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EIGENVALUES AND WEIGHTS OF INDUCED SUBGRAPHS

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Communicated by Slobodan Simić

This paper is dedicated to the memory of S. Poljak

Abstract. We apply eigenvalue techniques for cut evaluation to produce relations between the weight and order of induced subgraphs, and apply these results to bound the stability number.

1. Introduction

S. Poljak suggested that an upper bound for the stability number $\alpha(G)$ of a graph G could be obtained in the following way. Add a new vertex ω to the graph G, and connect it to all vertices of G by an edge; this yields a graph G'; give the weight 1 to the edges of the graph and for each vertex x of G give the weight $1 - d_x$ to the edge ωx if d_x is the degree of x in G. If a subset S of the set of vertices of G is chosen, we evaluate the cut $\mathrm{c}(S)$, that is the sum of the weights of the edges between S and the remaining part of G'. Then the maximum $\mathrm{mc}(G') = \max_{S\subset G} \mathrm{c}(S)$ is $\alpha(G)$, because if we add a vertex y from $V(G) \setminus S$ to S, such that the induced degree of y in $S \cup \{y\}$ is δ, then the cut is incremented by $1 - 2\delta$. Hence the maximum cut is obtained when S is a stable set as large as possible, in other words this maximum cut is α.

Clearly, this method works also with weights $t - d$, with $0 < t < 2$ (instead of $1 - d$), which suggests another upper bound for α, namely $\min_{0<t<2} \mathrm{mc}(G')/t$. And this quantity receives an upper bound with eigenvalue techniques.

We develop that idea to obtain further results.

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2. Max-cut and optimized eigenvalues

We have a weighted graph G, with the weight w_{ij} on the edge ij. The cocycle of G defined by a subset S of its vertex set is the set W_S of edges having an endpoint in S and the other one out of S. The value of the cocycle is the sum of the weights of the edges in W_S. The max-cut of G is the maximum value of its cocycles. Using the laplacian matrix of G, that is the $n \times n$ matrix with entry $L_{ij} = -w_{ij}$ and $L_{ii} = \sum_{j \neq i} w_{ij}$, we observe that $4W_S = X_S^* L X_S$, where X_S is the column whose entries x_i are given by

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S \end{cases}.$$

Moreover, if D is a diagonal matrix with null trace, then $0 = X_S^* D X_S$; hence the max-cut of G is at most $n\lambda(L + D)/4$, with λ the highest eigenvalue of $L + D$. Of course one may choose D to make this bound as low as possible. Let us call $\phi(G) = n \min_D \lambda(L + D)/4$ this minimized upper bound.

If all weights are positive or null, the bound is quite good: Goemans and Williamson [**4**] show that the actual max-cut is at least 0.878 $\phi(G)$. But this does not work as well if there are positive and negative weights. For example, the max-cut for a triangle with weights 1, -1, -1 is 0 and the bound is $1/3 > 0$.

2.1. Relation with the spectrum of the adjacency matrix. The adjacency matrix A and the laplacian matrix L of a (weighted) graph satisfy $L+A = M$, where M is the diagonal matrix with the same diagonal entries as L; the sum of these entries is $2w$, where w is the sum of the weights of all edges. Let us compare the minimized highest eigenvalue $\lambda = \min_D \max(\mathrm{Sp}(L + D))$ and the maximized lowest eigenvalue $\kappa = \max_{D'} \min(\mathrm{Sp}(A + D'))$, with D and D' diagonal matrices having null trace. They satisfy

$$\lambda + \kappa = 2w/n \tag{1}$$

where n is the number of vertices. We prove this in a few lines: the matrix $A + L$ is a diagonal matrix, say Δ, with entries the sums of weights of edges incident to one vertex, the trace of Δ is $2w$; we see that the lowest eigenvalue of $A + D$ and the highest eigenvalue of $L - D - \Delta + (2w/n)I$ sum up to $2w/n$, because the sum of these two matrices is $(2w/n)I$. The mapping $D \mapsto (-D - \Delta + (2w/n)I)$ maps one-to-one and onto the set of diagonal matrices with null trace onto that set itself.

3. Other cocycles and optimized eigenvalues

Obviously, if all weights are ≥ 0, the minimum value of a cocycle is null: it suffices to take $S = \emptyset$, or $S = V$.

There are (at least) two ways of making the problem more interesting: we may remove assumptions on the signs of the weights of the edges — and thus we are led back to the problem of max-cut, with all weights replaced by their opposite — or we may prescribe the number k of vertices in S (or both).

