceq B$, or equivalently $B \succeq A$, when there is a constant C such that $A \leq CB$, and $A \cong B$ when $A \preceq B$ and $B \preceq A$.

Lemma 2.1. [**12**], [**1**] *Let $0 < p < \infty$ and $0 < r < 1$. There exist constants $C_1 > 0$ and $C_2 > 0$ so that if $u \in \mathcal{M}$, then*

(*a*) $|\widetilde{\nabla}u(z)|^p \leq \frac{C_1}{(1-|z|^2)^{n+1}} \int_{E_r(z)} |\widetilde{\nabla}u(w)|^p d\nu(w), \quad z \in B$, *and*

(*b*) $|\nabla u(z)|^p \leq \frac{C_2}{(1-|z|^2)^{n+1}} \int_{E_r(z)} |\nabla u(w)|^p d\nu(w), \quad z \in B.$

Lemma 2.2. [**9**], [**8**] *Let $0 < r < 1$. There is a constant C sach that if $u \in \mathcal{M}$ then*

(*a*) $|T_{i,j}Ru(z)| \leq C(1 - |z|^2)^{-1/2} \int_{E_r(z)} |Ru(w)| d\tau(w), \quad z \in B,$

(*b*) $|T_{i,j}\bar{R}u(z)| \leq C(1 - |z|^2)^{-1/2} \int_{E_r(z)} |\bar{R}u(w)| d\tau(w), \quad z \in B,$

(*c*) $|T_{i,j}u(z)| \leq C(1 - |z|^2)^{-1/2} \int_{E_r(z)} |u(w)| d\tau(w), \ z \in B.$

Here, $d\tau(z) = (1 - |z|^2)^{-n-1} d\nu(z)$ is the Mobius invariant measure on B.

The following lemma gives some basic properties of G which will be needed later.

Lemma 2.3. *Let* $0 < \delta < \frac{1}{2}$ *be fixed. Then* $G(|z|, 1)$ *satisfies the following:*

(*a*) $G(|z|, 1) \geq c_n(1 - |z|^2)^n$, *for all* $z \in B$.

(*b*) $G(|z|, 1) \leq c_\delta(1 - |z|^2)^n$ *for all* $z \in B$, $|z| \geq \delta$, *where* c_δ *is a positive constant depending only on* δ.

Furthermore, for all z, $|z| \leq \delta$, $G(|z|, 1) \cong |z|^{2-2n}$, $n > 1$.

The proof is a routine estimation of the integral in (1.9), and thus is omitted.

Lemma 2.4. *Let* $0 < \delta < 1$. *There exists a constant* $K = K(\delta) > 0$ *such that* $|\varphi_a(z)| \geq K|z - a|$, *for all* a, $|a| \leq \delta$, *and* $z \in B$.

Proof. By [**13**, p. 26]

$$|\varphi_a(z)|^2 = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - z\bar{a}|^2} = \frac{|z - a|^2 + |z\bar{a}|^2 - |z|^2|a|^2}{|1 - z\bar{a}|^2}$$

Thus

$$\begin{aligned} |\varphi_a(z)|^2 &\geq \frac{|z - a|^2 + |z\bar{a}|^2 - |z|^2|a|^2}{(1 + \delta)^2} \\ &= \frac{|z - a|^2 + |(z - a)\bar{a}|^2 - |a|^2|z - a|^2}{(1 + \delta)^2} \\ &\geq \frac{(1 - \delta^2)|z - a|^2}{(1 + \delta)^2} \end{aligned}$$

3. Characterizations of the Hardy space $\mathcal{H}^p$

Proof of Theorem 1.3. Let $u_\epsilon = (u^2 + \epsilon^2)^{p/2}$, $\epsilon > 0$. Then $u_\epsilon \in C^2(B)$ and

$$\widetilde{\Delta} u_\epsilon = p(p - 2)(u^2 + \epsilon^2)^{p/2 - 2} u^2 |\widetilde{\nabla} u|^2 + p(u^2 + \epsilon^2)^{p/2 - 1}(|\widetilde{\nabla} u|^2 + u\widetilde{\Delta} u).$$

