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A FIRST ORDER PROBABILITY LOGIC - LP_Q

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Abstract. A conservative extension of the classical first order logic which allows making statements about probability is introduced. Some classes of probability models are described. An infinitary axiomatic system which is sound and complete with respect to these classes of models is given.

1. Introduction

The first order probability logic LP is given in [**3**,**4**] Its language is obtained by adding probability operators of the form $P_{\geq s}$ to the classical first order language, where s belongs to a set Index which is a finite subset of $[0,1]$. Formulas in the scope of a probability operator are classical first order formulas. LP allows making formulas such as $P_{\geq s}\alpha$, with the intended meaning "the probability of α is greater then or equal to s". In [**3**,**4**] a finitary axiomatic system is provided, and the corresponding extended completeness theorem is proved.

In this paper we investigate another first order probability logic, denoted LP_Q, whose language contains a list of probability operators of the form mentioned above, but the set Index is the set of all rational numbers from $[0,1]$. It turns out that such an assumption makes LP_Q different from LP. Namely, the compactness theorem does not hold for LP_Q, while it holds for LP: consider an arbitrary classical sentence α and the set $T = \{\neg P_{=0}\alpha\} \cup \{P_{<1/n}\alpha : n \text{ is a positive integer}\}$; although every finite subset of T is satisfiable, the set T is not. A consequence is that, if we want the extended completeness theorem, we cannot obtain a finitary axiomatization. In this paper, we describe some classes of probability models, give an axiomatization with an infinitary rule, and prove the corresponding extended completeness theorems. We also discuss (un)decidability of LP_Q.

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2. Syntax

Let Index be the set of all rational numbers from $[0,1]$. The language of LP_Q is a countable classical first order language extended by a list of probability operators $P_{\geq s}$, for every $s \in$ Index. Let us denote the set of all classical first order formulas by For_C. The formulas from the set For_C will be denoted by α, $\beta, \dots$. If $\alpha \in \mathrm{For}_C$, and $s \in$ Index, then $P_{\geq s}\alpha$ is a basic probability formula. The set of all probability formulas is the least set For_P containing all basic probability formulas, and closed under formation rules: if $A, B \in \mathrm{For}_P$, then $\neg A$, $A \wedge B \in \mathrm{For}_P$. The formulas from the set For_P will be denoted by A, $B, \dots$ Let $\mathrm{For}_C \cup \mathrm{For}_P$ be denoted by For, and the set of all sentences from For by Sentences. The formulas from the set For will be denoted by Φ, $\Psi, \dots$ We use the usual abbreviations for the other classical connectives, and also denote $\neg P_{\geq s}(\alpha)$ by $P_{<s}(\alpha)$, $P_{\geq 1-s}(\neg\alpha)$ by $P_{\leq s}(\alpha)$, $\neg P_{\leq s}(\alpha)$ by $P_{>s}(\alpha)$, and $\Phi \wedge \neg\Phi$ for an arbitrary $\Phi \in$ For by $\perp$.

3. Semantics

We use the possible-worlds approach to give semantics to formulas and interpret formulas such that they remain either true or false. An LP_Q-model is a structure $M = \langle W, D, I, A, \mu\rangle$ where:

- W is a non empty set of objects called worlds,
- D is a function which assigns to every $w \in W$ a domain $D(w)$,
- I is a function which assigns to every $w \in W$ a classical interpretation $I(w)$,
- A is an algebra of subsets of W, and
- μ is a finitely additive probability measure, $\mu : A \to [0,1]$.

Let $M = \langle W, D, I, A, \mu\rangle$ be an LP_Q-model. A variable valuation v assigns some element of the domain $D(w)$ to every world w and every variable x, i.e., $v(w)(x) \in D(w)$. If $D(w)$ is a domain, $d \in D(w)$, and v is a valuation, then $v_w[d/x]$ is a valuation like v except that $v_w[d/x](w)(x) = d$. The values of terms and classical formulas in a world is defined as usual. For example, the value of a classical formula $(\forall x)\alpha$ in $w \in W$ for a given valuation v (denoted by $I(w)((\forall x)\alpha)_v$) is true if and only if for every $d \in D(w)$, $I(w)(\alpha)_{v_w[d/x]}$ is true. A classical formula holds in a world w of an LP_Q model M (denoted by $(M,w) \models \alpha$) if for every valuation v, $I(w)(\alpha)_v$ is true.