We thus make no assumption on signs of weights, and impose $S = k$, with $2k \leq n$ without loss of generality. We have then two approaches.

Splitting $X_S = (2k - n)X_V/n + Y_S$, we obtain a column Y_S, with entries $2(n-k)/n$ on S and $-2k/n$ out of S; thus Y_S is orthogonal to X_V, with length given by $Y^2 = 4k(n-k)/n$. Then the value of the cocycle is at most $k(n-k)\lambda^*(L)/n$ and also at least $k(n-k)\mu^*(L)/n$, where $\lambda^*(L)$ and $\mu^*(L)$ are the highest and lowest eigenvalue in $\mathrm{Sp}^*(L)$, that is the spectrum of L with one occurrence of 0 removed.

We can also use two eigenvalues of L, like Donath and Hoffman [**3**]. The two columns X_S and X_V give by an orthogonal combination two orthogonal columns with entries 0 and $\sqrt{2}$, and squared lengths $2k$ and $2(n-k)$, namely $(X_S + X_V)/\sqrt{2}$ and $(X_V - X_S)/\sqrt{2}$, say Z_S and $Z_{V\setminus S}$. Then the 2×2 matrix $\begin{bmatrix} Z_S^* \\ Z_{V\setminus S}^* \end{bmatrix}(L+D)$ $\times\,[Z_S\ Z_{V\setminus S}]$ has the same trace as $\begin{bmatrix} X_V^* \\ X_S^* \end{bmatrix}(L+D)[X_V\ X_S]$, namely $4W_S$. Hence W_S is at most $(k\lambda'(L+D) + (n-k)\lambda(L+D))/2$, where $\lambda'(L+D) \leq \lambda(L+D)$ are the two highest eigenvalues of $L+D$. Similarly W_S is at least $(k\mu'(L+D) + (n-k)\mu(L+D))/2$, where $\mu(L+D) \leq \mu'(L+D)$ are the two lowest eigenvalues of $L+D$.

Of course, it is now useful to optimize these expressions, which is not too hard, since $k\lambda'(L+D) + (n-k)\lambda(L+D)$ is convex and $(k\mu'(L+D) + (n-k)\mu(L+D)$ is concave.

3.1. Application: stability number of regular graphs. For a regular graph of degree $d > 0$, we obtain bounds for the number k of vertices inducing a cocycle of weight kd. The sum of the weights of the set of edges inside such a set is null. Hence, if all weights are > 0, such a set is stable. The two ways of bounding the cut give two upper bounds for the order of a stable set:

$$\begin{aligned} \alpha &\leq n(\lambda - d)/\lambda \\ \alpha &\leq n\lambda/(2d + \lambda - \lambda') \end{aligned} \tag{2}$$

since the cocycle of a stable set with α elements in $d\alpha$.

3.2. Example. We consider Petersen graph, with all edges bearing value 1. Because Petersen graph is vertex-transitive, the optimisation of eigenvalues is realized with $D = 0$; the eigenvalues are then 0, 2 (with multiplicity 5) and 5 (with multiplicity 4), thus $\lambda = \lambda' = \lambda^* = 5$ and $\mu = 0$ and at last $\mu' = \mu^* = 2$. See [**1**] for details. The bounds and actual values of the cocycles appear in the following table.

k	$\frac{\lambda^* k(n-k)}{n}$	$\frac{\mu^* k(n-k)}{n}$	$\frac{\lambda(n-k)+\lambda' k}{2}$	$\frac{\mu(n-k)+\mu' k}{2}$	actual cuts
0	0	0	25	0	0
1	4.5	1.8	25	1	3
2	8	3.2	25	2	4, 6
3	10.5	4.2	25	3	5, 7, 9
4	12	4.8	25	4	6, 8, 10, 12
5	12.5	5	25	5	5, 7, 9, 11

We obtain $\alpha \leq 4$ (it is the actual value).