By the Green formula we have

$$\frac{d}{dr} \int_S u_\epsilon(r\xi)\, d\sigma(\xi) = \frac{r^{1-2n}(1 - r^2)^{n-1}}{2n} \int_{rB} \widetilde{\Delta} u_\epsilon(z)\, d\tau(z), \quad 0 < r < 1, \tag{3.1}$$

(see [**12**])

Since u is $\mathcal{M}$-harmonic, we have

$$\widetilde{\Delta} u_\epsilon = p(p - 2)(u^2 + \epsilon^2)^{p/2 - 2} u^2 |\widetilde{\nabla} u|^2 + p(u^2 + \epsilon^2)^{p/2 - 1} |\widetilde{\nabla} u|^2, \tag{3.2}$$

and

$$\lim_{\epsilon \to 0} \frac{d}{dr} \int_S u_\epsilon(r\xi)\, d\sigma(\xi) = \frac{d}{dr} \int_S |u(r\xi)|^p d\sigma(\xi), \tag{3.3}$$

The latter holds because the function

$$(\epsilon, z) \to (u^2(z) + \epsilon^2)^{p/2} = |u(z) + i\epsilon|^p$$

is of class C^1, $(\epsilon, z) \in (-\infty, \infty) \times B$.

If $1 < p \le 2$, then it follows from (3.2) that

$$\begin{aligned} \widetilde{\Delta} u_\epsilon &\ge p(p-2)(u^2+\epsilon^2)^{p/2-2}(u^2+\epsilon^2)|\widetilde{\nabla} u|^2 \\ &\quad + p(u^2+\epsilon^2)^{p/2-1}|\widetilde{\nabla} u|^2 \\ &= p(p-1)(u^2+\epsilon^2)^{p/2-1}|\widetilde{\nabla} u|^2 \end{aligned} \tag{3.4}$$

Hence, by (3.1),

$$c_p \frac{r^{1-2n}(1-r^2)^{n-1}}{2n} \int_{rB} (u(z)^2+\epsilon^2)^{p/2-1}|\widetilde{\nabla} u(z)|^2 d\tau(z) \le \frac{d}{dr}\int_S u_\epsilon(r\xi)\, d\sigma(\xi)$$

where $c_p = p(p-1)$.

Using Fatou's lemma we find that

$$\frac{p(p-1)}{2n} r^{1-2n}(1-r^2)^{n-1} \int_{rB} |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 d\tau(z) \le \frac{d}{dr}\int_S |u(r\xi)|^p d\sigma(\xi). \tag{3.5}$$

If $2 \le p < \infty$, we find that

$$\frac{p}{2n}(1-r^2)^{n-1} r^{1-2n} \int_{rB} |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 d\tau(z) \le \frac{d}{dr}\int_S |u(r\xi)|^p d\sigma(\xi). \tag{3.6}$$

Hence by (3.5) and (3.6) we conclude that the function $|u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2$ is locally integrable.

By (3.4) and (3.2) $\widetilde{\Delta} u_\epsilon \ge 0$, for every $z \in B$. (Note that $p > 1$).

It is easy to see that

$$\widetilde{\Delta} u_\epsilon(z) \le c_p p (u^2(z)+\epsilon^2)^{p/2-1}|\widetilde{\nabla} u(z)|^2, \quad z \in B,$$

where $c_p = 1$ if $1 < p \le 2$ and $c_p = p-1$ if $p \ge 2$.