Let M be an LP_Q model and α a classical sentence. The set $\{w \in W : (M,w) \models \alpha\}$ is denoted by $[\alpha]_M$. We will omit the subscript M from $[\alpha]_M$ and write $[\alpha]$, if M is clear from the context. An LP_Q-model M is measurable if $[\alpha]$ is measurable for every classical sentence α. In this paper we will focus on the class $LP_{Q,\mathrm{Meas}}$ of all measurable LP_Q-models, as well as on its subclasses: $LP_{Q,\mathrm{All}}$, the class of all $LP_{Q,\mathrm{Meas}}$-models such that a model $M = \langle W, D, I, A, \mu, \rangle$ belongs to $LP_{Q,\mathrm{All}}$ if A is the power set of W, and $LP_{Q,\sigma}$, the class of all $LP_{Q,\mathrm{Meas}}$-models with σ-additive measure.

Let L be one of the above class of models. The satisfiability relation $\models \subset L \times$ Sentences fulfills the following conditions:

- if $\alpha \in \mathrm{For}_C$, $M \models \alpha$ if $(\forall w \in W)(M, w) \models \alpha$,
- $M \models P_{\geq s}\alpha$ if $\mu([\alpha]) \geq s$,
- if $A \in \mathrm{For}_P$, $M \models \neg A$ if $M \not\models A$, and
- if $A, B \in \mathrm{For}_P$, $M \models A \wedge B$ if $M \models A$ and $M \models B$.

A set T of sentences is L-satisfiable if there is an L-model M such that every sentence from T is satisfied in M. A sentence $\Phi \in \mathrm{For}$ is L-valid if it is satisfied in every L-model.

4. Complete Axiomatization

The axiom schemata for LP_Q are:

(1) axiom schemata of the classical first order logic
(2) $P_{\geq 0}\alpha$
(3) $P_{\leq r}\alpha \to P_{<s}\alpha$, $s > r$
(4) $P_{<s}\alpha \to P_{\leq s}\alpha$
(5) $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg(\alpha \wedge \beta))) \to P_{\geq \min(1, r+s)}(\alpha \vee \beta)$
(6) $(P_{\leq r}\alpha \wedge P_{<s}\beta) \to P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$

while the inference rules are:

(1) From Φ and $\Phi \to \Psi$ infer Ψ.
(2) From α infer $(\forall x)\alpha$
(3) From α infer $P_{\geq 1}\alpha$.
(4) From $A \to P_{\geq s-1/k}\alpha$, for every integer $k \geq 1/s$, and $s > 0$ infer $A \to P_{\geq s}\alpha$.

The main difference between the axiomatic system for the logic LP and the one given above is that the inference rule 4 does not appear in the former system. Note that formulas obtained by applications of the inference rules must obey the formation rules, i.e., in the inference rules 2 and 3, α must be a classical formula. A formula $\Phi \in \mathrm{For}$ is deducible from a set T of sentences ($T \vdash \phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \dots, \Phi$, such that every formula in the sequence is an axiom or a formula from the set T, or it is derived from the preceding formulas by an application of an inference rule. A set T of sentences is inconsistent if $T \vdash \perp$, otherwise it is consistent.

In the proof of the completeness theorem the Henkin procedure will be used. We begin with some auxiliary statements.

THEOREM 4.1 (Deduction theorem) *If* $T \subset$ Sentences, $\Phi \in$ Sentences, *and* $T \cup \{\Phi\} \vdash \Psi$, *then* $T \vdash \Phi \to \Psi$, *where* Φ *and* Ψ *are either both classical or both probability formulas.*

Proof. We use the transfinite induction on the length of the proof of Ψ from $T \cup \{\Phi\}$. For example, we consider the case where $B = C \to P_{\geq s}\delta$ is obtained from $T \cup \{A\}$ by an application of the inference rule 4, and A is a probability sentence. Then:

$T, A \vdash C \to P_{\geq s-1/k}\delta$, for every integer $k \geq 1/s$

$T \vdash A \to (C \to P_{\geq s-1/k}\delta)$, for every integer $k \geq 1/s$, by the induction hypothesis

$T \vdash (A \wedge C) \to P_{\geq s-1/k}\delta$, for every integer $k \geq 1/s$

$T \vdash (A \wedge C) \to P_{\geq s}\delta$, by the inference rule 4

$T \vdash A \to B$

The other cases follow similarly. $\Box$

THEOREM 4.2. *Let α and β be classical sentences. Then:*

(1) $\vdash P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$
(2) $\vdash P_{\geq r}\alpha \to P_{\geq s}\alpha$, $r > s$