3.3. Example We consider the 4-cycle with weights 1 and -1 alternating. Thus it is a vertex-transitive weighted graph.

The eigenvalues of the Laplacian matrix, or the adjacency matrix as well, are then 0 (twice), 2, and -2. The bounds and actual values of the cocycles appear in the following table.

k	$\frac{\lambda^* k(n-k)}{n}$	$\frac{\mu^* k(n-k)}{n}$	$\frac{\lambda(n-k)+\lambda' k}{2}$	$\frac{\mu(n-k)+\mu' k}{2}$	actual cuts
0	0	0	4	-4	0
1	1.5	-1.5	3	-3	0
2	2	-2	2	-2	$-2, 0, 2$

3.4. Example. The complete bipartite graph $K_{2,6}$. The diagonal D has entries $-s$ and $3s$ on the two stable components. The eigenvalues are then $2-s$ (5 times), $3s+6$, and $4+s \pm 2\sqrt{(4+2s+s^2)}$.

k	$\frac{\lambda^* k(n-k)}{n}$	$\frac{\mu^* k(n-k)}{n}$	$\frac{\lambda(n-k)+\lambda' k}{2}$	$\frac{\mu(n-k)+\mu' k}{2}$	actual cuts
0	0	0	24	0	0
1	7	1.75	22.96	1.04	$2, 6$
2	12	3	21.80	2.20	$4, 6, 12$
3	15	3.75	20.49	3.51	$6, 10$
4	16	4	20	4	$6, 8$

4. Weights of induced subgraphs and eigenvalues

We have a weighted graph G on N vertices. We build a graph $G'(t)$ that is G with an extra vertex ω and we give the weights $t - d_x$ to the edges $x\omega$, where d_x is the degree (the sum of the weights of edges incident to x) in G, and we allow the variable t to take any real value.

Then each subset S of G induces a cocycle in G': the two parts are S and $V(G') \setminus S$; if S has n vertices and the sum of the weights of its induced edges is m, then the corresponding value of the cut in G' is $W_S(t) = nt - 2m$.

On the other hand, owing to the tools previously recalled we can bound $W_S(t)$ with expressions using the cardinality of S and eigenvalues involving the laplacian matrix $L(t)$ of G'.

This matrix $L(t)$ is symmetric and has entries

$$\begin{cases} L(t)_{xy} = -1 & \text{if } x \text{ and } y \text{ are adjacent vertices of } G \\ L(t)_{xy} = 0 & \text{if } x \text{ and } y \text{ are non-adjacent different vertices of } G \\ \left.\begin{array}{l} L(t)_{x\omega} = d_x - t \\ L(t)_{xx} = t \end{array}\right\} & \text{for each vertex } x \text{ in } G \\ L(t)_{\omega\omega} = Nt - 2M \end{cases} \tag{3}$$

where N is the number of vertices of G, and M the sum of weights of edges in G.

A common upper bound for all affine functions corresponding to cocycles in G' is

$$nt - 2m \leq (N+1)\lambda(L(t)+D)/4.$$

Thus an other common upper bound is

$$nt - 2m \leq (N+1)\lambda(t)/4,$$

where $\lambda(t)$ is obtained by minimizing $\lambda(L(t)+D)$ on the N-dimensional space of diagonal matrices D with null trace.

In the same vein, we have

$$nt - 2m \geq (N+1)\mu(L(t)+D)/4,$$

and

$$nt - 2m \geq (N+1)\mu(t)/4,$$

where $\mu(t)$ is obtained by maximizing the lowest eigenvalue $\mu(L(t)+D)$ of the matrix $L(t)+D$ on the N-dimensional space of diagonal matrices D with null trace.

4.1. Convexity properties. We recall that the highest eigenvalue of a symmetric matrix is a convex function of the matrix.

Thus the function $D \mapsto \lambda(t, D)$ is convex, and we can assume without loss of generality that vertices of G lying in the same orbit under the automorphism group of G give equal diagonal entries in D. This may significantly decrease the dimension of the space where minimization should be carried.

The function $t \mapsto \lambda(t)$ is convex.

Let us give a short proof. If t_1 and t_2 are given, and D_1 and D_2 are diagonals that minimize λ for these two values, then for a new value $t_3 = t_1 + \tau(t_2 - t_1)$, with $0 \leq \tau \leq 1$, we have $\lambda(t_3) \leq \ell \leq \lambda(t_1) + \tau(\lambda(t_2) - \lambda(t_1))$, where ℓ is the highest eigenvalue of the matrix $L(t_1)+D_1+\tau(L(t_2)+D_2-(L(t_1)+D_1))$; the first inequality comes from the optimization, and the second one from the convexity.

This could be seen also as a corollary of the following fact: the image by an affine application of a convex set is also convex.

In the same vein, it is easily proven that $\mu(t)$ is a concave function of t.