Hence by (3.1), (3.2), (3.3) and the Lebesgue theorem, we obtain

$$\begin{aligned} \frac{d}{dr}\int_S |u(r\xi)|^p d\sigma(\xi) &= \lim_{\epsilon\to 0}\int_{rB} (\widetilde{\Delta} u_\epsilon(z))\, d\tau(z) \\ &= \frac{p(p-1)}{2n}(1-r^2)^{n-1} r^{1-2n} \int_{rB} |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 d\tau(z) \end{aligned}$$

To prove Theorem 1.4 we need the following proposition

PROPOSITION 3.1. *Let u be $\mathcal{M}$-harmonic and $p > 1$. Then $u \in \mathcal{H}^p$ if and only if*

$$\overline{\lim_{r\to 1}}\big[M_p^p(r,u) - r^{-2n}\int_{rB}|u(z)|^p d\nu(z)\big] < \infty$$

Proof. "Only if" is trivial. To prove the "if" let

$$\varphi(r) = \int_{rB}|u(z)|^p d\nu(z) = 2n\int_0^r \rho^{2n-1}M_p^p(\rho,u)\,d\rho.$$

Hence,

$$M_p^p(r,u) - r^{-2n}\int_{rB}|u(z)|d\nu(z) = \frac{r^{1-2n}}{2n}\varphi'(r) - r^{-2n}\varphi(r) = \frac{r}{2n}\big(\varphi(r)r^{-2n}\big)'.$$

From this the result follows easily.

Proof of Theorem 1.4. By (1.7) we have

$$r^{2n}\frac{d}{dr}M_p^p(r,u) = \frac{p(p-1)}{2n}(1-r^2)^{n-1}r\int_{rB}|u(z)|^{p-2}|\widetilde{\nabla}u(z)|^2 d\tau(z)$$

Integrating this from 0 to ϵ we get, by integration by parts and Fubini's theorem,

$$\begin{aligned}
&\epsilon^{2n}M_p^p(\epsilon,u) - \int_{\epsilon B}|u(z)|^p d\nu(z)\\
&\qquad = \frac{c_p}{2n}\int_0^\epsilon\big[r(1-r^2)^{n-1}\int_{rB}|u(z)|^{p-2}|\widetilde{\nabla}u(z)|^2 d\tau(z)\big]dr\\
&\qquad = \frac{c_p}{2n}\int_{\epsilon B}|u(z)|^{p-2}|\widetilde{\nabla}u(z)|^2 d\tau(z)\int_{|z|}^\epsilon(1-r^2)^{n-1}r\,dr\\
&\qquad = \frac{c_p}{2n}\int_{\epsilon B}|u(z)|^{p-2}|\widetilde{\nabla}u(z)|^2\frac{(1-|z|^2)^n - (1-\epsilon^2)^n}{2n}d\tau(z), \qquad (3.7)
\end{aligned}$$

where $c_p = p(p-1)$.

If $u \in \mathcal{H}^p$, by taking limit in (3.7), $\epsilon \to 1$, we obtain (1.8).

Conversely, if

$$\int_B |u(z)|^{p-2}|\widetilde{\nabla}u(z)|^2(1-|z|)^{-1}d\nu(z) < \infty$$

then

$$\overline{\lim_{\epsilon\to 1}}\big[M_p^p(\epsilon,u) - \epsilon^{-2n}\int_{\epsilon B}|u(z)|^p d\nu(z)\big] < \infty$$

and $u \in \mathcal{H}^p$, by Proposition 3.1.

Proof of Theorem 1.5. We have, by (1.7) and Fubini's theorem,

$$M_p^p(r,u) = |u(0)|^p + p(p-1)\int_{rB} |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 G(|z|,r)\, d\tau(z). \tag{3.8}$$

If $M_p^p(r,u) \leq C < \infty$, then we apply Fatou's lemma to get that

$$|u(0)|^p + p(p-1)\int_{B} |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 G(|z|,1) d\tau(z) \leq C$$

In the opposite direction we use the Lebesgue theorem and the inequality

$$G(|z|,r) \leq G(|z|,1).$$

Remark 2. An upper-semicontinuous function $u : B \to [-\infty,\infty)$ is $\mathcal{M}$-subharmonic or invariant subharmonic if for each $a \in B$

$$u(a) \leq \int_S u(\varphi_a(r\xi))\, d\sigma(\xi), \quad 0 < r < 1,$$

provided that none of the integrals is $-\infty$.

