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INITIAL MODELS AND HORN CLAUSE AXIOMATIZABILITY

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Abstract. Maljcev introduced a notion of free presentation for arbitrary classes of models and characterized classes of structures admitting such presentations. The connection with Horn clause axiomatizability was shown by Tabata in case of first order theories with equality. Here we prove a generalization of this result for infinitary theories without equality.

In his papers [**3**] and [**4**] Maljcev generalized the notion of free algebra by defining presentations for arbitrary classes of models and characterized classes of structures which admit such presentations. The connection with Horn clause axiomatizability was shown by Tabata [**5**] for first order languages with equality. The more general formulation using the notion of initial model is in [**2**, p. 472] but without proof and again for the first order case. There is, on the other hand, an exposition of these results for infinitary theories in [**1**, 9.2] but again for languages with equality. Our intention is to go step further and consider *infinitary theories without equality.*

Let us first recall some notation and definitions. By L_A we shall denote the expansion of the language L formed by adding the (names of) elements of A as new constant symbols. Similarly, $(A, \bar{a})$ will denote the expansion of a structure A obtained by taking all elements of (the domain of) A as constants. By a strict Horn clause we mean a sentence of the form $\forall \bar{x}(\bigwedge \Gamma \to \phi)$, where both ϕ and all members of Γ are atomic. As usual theories are sets of sentences and closed terms are terms without variables.

Definition 1. A structure $I \models T$ is an *initial model* of a theory T if:
a) every element of (the domain of) I is a value t^I of some closed term t of $L(T)$;
b) for all atomic sentences α of $L(T)$, $\quad I \models \alpha$ implies $T \models \alpha$.

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One of the principal tools for building models of theories is a notion of *Hintikka set*. In order to make a paper more self contained we shall repeat its definition in the form to be used here.

Definition 2. A theory H is a *Hintikka set* if it satisfies the following conditions.

1. for all atomic sentences α at most one of $\alpha, \neg\alpha$ is in H;
2. if $\neg\neg\phi \in H$ then $\phi \in H$;
3. if $\bigwedge \Gamma \in H$ then $\Gamma \subseteq H$;
   if $\neg \bigvee \Gamma \in H$ then $\{\neg\phi \mid \phi \in \Gamma\} \subset H$;
4. if $\bigvee \Gamma \in H$ then $\phi \in H$ for some $\phi \in \Gamma$;
   if $\neg \bigwedge \Gamma \in H$ then $\neg\phi \in H$ for some $\phi \in \Gamma$;
5. if $\forall x\phi \in H$ then $\phi(t) \in H$ for all closed terms t;
   if $\neg\exists x\phi \in H$ then $\neg\phi(t) \in H$ for all closed terms t;
6. if $\exists x\phi \in H$ then $\phi(t) \in H$ for some closed term t;
   if $\neg\forall x\phi \in H$ then $\neg\phi(t) \in H$ for some closed term t.

We shall use the fact that every Hintikka set has a model (see [**1**, Theorem 2.3.3]) as well as the following.

Lemma *If $A \models T$ then T can be extended to a Hintikka set T^* in the expanded language $L_A(T)$ so that $(A, \bar{a}) \models T^*$.*

Sketch of proof. The construction is done in stages. Put $T_0 = T$ and at limit stages let $T_\sigma = \bigcup_{\tau<\sigma} T_\tau$. At successor stages we choose one of the sentences from T_σ and follow the definition. For instance if $\bigwedge \Gamma \in T_\sigma$, put $T_{\sigma+1} = T_\sigma \cup \Gamma$ and if $\forall x\phi \in T_\sigma$, put $T_{\sigma+1} = T_\sigma \cup \{\phi(t) \mid t$ is a closed term of $L_A(T).\}$ Next if $\bigvee \Gamma \in T_\sigma$, choose $\phi \in \Gamma$ such that $(A, \bar{a}) \models \phi$ and put $T_{\sigma+1} = T_\sigma \cup \{\phi\}$. Finally if $\exists x\phi \in T_\sigma$ choose an $a \in |A|$ such that $(A, \bar{a}) \models \phi(a)$ and put $T_{\sigma+1} = T_\sigma \cup \{\phi(a)\}$. Other cases are treated similarly. The process of saturation must end at some stage ρ so we put $T^* = \bigcup_{\sigma<\rho} T_\sigma$. Conditions 2.-6. of Definition 2 are obviously fulfilled. By induction we can prove that for all $\sigma < \rho$ $(A, \bar{a}) \models T_\sigma$, hence $(A, \bar{a}) \models T^*$ so that the condition 1. of the definition holds too, i.e., T^* is a Hintikka set.