4.2. Application: stability number. The graphical representation of the two functions $t \mapsto (N+1)\mu(t)/4$ and $t \mapsto (N+1)\lambda(t)/4$ gives some indications about the average degree, number of vertices and weights of induced subgraphs of G. These subgraphs correspond to lines between the two curves, the slope is the number of vertices, the t-intercept is the average degree and the f-intercept is -2 times the weight.

Thus we obtain visual bounds for the number of vertices that induce a subgraph with a given weight w (from the slopes of the lines through the point $(t = 0, f = -2w)$ between the two curves), or a given average degree d (slopes of lines through point $(t = d, f = 0)$) and the weights of subgraphs with given order n (f-intercepts of lines with slope n).

In particular, for graphs with positive weights only, the stability number α is at most the maximum slope of lines through the origin.

4.3. Example. Figure 1 shows the lines that correspond to actual cuts of G' and the bounds from λ and μ for G a 5-cycle with all edges bearing weight 1, as well as the tangent indicating the bound $\sqrt{5}$ for the stability number

5. Partial optimisation for regular graphs

We consider a graph G, regular with degree d.

We may decide to give the same value a to the N entries of D corresponding to the N vertices of G, and $-Na$ to the entry that corresponds to ω. Thus we obtain a partial optimization for the highest and lowest eigenvalues, that provides already some information, and very simple calculations.

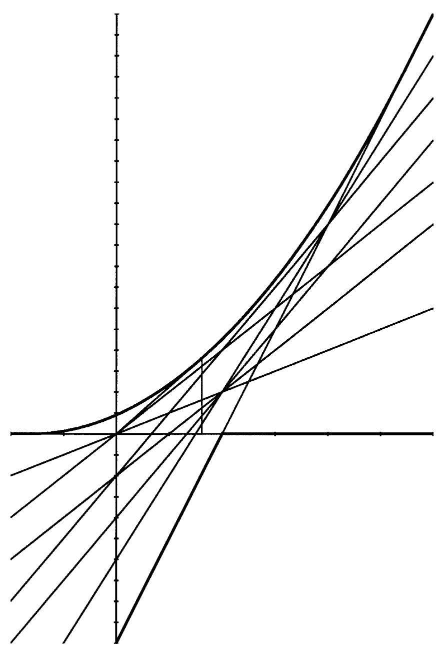

Figure 1: Cuts and bounds for C_5

Let $\mathrm{Sp}^*(H) = \{u_i,\, 1 \leq i \leq N-1\}$ be the spectrum of the laplacian matrix H of G with one occurrence of 0 removed. The highest value is thus $\lambda^*(H)$ and the lowest $\mu^*(H)$.

The eigenvalues of $L+D$ are then the $u_i + t - d + a$, and the two eigenvalues of $\begin{bmatrix} t+a-d & d-t \\ N(d-t) & -Na+Nt-Nd \end{bmatrix}$, in other words the roots of

$$X^2 - X((N+1)(t-d) - (N-1)a) - Na^2.$$

The (partially) minimized highest eigenvalue of $L+D$ is

$$\lambda = \begin{cases} 0 & \text{if } t \leq d-\ell, \text{ with } a = 0 \\ \frac{4N(t-d)}{N+1} & \text{if } t \geq d+\ell, \text{ with } a = \frac{2(N+1)(t-d)}{N-1} \\ \frac{N(\ell+t-d)^2}{(N+1)\ell} & \text{if } d-\ell \leq t \leq d+\ell, \text{ with } a = \lambda - \ell - t + d \end{cases} \tag{4}$$

where $\ell = \lambda^*(H)$ if it is ≥ 0, and it is

$$\lambda = \begin{cases} 0 & \text{if } t \leq d, \text{ with } a = 0 \\ \frac{4N(t-d)}{N+1} & \text{if } t \geq d, \text{ with } a = \frac{2(N+1)(t-d)}{N-1} \end{cases} \tag{5}$$

if $\lambda^*(H) \leq 0$.

Similarly the (partially) maximized lowest eigenvalue of $L+D$ is

$$\mu = \begin{cases} 0 & \text{if } t \geq d-p, \text{ with } a = 0 \\ \frac{4N(t-d)}{N+1} & \text{if } t \leq d-p, \text{ with } a = \frac{2(N+1)(t-d)}{N-1} \\ \frac{N(p+t-d)^2}{(N+1)p} & \text{if } d+p \leq t \leq d-p, \text{ with } a = \lambda - p - t + d \end{cases}$$

where $p = \mu^*(H)$ if it ≤ 0, and it is

$$\mu = \begin{cases} 0 & \text{if } t \geq d, \text{ with } a = 0 \\ \frac{4N(t-d)}{N+1} & \text{if } t \leq d, \text{ with } a = \frac{2(N+1)(t-d)}{N-1} \end{cases}$$

if $\mu^*(H) \geq 0$.