Recall that the Riesz measure of $\mathcal{M}$-subharmonic function u on B is the non-negative regular Borel measure μ_u satisfying $\int_B \phi d\mu_u = \int_B u\widetilde{\Delta}\phi d\tau$, for all $\phi \in C_c^2(B)$, the class of twice continuosly differentiable functions on B with compact support.

If u is $\mathcal{M}$-harmonic on B and $1 < p < \infty$, then the Riesz measure of $|u(z)|^p$ is given by $u_p^\star d\tau$, where $u_p^\star(z) = p(p-1)|u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2$, (see [**14**]).

If $u \in \mathcal{H}^p(B)$, $1 < p < \infty$, then the least $\mathcal{M}$-harmonic majorant of $|u|^p$ is given by

$$P[u^\star](z) = \int_S P(z,\xi) u^\star(\xi)\, d\sigma(\xi),$$

where $P(z,\xi) = |1 - z\bar{\xi}|^{-2n}(1-|z|^2)^n$ is the invariant Poisson kernel on B, and $u^\star(\xi) = \lim_{r\to 1} u(r\xi)$ a.e. on S, [**13**]. Thus by the Riesz Decomposition Theorem [**14**], [**15**] with $z = 0$, we again have the equality (1.10):

$$\|u\|_{\mathcal{H}^p}^p = |u(0)|^p + p(p-1)\int_B |u(z)|^{p-2}|\widetilde{\nabla} u(z)|^2 G(|z|,1)\, d\tau(z)$$

For a similar argument see [**4**], [**14**]. Our argument in proving (1.10) is much simpler. We only use Green's theorem, as well as in the proof of Theorem 1.4.We note that the characterization of the Hardy space $\mathcal{H}^p$ given in Theorem 1.4 is a part of Proposition 5, [**14**]. The proof given in [**14**] is also based on the computation of the Riesz measure of the function $|u|^p$ and the Riesz decomposition theorem.

4. A Littlewood-Paley type inequalities

Proof of Theorem 1.1. Without loss of generality we may suppose that f is real. It is well known that a $\mathcal{M}$-harmonic function v belongs to $\mathcal{H}^p$, $1 < p < \infty$, if and only if $v = P[g]$, for some $g \in L^p(S)$, $1 < p < \infty$. Furthemore, $\|v\|_{\mathcal{H}^p} = \|g\|_{L^p(S)}$. Thus to prove (1.1), by (1.8), it is sufficient to show that

$$I_1 = \int_B |u(z)|^p d\nu(z) \preceq \left(|u(0)|^p + \int_B |\widetilde{\nabla} u(z)|^p (1-|z|^2)^{-1} d\nu(z)\right)$$

and

$$I_2 = \int_B |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} (1-|z|^2)^{-1} d\nu(z) \preceq \int_B |\widetilde{\nabla} u(z)|^p (1-|z|^2)^{-1} d\nu(z).$$

Let I denote the last integral. Using Lemma 2.1, (b), we find that

$$\begin{aligned}
|u(z)|^p &= \left|\int_0^1 \frac{d}{dt} u(tz)\, dt + u(0)\right|^p \\
&\le C\left(\int_0^1 |\nabla u(tz)|^p dt + |u(0)|^p\right) \\
&\le C\left(\int_0^1 \left(\int_{E_r(tz)} |\nabla u(w)|^p d\tau(w)\right) dt + |u(0)|^p\right)
\end{aligned}$$