Proof. 1. If $s = 0$, the statement obviously holds. So, let s be a rational number from $(0, 1]$. First note that by an application of the inference rule 3, we obtain

$$\vdash P_{\geq 1}(\neg\alpha \vee \neg\bot) \tag{1}$$

from $\vdash \neg\alpha \vee \neg\bot$. Similarly, from $\vdash (\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha$ we have

$$\vdash P_{\geq 1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha). \tag{2}$$

By the axiom 5, we have $\vdash (P_{\geq s}\alpha \wedge P_{\geq 0}\neg\bot \wedge P_{\geq 1}(\neg\alpha \vee \neg\bot)) \to P_{\geq s}(\alpha \vee \bot)$. Since $\vdash P_{\geq 0}\neg\bot$ by the axiom 2, from (1) it follows that

$$\vdash P_{\geq s}\alpha \to P_{\geq s}(\alpha \vee \bot). \tag{3}$$

The expressions $P_{\geq s}(\alpha \vee \bot)$ and $\neg P_{\geq s}\neg\neg\alpha$ denote $P_{\leq 1-s}(\neg\alpha \wedge \neg\bot)$, and $P_{<s}\neg\neg\alpha$, respectively. By the axiom 6, we have $\vdash (P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha) \to P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$. From (2) we obtain that $\vdash (P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha) \to (P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha) \wedge \neg P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$. It follows that $\vdash P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \to \neg P_{<s}\neg\neg\alpha$, i.e.

$$\vdash P_{\geq s}(\alpha \vee \bot) \to P_{\geq s}\neg\neg\alpha. \tag{4}$$

From (3) and (4) we obtain:

$$\vdash P_{\geq s}\alpha \to P_{\geq s}\neg\neg\alpha. \tag{5}$$

The negation of the formula $P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$ is equivalent to $P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\geq s}\alpha \wedge P_{<s}\beta$. By (5) this formula implies $P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\geq s}\neg\neg\alpha \wedge P_{<s}\beta$ which can be rewritten as $P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\leq 1-s}\neg\alpha \wedge P_{<s}\beta$. From the axiom 6, $P_{\leq 1-s}\neg\alpha \wedge P_{<s}\beta \to P_{<1}(\neg\alpha \vee \beta)$, and $P_{<1}\alpha = \neg P_{\geq 1}\alpha$, we have $\vdash \neg(P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)) \to P_{\geq 1}(\neg\alpha \vee \beta) \wedge \neg P_{\geq 1}(\neg\alpha \vee \beta)$, a contradiction. It follows that $\vdash P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$.

2. By the axioms 3 and 4, we have $\vdash P_{\geq r}\alpha \to P_{>s}\alpha$, for $r > s$, and $\vdash P_{>s}\alpha \to P_{\geq s}\alpha$. Thus, $\vdash P_{\geq r}\alpha \to P_{\geq s}\alpha$, for $r > s$. $\square$

THEOREM 4.3. (Completeness theorem for $LP_{Q,\mathrm{Meas}}$) *Let $T \subset$ Sentences. Then, T is consistent if and only if T has an $LP_{Q,\mathrm{Meas}}$-model.*

Proof. The ($\Leftarrow$)-direction follows from the soundness of the above axiomatic system. In order to prove the ($\Rightarrow$)-direction let us suppose that T is a consistent set of sentences, that clconseq(T) is the set of all classical sentences that are consequences of T and that $A_0, A_1, \ldots$ is an enumeration of all probability sentences. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$ such that:

(1) $T_0 = T \cup \mathrm{clconseq}(T) \cup \{P_{\geq 1}\alpha : \alpha \in \mathrm{clconseq}(T)\}$
(2) for every $i \geq 0$, if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$, otherwise, $T_{i+1} = T_i \cup \{\neg A_i\}$,
(3) if the set T_{i+1} is obtained by adding a formula of the form $\neg(B \to P_{\geq s}\gamma)$, then for some positive integer n, $B \to \neg P_{\geq s-1/n}\gamma$, is also added to T_{i+1}, so that T_{i+1} is consistent.