5.1. Regular graphs with positive weights. Consider now the case where all weights are > 0; then $\ell = \lambda^*(H) \geq 0$ and $\mu^*(H) \geq 0$. Hence the maximum cut is at most 0, or $N(t-d)$, or $N(\ell+t-d)^2/(4\ell)$, according to the position of t with respect to $d-\ell$ and $d+\ell$.

The second tangent from the origin to the parabola $(t, N(\ell{+}t{-}d)^2/(4\ell)), t \in \mathbb{R}$ touches it at the point $t = \ell - d$ and has slope $N(\ell-d)/\ell$.

Hence the stable sets have at most $N(\ell-d)/\ell$ vertices. This is the bound obtained previously.

On the other hand, the lower bound tells only that the average degree is at most $2M/N$, a rather dull result.

However, if the graph is not vertex transitive, one may improve the bounds.

5.2. Example. The disjoint union of a 3-cycle and a 4-cycle constitutes a regular graph of degree 2. The Laplacian eigenvalues are 0,2,3,4. The partial optimization gives the bound

$$nt - 2m \leq \begin{cases} 0 & \text{if } t \leq -2 \\ 7t - 14 & \text{if } t \geq 6 \\ \frac{7(t+2)^2}{16} & \text{if } -2 \leq t \leq 6. \end{cases}$$

It gives only $\alpha \leq 7/2$.

The optimization uses the diagonal matrix with a for the vertices of the 3-cycle, b for those of the 4-cycle and $-3a-4b$ for the extra vertex ω. The spectrum of $L+D$ is made from $b+t$, $b+t+2$ (twice), $a+t+1$ (twice) and the eigenvalues of $\begin{bmatrix} a+t-2 & 0 & 2-t \\ 0 & b+t-2 & 2-t \\ 3(2-t) & 4(2-t) & -3a-4b+7t-14 \end{bmatrix}$.

Its optimized maximum, multiplied by $2 = (7+1)/4$ to fit with cuts is given in the table

t		-2		-1		5		6	
bound	0	0	$\frac{(t+2)^2}{4}$	$\frac{1}{4}$	$\frac{2t^2+6t+5}{4}$	$\frac{85}{4}$	$\frac{t^2+16t-20}{4}$	28	$7t-14$

that can be obtained with the remark that the graph is the union of two graphs with only ω as common vertex, and the theorem 4 of [**2**]. The maximum slope of a line through the origin inside the allowed region is now $(3+\sqrt{10})/2 = 3.081\ldots$, that is closer to the true value 3.

6. More about stability number

For the stability number, counting edges is irrelevant, we just want to find a maximum number of vertices inducing a null weight. Therefore, we can assign arbitrary weights to edges and minimize as far as possible the slopes of the lines inside the allowed region going through the origin. This fiddling with the weights of edges was already practiced by Lovász [**5**].

Since the highest eigenvalue is a convex function with respect to the entries, it is not necessary to give different weights to edges in a same orbit from the automorphism group of the (non weighted) graph.

6.1. Example. Let us use again the disjoint union of C_3 and C_4 as an example. Giving the weight 2 to the edges of the C_3 and 1 to the edges of C_4 gives the bound for the maxcut of G'.

t		-2		6		10	
bound	0	0	$\frac{3(t+2)^2}{8}$	24	$\frac{t^2+36t-60}{8}$	50	$7t-20$

Hence the line with slope 3 from the origin is tangent to the curve at the point $t = 2, f = 6$. This gives the actual value of the stability number.

6.2. Remark. Let us call $\alpha'(G)$ the upper bound of the stability number $\alpha(G)$ of a graph G obtained above.

It is easy to check that the function $f(G,t) = (N+1)\lambda(t)/4$ satisfies $uf(G,t) = f(G_u, tu)$, where G_u is obtained from G by multiplying all weights by u. Therefore, the disjoint union $G + H$ of two graphs G and H the following equality holds:

$$\alpha'(G+H) = \alpha'(G) + \alpha'(H) \tag{6}$$

since it is possible to adjust the weights in such a way that the contacts of the tangents from the origin to the curves $f(G,t)$ and $f(H,t)$ have the same positive coordinate t. Note that a set of a isolated vertices causes no problem, since the curve is then included into the lines $f = 0$ and $f = at$. This property implies

$$\alpha'(G) \leq \theta(G) \tag{7}$$

where $\theta(G)$ is the minimal number of complete subgraphs that covers all the vertices of G.