Since $1-|w|^2 \cong 1-t^2|z|^2 \cong |1-tz\bar{w}|$ for $w \in E_r(tz)$, we obtain

$$\begin{aligned}
|u(z)|^p &\le C\left(\int_0^1 \left(\int_{E_r(tz)} \frac{|\nabla u(w)|^p (1-|w|^2)^{\alpha-n-1}}{|1-tz\bar{w}|^\alpha} d\nu(w)\right) dt + |u(0)|^p\right) \\
&\le C\left(\int_0^1 \left(\int_B \frac{|\nabla u(w)|^p (1-|w|^2)^{\alpha-n-1}}{|1-tz\bar{w}|^\alpha} d\nu(w)\right) dt + |u(0)|^p\right) \\
&\le C\left(\int_B \frac{|\nabla u(w)|^p (1-|w|^2)^{\alpha-n-1}}{|1-z\bar{w}|^{\alpha-1}} d\nu(w) + |u(0)|^p\right)
\end{aligned}$$

We may suppose that $\alpha > n+2$.

By (2.1) $|\nabla u(w)| \le \frac{|\widetilde{\nabla} u(w)|}{1-|w|^2}$. Thus

$$\begin{aligned}
I_1 &\le C\left(|u(0)|^p + \int_B d\nu(z) \int_B \frac{|\widetilde{\nabla} u(w)|^p (1-|w|^2)^{\alpha-n-1-p}}{|1-z\bar{w}|^{\alpha-1}} d\nu(w)\right) \\
&= C\left(|u(0)|^p + \int_B |\widetilde{\nabla} u(w)|^p (1-|w|^2)^{\alpha-n-1-p} d\nu(w) \int_B \frac{d\nu(z)}{|1-z\bar{w}|^{\alpha-1}}\right)
\end{aligned}$$

By standard estimates (see [**13**, p. 17])

$$\int_B \frac{d\nu(z)}{|1-z\bar{w}|^{\alpha-1}} \le \frac{C}{(1-|w|^2)^{\alpha-n-2}}.$$

Hence,

$$I_1 \le C\Big(|u(0)|^p + \int_B |\widetilde{\nabla} u(w)|^p (1-|w|^2)^{1-p} d\nu(w)\Big)$$

Since $1 < p \le 2$, we have $(1-|w|^2)^{1-p} \le (1-|w|^2)^{-1}$. Therefore, $I_1 \preceq (|u(0)|^p + I)$.

Let $r \in (0, 1/2)$ be fixed. Set $\epsilon = r/4$ and $\delta = r/2$. By Fubini's theorem we have

$$I_2 \le C \int_B (1-|a|^2)^n d\tau(a) \int_{E_\epsilon(a)} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z). \tag{4.1}$$

It is easy to see that if $|z| \le r/4$ then $G(|z|, r/2) \ge C > 0$ and therefore

$$\int_{\epsilon B} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z) \le C \int_{\delta B} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|z|, r/2)\, d\tau(z).$$

Applying this to $u \circ \varphi_a$ and using the $\mathcal{M}$-invariance of $|\widetilde{\nabla}|$, we see that

$$\int_{E_\epsilon(a)} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z) \le C \int_{E_\delta(a)} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|\varphi_a(z)|, \delta) d\tau(z). \tag{4.2}$$

Let $c_p = p(p-1)$. Using (3.8) and Jensen‘s inequality we get

$$\begin{aligned} c_p \int_{\delta B} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|z|, r/2)\, d\tau(z) &= \int_S |u(\delta\xi)|^p d\sigma(\xi) - |u(0)|^p \\ &= \int_S \big(|u(\delta\xi)|^2\big)^{p/2} - \big(|u(0)|^2\big)^{p/2} \\ &\le \Big(\int_S |u(\delta\xi)|^2 d\sigma(\xi)\Big)^{p/2} - \big(|u(0)|^2\big)^{p/2}. \end{aligned} \tag{4.3}$$