As remarked in the introduction the main result to be proved here is a certain generalization of the one in stated in [**1**, Theorem 9.2.2], and that is achieved by abandoning equality and using the notion of initial model. The point is that in absence of equality the condition b) of Definition 1 becomes weaker than the one regularly used in definitions of free structures and involving homomorphisms.

Definition 3. A theory T *admits presentations* if for any expansion of the language $L(T)$ by a set of new constants and any set Δ of atomic sentences of the expanded language, the theory $T \cup \Delta$ has an initial model.

Theorem *If a theory T admits presentations then it is axiomatizable by a strict Horn clause theory.*

Proof. Let T^{HC} the set of all strict Horn clauses of $L(T)$ which are consequences of T so that $T \models T^{HC}$. We need to prove that $T^{HC} \models T$ as well. Given any $B \models T^{HC}$, let $\Delta(\bar{b})$ be a positive diagram of B i.e. the set of all atomic sentences of $L_B(T)$ that hold in $(B, \bar{b})$. Then by assumption the theory $T \cup \Delta(\bar{b})$ has an initial model I. Use Lemma 1 to get a Hintikka set $(T \cup \Delta(\bar{b}))^*$ extending $T \cup \Delta(\bar{b})$ and satisfied by I. Notice that, since I is initial, the construction can be done in $L_B(T)$, i.e. no further expansion of language is necessary. We shall prove that $(B, \bar{b}) \models (T \cup \Delta(\bar{b}))^*$. Let $\alpha(\bar{b})$ be any atomic sentence in $(T \cup \Delta(\bar{b}))^*$. Then $I \models \alpha(\bar{b})$ so $T \cup \Delta(\bar{b}) \models \alpha(\bar{b})$, hence $T \models \bigwedge \Delta(\bar{b}) \to \alpha(\bar{b})$. Since constants $\bar{b}$ do not belong to $L(T)$, we can infer $T \models \forall \bar{x}(\bigwedge \Delta(\bar{x}) \to \alpha(\bar{x}))$. From $B \models T^{HC}$ we get that $B \models \forall \bar{x}(\bigwedge \Delta(\bar{x}) \to \alpha(\bar{x}))$ and consequently $(B, \bar{b}) \models \bigwedge \Delta(\bar{b}) \to \alpha(\bar{b})$. By the definition of $\Delta(\bar{b})$, $(B, \bar{b}) \models \Delta(\bar{b})$ so $(B, \bar{b}) \models \alpha(\bar{b})$. Next if $\neg\alpha \in (T \cup \Delta(\bar{b}))^*$ with α atomic then certainly $\alpha \notin (T \cup \Delta(\bar{b}))^*$ so $\alpha \notin \Delta(\bar{b})$, hence $(B, \bar{b}) \not\models \alpha$ i.e., $(B, \bar{b}) \models \neg\alpha$. The rest is proved by induction on the complexity of sentences in $(T \cup \Delta(\bar{b}))^*$. For instance if $\forall x\phi \in (T \cup \Delta(\bar{b}))^*$ then for all terms t from $L_B(T)$, $\phi(t) \in (T \cup \Delta(\bar{b}))^*$ and all of them hold in $(B, \bar{b})$ by induction hypothesis. In particular $(B, \bar{b}) \models \phi(b)$ for all $b \in B$ so $(B, \bar{b}) \models \forall x\phi$. This suffices to conclude that $(B, \bar{b}) \models (T \cup \Delta(\bar{b}))^*$ and in particular $B \models T$.

In fact a stronger result follows by the same argument, namely that $(B, \bar{b}) \models Th(I)$, hence $I \equiv (B, \bar{b})$.

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