Every T_i is a consistent set. T_0 is consistent because it is a set of consequences of a consistent set. Suppose that T_i is obtained by the step 2 of the above construction and that neither $T_i \cup \{A_i\}$, nor $T_i \cup \{\neg A_i\}$ are consistent. It follows by the deduction theorem that $T_i \vdash A_i \wedge \neg A_i$, which is a contradiction. Consider the step 3 of the construction. If $T_i \cup \{B \to P_{\geq s}\gamma\}$ is not consistent, then the set T_i can be consistently extended as above. Suppose that it is not the case. Then:

(1) $T_i, \neg(B \to P_{\geq s}\gamma), B \to \neg P_{\geq s-1/k}\gamma \vdash \perp$, for every $k > 1/s$, by the hypothesis
(2) $T_i, \neg(B \to P_{\geq s}\gamma) \vdash \neg(B \to \neg P_{\geq s-1/k}\gamma)$ for every $k > 1/s$, by the deduction theorem
(3) $T_i, \neg(B \to P_{\geq s}\gamma) \vdash B \to P_{\geq s-1/k}\gamma$ for every $k > 1/s$, from 2, by the classical tautology $\neg(\alpha \to \gamma) \to (\alpha \to \neg\gamma)$
(4) $T_i, \neg(B \to P_{\geq s}\gamma) \vdash B \to P_{\geq s}\gamma$, from 3, by the inference rule 4
(5) $T_i \vdash \neg(B \to P_{\geq s}\gamma) \to B \to P_{\geq s}\gamma$, from 4, by the deduction theorem
(6) $T_i \vdash B \to P_{\geq s}\gamma$

Since $T_i \cup \{B \to P_{\geq s}\gamma\}$ is not consistent, from $T_i \vdash B \to P_{\geq s}\gamma$ it follows that T_i is not consistent, a contradiction.

Let $T^* = \cup_i T_i$. The set T^* is a deductively closed set that does not contain all sentences. First note that for every $\Phi \in$ Sentences, if $T_i \vdash \Phi$, then it must be $\Phi \in T^*$. If Φ is a classical sentence, then $T \vdash \Phi$, and $\Phi \in T_0$. If $\Phi = A_k$ is a probability sentence, and $\Phi \notin T^*$, then $T_{\max\{i,k\}+1} \vdash \Phi$ and $T_{\max\{i,k\}+1} \vdash \neg\Phi$, a contradiction. Since T is a consistent set, there is at least a classical sentence α such that $T \nvdash \alpha$. If A is a probability sentence, it cannot be $A = A_k \in T^*$, and $\neg A = A_m \in T^*$, because $T_{\max\{k,m\}+1}$ is consistent. Finally, we can prove that if A is a probability sentence, and $T^* \vdash A$, then $A \in T^*$. Suppose that the sequence $\Phi_1, \Phi_2, \ldots, A$ of formulas which forms the proof of A from T^* is countably infinite (otherwise there must be some k such that $T_k \vdash A$, and it must be $A \in T^*$). We can show that for every i, if Φ_i is obtained by an application of an inference rule, and all the premises of Φ_i belong to T^*, then $\Phi_i \in T^*$. Suppose Φ_i is obtained by the inference rule 1 (modus ponens) and its premises Φ_i^1 and Φ_i^2 belong to T^*. There must be some k such that $\Phi_i^1, \Phi_i^2 \in T_k$. From $T_k \vdash \Phi_i$, it follows $\Phi_i \in T^*$. If Φ_i is obtained by the inference rules 2 and 3, then $T_0 \vdash \Phi_i$, and $\Phi_i \in T^*$. Suppose that $\Phi_i = B \to P_{\geq s}\gamma$ is obtained by the infinitary inference rule 4, and that the premises $\Phi_i^1 = B \to P_{\geq s-1/k}\gamma, \Phi_i^2 = B \to P_{\geq s-1/(k+1)}\gamma, \ldots$ belong to T^*. If $\Phi_i \notin T^*$, by the step 3 of the construction of T^*, there is some $j > 1/s$, such that $B \to \neg P_{\geq s-1/j}\gamma \in T^*$. Let $l = \max\{k, j\}$. By the axioms 3 and 4, $B \to P_{\geq s-1/l}\gamma \in T^*$, and $B \to \neg P_{\geq s-1/l}\gamma \in T^*$. There must be a set T_m which also contains these formulas. It follows that $T_m \cup \{B\}$ is not consistent. Thus, $B \notin T^*$, and there is some j such that $\neg B \in T_j$, $T_j \vdash B \to \bot$, $T_j \vdash B \to P_{\geq s}\gamma$, and $B \to P_{\geq s}\gamma \in T^*$, which is a contradiction. Hence, from $T^* \vdash A$, it follows $A \in T^*$.