If $0 < \alpha \le 1$ and $0 \le b \le a$ then $a^\alpha - b^\alpha \le (a-b)^\alpha$. Thus

$$\begin{aligned} \Big(\int_S |u(\delta\xi)|^2 d\sigma(\xi)\Big)^{p/2} - \big(|u(0)|^2\big)^{p/2} &\le \Big(\int_S |u(\delta\xi)|^2 d\sigma(\xi) - |u(0)|^2\Big)^{p/2} \\ &= \Big(2\int_{\delta B} |\widetilde{\nabla} u(z)|^2 G(|z|, r/2) d\tau(z)\Big)^{p/2} \\ &\le C \sup_{z \in \delta B} |\widetilde{\nabla} u(z)|^p \Big(\int_{\delta B} G(|z|, r/2) d\tau(z)\Big)^{p/2} \\ &\le C \sup_{z \in \delta B} |\widetilde{\nabla} u(z)|^p. \end{aligned} \tag{4.4}$$

Using $\mathcal{M}$-harmonic behaviour of $|\widetilde{\nabla} u(z)|^p$, (Lemma 2.1), we find that

$$\sup_{z\in\delta B}|\widetilde{\nabla} u(z)|^p \le C \sup_{z\in\delta B}\int_{E_\delta(z)}|\widetilde{\nabla} u(w)|^p d\tau(w) \le C\int_{rB}|\widetilde{\nabla} u(w)|^p d\tau(w). \tag{4.5}$$

Combining (4.3), (4.4) and (4.5) we obtain

$$\int_{\delta B}|\widetilde{\nabla} u(z)|^2|u(z)|^{p-2}G(|z|,r/2)\,d\tau(z) \le C\int_{rB}|\widetilde{\nabla} u(z)|^p d\tau(z)$$

Applying this again to $u\circ\varphi_a$ we get

$$\int_{E_\delta(a)}|\widetilde{\nabla} u(z)|^2|u(z)|^{p-2}G(|\varphi_a(z)|,r/2)d\tau(z) \le C\int_{E_r(a)}|\widetilde{\nabla} u(z)|^p d\tau(z) \tag{4.6}$$

Finally, from (4.1), (4.2) and (4.6) we see that

$$\begin{aligned} I_2 &\le C\int_B (1-|a|^2)^n d\tau(a)\int_{E_r(a)}|\widetilde{\nabla} u(z)|^p d\tau(z)\\ &= C\int_B |\widetilde{\nabla} u(z)|^p(1-|z|^2)^{-1}d\nu(z) = CI, \end{aligned}$$

by Fubini's theorem.

This finishes the proof of the inequality (1.1).

Since the function $|u|^p$ is $\mathcal{M}$-subharmonic we have

$$|u(0)|^p \le \int_B |u(z)|^p d\nu(z).$$

Hence by (1.8) to prove (1.2) it sufficies to show that $I_2 \ge CI$.

By Fubini's theorem

$$I_2 \cong \int_B (1-|a|^2)^n d\tau(a)\int_{E_r(a)}|\widetilde{\nabla} u(z)|^2|u(z)|^{p-2}d\tau(z) \tag{4.7}$$

Using (3.8) and Jensen's inequality we find that

$$\begin{aligned} c_p\int_{\delta B}|\widetilde{\nabla} u(z)|^2|u(z)|^{p-2}G(|z|,\delta)\,d\tau(z) &= \int_S |u(\delta\xi)|^p d\sigma(\xi) - |u(0)|^p\\ &= \int_S \left(|u(\delta\xi)|^2\right)^{p/2} d\sigma(\xi) - |u(0)|^p\\ &\ge \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi)\right)^{p/2} - (|u(0)|^2)^{p/2}. \end{aligned}$$

For $1 \le \alpha < \infty$ and $0 \le b \le a$, we have $(a-b)^\alpha \le a^\alpha - b^\alpha$.