I do not know whether the bound α' coincides with the bound ϑ described by Lovász [**5**, Theorem 9], although it obviously does for vertex-transitive graphs and graphs satisfying $\theta = \alpha$.

7. Subgraphs again

It is possible to extend the techniques of section 3.

In the case of regular graphs, we observe that the partial optimisation gives the eigenvalues between $\lambda^*(G) + t - d + a$ and $\mu^*(G) + t - d + a$; the corresponding eigenvectors are copied from those of G with a null component for ω. Besides, we have the two eigenvalues roots of $X^2 - X((N+1)(t-d) - (N-1)a) - Na^2$, that are associated to the vector space T of vectors where all components are equal, with the possible exception of the one of ω.

Then one splits the vector X_S that corresponds to a set S of k vertices in G on Y_S of squared length $4k(N-k)/N$, orthogonal to T (i.e. the component relative to ω is null and the sum of components is null), and a vector in T, whose components are -1 on ω and $(2k-N)/N$ elsewhere. The cut is thus between $A + 4k(N-k)(\lambda^*(G) + t - d + a)/N$ and $A + k(N-k)(\mu^*(G) + t - d + a)/N$, with $A = -4ak(N-k)/N + (t-d)4k$. Some terms cancel, and we obtain $kt - 2m$ is between $(t-d)k + k(N-k)(\lambda^*(G))$ and $(t-d)k + k(N-k)(\mu^*(G))$; in other words, $2m$ is between $dk - k(N-k)(\lambda^*(G))$ and $dk - k(N-k)(\mu^*(G))$. This is not unexpected, since that result is also obtained by relating the weights of edges inside S and the weight of the cocycle in G defined by S and the degree d of the k vertices of S.

Of course, it is possible to apply both methods to graphs that are not regular, thus bounds on the cuts between subsets of orders k and $N+1-k$ in G' provide bounds on the weights of subgraphs of G induced by k or $N+1-k$ vertices.

7.1. Example. For the graph $K_{1,2}$, this method gives the following bounds: the highest eigenvalue of $L(t,G)$ is $\lambda^* = (5t-4+\sqrt{(9t^2-40t+48)})/2$, that is close to $4t-16/3$ if t is large. The lowest one is t if $t \geq 2$, and $(5t-4-\sqrt{(9t^2-40t+48)})/2$ otherwise. Figure 2 allows the comparison of the bounds for $k=2$ (the eigenvalues above) and $k=1$ or 3 (3/4 of the eigenvalues above).

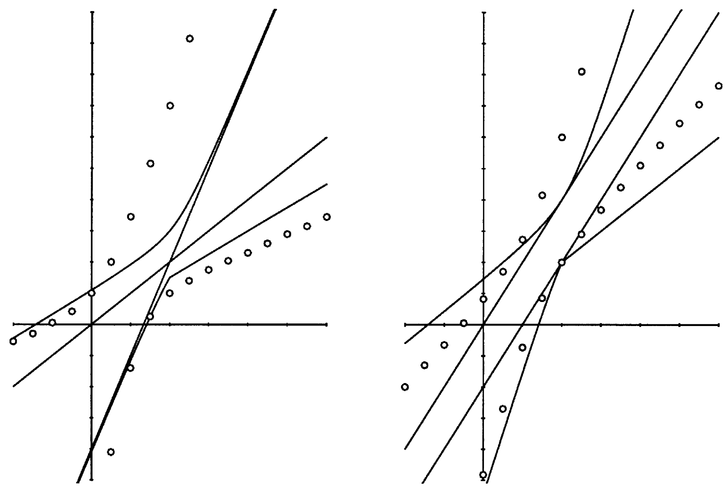

Figure 2: Cuts and bounds for $K_{3,3}$

The dots show some results for the same graph, with the method of Donath and Hoffman, that is use two highest or two lowest eigenvalues with coefficients 1 and 1 (for $k=2$) or 3/2 and 1/2 (for $k=1$ or 3).

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