The set T^* is used to construct a tuple $M = \langle W, D, I, \{[\alpha] : \alpha \text{ is a classical sentence}\}, \mu, \rangle$, where:

- $W = \{w : w \models \text{clconseq}(T)\}$ contains all the classical first order interpretations with at most countable domains that satisfy the set clconseq(T) of all classical consequences of the set T; the corresponding domains are denoted by $D(w)$,
- D maps every $w \in W$ to $D(w)$,
- $I(w)$ is the interpretation w,
- $\mu : \{[\alpha] : \alpha \text{ is a classical sentence}\} \to [0, 1]$ such that $\mu([\alpha]) = \sup\{s : P_{\geq s}\alpha \in T^*\}$.

The axioms guarantee that everything is well defined. For example, by the classical reasoning we can show that $\{[\alpha] : \alpha$ is a classical sentence$\}$ is an algebra of subsets of W. The theorem 4.2.1 implies that if $[\alpha] = [\beta]$, then $\mu([\alpha]) = \mu([\beta])$. From the axioms 2–6 about probability it follows that μ is a finitely additive probability measure.

By the induction on the complexity of formulas we can prove that for every sentence Φ, $M \models \Phi$ iff $\Phi \in T^*$. For example, let Φ be a classical sentence. If $\Phi \in \text{clconseq}(T)$, then by the definition of M, $M \models \Phi$. Conversely, let $M \models \Phi$. Then, by the completeness of the classical first order logic, $\Phi \in \text{clconseq}(T)$. If $\Phi = P_{\geq s}\alpha \in T^*$, then $\sup\{r : P_{\geq r}(\alpha) \in T^*\} = \mu([\alpha]) \geq s$, and $M \models \Phi$. For the other direction, suppose that $M \models \Phi$, i.e., that $\sup\{r : P_{\geq r}(\alpha) \in T^*\} \geq s$. If $\mu([\alpha]) > s$, then, by the well known property of supremum and monotonicity of μ (the theorem 4.2.2), $\Phi \in T^*$. Let $\mu([\alpha]) = s$. If $\Phi \notin T^*$, then by the step 3 of the construction of T^*, for some integer $k > 1/s$, $\neg P_{\geq s-1/k}\alpha \notin T^*$. It follows that s cannot be the supremum of the set $\{r : P_{\geq r}\alpha \in T^*\}$, which is a contradiction. The other cases follow easily. $\square$

THEOREM 4.4. (Completeness theorem for $LP_{Q,\text{All}}$) *Let* $T \subset$ Sentences. *Then,* T *is consistent if and only if* T *has an* $LP_{Q,\text{All}}$*-model.*

Proof. The proof can be obtained by applying the extension theorem for additive measures [**1**] on the measure μ from the canonical model M described in the theorem 4.3. It is proved that there is an additive measure $\overline{\mu}$ defined on the power set of W which is an extension of the measure μ. $\square$

THEOREM 4.5. (Completeness theorem for $LP_{Q,\sigma}$) *Let* $T \subset$ Sentences. *Then, T is consistent if and only if T has an $LP_{Q,\sigma}$-model.*

Proof. By the Loeb process and a bounded elementary embedding [**2**] we can transform the canonical model M from the theorem 4.3 into a σ-additive probability model ${}^*M$ such that for every $\Phi \in$ Sentences, $M \models \Phi$ iff ${}^*M \models \Phi$. $\square$

5. Decidability

LP_Q-logic is undecidable since it contains the classical first order logic. However, some fragments of LP_Q are decidable. One of these fragments is the monadic first order probability logic (without function symbols except constants) in which the arity of all relation symbols is 1. By the Herbrand theorem, every first order classical sentence α is satisfiable if and only if the set $E(\alpha)$ of formulas that form the Herbrand expansion of α is satisfiable. Formulas from $E(\alpha)$ are without variables and can be understood as formulas in the classical propositional logic. In the monadic case, for every formula α the set $E(\alpha)$ is finite. Thus, the satisfiability of the monadic LP_Q-logic can be reduced to the satisfiability of the propositional probability logic. Since the propositional probability logic is decidable [**3**], the monadic LP_Q-logic is decidable.

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