Hence, by using (3.8) and Lemma 2.1,

$$\begin{aligned}\left(\int_S |u(\delta\xi)|^2 d\sigma(\xi)\right)^{p/2} - (|u(0)|^2)^{p/2} &\ge \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) - |u(0)|^2\right)^{p/2} \\ &= \left(2\int_{\delta B} |\widetilde{\nabla} u(z)|^2 G(|z|,\delta) d\tau(z)\right)^{p/2} \\ &\ge C\left(\int_{\delta B} |\widetilde{\nabla} u(z)|^2 d\tau(z)\right)^{p/2} \ge C|\widetilde{\nabla} u(0)|^p\end{aligned}$$

Therefore,

$$|\widetilde{\nabla} u(0)|^p \le C\int_{\delta B} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|z|,\delta)\, d\tau(z)$$

Since $G(|z|,\delta) \le C|z|^{2-2n}$, for $|z| \le \delta$, (note that $n > 1$), we get

$$|\widetilde{\nabla} u(0)|^p \le C\int_{\delta B} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} |z|^{2-2n} d\tau(z).$$

Applying this to $u \circ \varphi_a$, where $|a| < \delta$, we find that

$$|\widetilde{\nabla} u(a)|^p \le C\int_{E_\delta(a)} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} |\varphi_a(z)|^{2-2n} d\tau(z)$$

By Lemma 2.4 $|\varphi_a(z)| \ge C|z-a|$. Thus

$$|\widetilde{\nabla} u(a)|^p \le C\int_{E_\delta(a)} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} |z-a|^{2-2n} d\tau(z).$$

Integrating the last inequality over $|a| < \delta$ with respect to ν we obtain

$$\int_{\delta B} |\widetilde{\nabla} u(a)|^p d\nu(a) \le C\int_{rB} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z) \int_{\delta B} |z-a|^{2-2n} d\nu(a).$$

Since, for $z \in rB$,

$$\int_{\delta B} |z-a|^{2-2n} d\nu(a) \le \int_{\{a:|z-a|<2r\}} |z-a|^{2-2n} d\nu \le C$$

where C is independent of z, we have

$$|\widetilde{\nabla} u(0)|^p \le C\int_{\delta B} |\widetilde{\nabla} u(a)|^p d\tau(a) \le C\int_{rB} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z)$$

Applying this again to $u \circ \varphi_a$ we get

$$|\widetilde{\nabla} u(a)|^p \leq C \int_{E_r(a)} |\widetilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z) \tag{4.8}$$

From (4.7) and (4.8) it follows that $I_2 \geq CI$. This finishes the proof of Theorem 1.1

Now, using Theorem 1.1 we get another version of Littlewood-Paley theorem.

THEOREM 4.1. *Let $f \in L^p(S)$ and $u = P[f]$. Then*

$$\|f\|_p^p \preceq \left(|u(0)|^p + \int_B |\widetilde{\nabla} u(z)|^p G(|z|, 1)\, d\tau(z) \right), \quad 1 < p \leq 2 \tag{4.9}$$

and

$$\|f\|_p^p \succeq \left(|u(0)|^p + \int_B |\widetilde{\nabla} u(z)|^p G(|z|, 1) d\tau(z) \right), \quad 2 \leq p < \infty. \tag{4.10}$$

Proof. By Lemma 2.3 $G(|z|, 1) \geq c_n (1 - |z|)^n$ for all $z \in B$. Hence, (4.9) is a consequence of (1.1).

Let $0 < \delta < 1/2$ be fixed. Then using again Lemma 2.3 we find that

$$\begin{aligned} &\int_B |\widetilde{\nabla} u(z)|^p G(|z|, 1)\, d\tau(z) \\ &\qquad \leq C \left(\int_{\frac{1}{2}\delta B} |\widetilde{\nabla} u(z)|^p |z|^{2-2n} d\tau(z) + \int_{(\frac{1}{2}\delta B)^c} |\widetilde{\nabla} u(z)|^p (1 - |z|^2)^n d\tau(z) \right). \end{aligned}$$

Thus, (4.10) follows from (1.2) since

$$\int_{\frac{1}{2}\delta B} |\widetilde{\nabla} u(z)|^p |z|^{2-2n} d\tau(z) \leq C \int_{\delta B} |\widetilde{\nabla} u(z)|^p (1 - |z|^2)^{-1} d\nu(z).$$

Theorem 1.2 is a corollary of Theorem 1.1 and the following theorem

THEOREM 4.2. *Let $1 < p < \infty$ and $u \in \mathcal{M}$. Then the following statements are equivalent:*

(*i*) $\int_B |\widetilde{\nabla} u(z)|^p (1 - |z|^2)^{-1} d\nu(z) < \infty$.
(*ii*) $\int_B |\nabla u(z)|^p (1 - |z|^2)^{p-1} d\nu(z) < \infty$.
(*iii*) $\int_B |Ru(z)|^p (1 - |z|^2)^{p-1} d\nu(z) < \infty$.

Proof. Since $|\widetilde{\nabla} u(z)| \geq (1 - |z|^2)|\nabla u(z)| \geq (1 - |z|^2)|Ru(z)|$, we have $(i) \Rightarrow (ii) \Rightarrow (iii)$. So it remains to show that $(iii) \Rightarrow (i)$.

By (2.2) to prove that $(iii) \Rightarrow (i)$ it is sufficient to prove that

$$\int_B (1-|z|^2)^{p/2-1} |T_{i,j} u(z)|^p d\nu(z) \le C \int_B (1-|z|^2)^{p-1} |Ru(z)|^p d\nu(z), \;\; 1 \le i < j \le n.$$

An integration by parts show that

$$u(z) = \int_0^1 \big[Ru(tz) + \bar{R}u(tz) + u(tz) \big] dt$$

From this we conclude that it is sufficient to prove that

$$I = \int_B (1-|z|^2)^{p/2-1} \left(\int_0^1 |T_{i,j} v(tz)| dt \right)^p d\nu(z) \le C \int_B (1-|z|^2)^{p-1} |Ru(z)|^p d\nu(z)$$

where v is either Ru or $\bar{R}u$ or u. We prove this for $v = Ru$. The remaining cases may be treated analogously.

Let $J = \int_0^1 |T_{i,j} Ru(tz)| dt$. Using Lemma 2.2 and Fubini's theorem we find that for any $s > 0$

$$\begin{aligned} J &\le C \int_0^1 \left(\int_{E_r(tz)} \frac{|Ru(w)|(1-|w|^2)^s d\nu(w)}{|1 - tz\bar{w}|^{n+s+3/2}} \right) dt \\ &\le C \int_0^1 \left(\int_B \frac{|Ru(w)|(1-|w|^2)^s d\nu(w)}{|1 - tz\bar{w}|^{n+s+3/2}} \right) dt \\ &= \int_B |Ru(w)|(1-|w|^2)^s \left(\int_0^1 \frac{dt}{|1 - tz\bar{w}|^{n+s+3/2}} \right) d\nu(w) \\ &\le C \int_B \frac{|Ru(w)|(1-|w|^2)^s d\nu(w)}{|1 - z\bar{w}|^{n+s+1/2}}. \end{aligned}$$

Now we apply Lemma 4.1 [**3**] to conclude that

$$\begin{aligned} I &\le C \left(\int_B (1-|z|^2)^{p/2-1} \left(\int_B \frac{|Ru(w)|(1-|w|^2)^s d\nu(w)}{|1 - z\bar{w}|^{n+s+1/2}} \right)^p d\nu(z) \right) \\ &\le C \int_B (1-|w|^2)^{p-1} |Ru(w)|^p d\nu(w) \end{aligned}$$

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