PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 65 (79), 1999, 31–51

CONVOLUTIONS OF FOURIER COEFFICIENTS OF CUSP FORMS

M. Jutila

Communicated by Aleksandar Ivić

Abstract. Analogues of classical binary additive divisor problems for Fourier coefficients of (holomorphic or non-holomorphic) cusp forms are discussed in a new way by a variant of the circle method. The results are either new or coincide with earlier ones.

1. Introduction

Our aim in this paper is to develop a new approach to analogues of binary additive divisor problems for Fourier coefficients of cusp forms. We outlined the underlying argument - a version of the circle method - in our recent paper [J4], and the present application may serve as a test of its scope.

Recall that the binary additive divisor problems are concerned with sums of the type

$$\sum_{n=1}^{N} d(n)d(n+f) \qquad (f \geq 1), \tag{1.1}$$

$$\sum_{n=1}^{N-1} d(n)d(N-n), \tag{1.2}$$

where $d(n)$ is the usual divisor function. The classical approach to these sums was via Kloosterman's refinement of the circle method leading ultimately to Kloosterman sums, and if these are estimated by Weil's bound (best possible for individual sums), then the error terms in the respective asymptotic formulae are $O(N^{5/6+\varepsilon})$ and $O(N^{3/4+\varepsilon})$ (see [M1] for a discussion of the history of these problems).

AMS Subject Classification (1991): Primary 11F30, 11N37, 11P55.
Keywords: cusp forms, circle method, spectral theory.

The significantly improved error term $O(N^{2/3+\varepsilon})$ for the sum (1.1) was obtained by Deshouillers and Iwaniec [DI2] who introduced spectral methods into this problematics. The main novelty was Kuznetsov's [K1] trace formula allowing nontrivial summation of Kloosterman sums over the "denominators".

A unified spectral theoretic treatment of both sums (1.1) and (1.2) was indicated by Kuznetsov [K2]. A detailed and penetrating study of these topics with a number of concrete results was recently carried out by Motohashi [M1].

The Fourier coefficients, say $b(n)$, of a (holomorphic or non-holomorphic) cusp form behave somewhat similarly to the divisor function, up to the positivity, so it is natural to expect that estimates for the sums

$$\sum_{n=1}^{N} b(n)b(n+f) \qquad (f \geq 1), \tag{1.3}$$

$$\sum_{n=1}^{N-1} b(n)b(N-n) \tag{1.4}$$

should be comparable with the *error terms* for the corresponding sums (1.1) and (1.2). Another aspect of this analogy is methodical unification of the treatment of all the above sums. These are, all in all, of six types, for (1.3) and (1.4) are subdivided into two cases according to the holomorphicity or non-holomorphicity of the form in question.

The sums with a "shift", that is (1.1) and (1.3), are sum functions of certain arithmetic functions, so it is immediate to invoke the generating Dirichlet series method. A tool for the study of the sum (1.3) could thus be the Dirichlet series

$$\sum_{n=1}^{\infty} b(n)b(n+f)(n+f)^{-s},$$

and likewise for the sum (1.1), though the above mentioned works of Deshouillers-Iwaniec, Kuznetsov and Motohashi on the latter sum followed different lines of argument. The sum (1.3) in the holomorphic case was treated in this way by Good [G1-2] with a result comparable with that of Deshouillers and Iwaniec for the sum (1.1). The last mentioned sum is also amenable to the Dirichlet series method, as shown by Tahtadjan and Vinogradov [TV]. Recently, in [J2-3], we extended Good's theory to non-holomorphic cusp forms, so after all a full analogy between the three types of sums included in (1.1) and (1.3) is now established as to the results and methods.

Consider next the "dual" sums (1.2) and (1.4). First note that the latter represents in the holomorphic case simply the Nth Fourier coefficient of the square of our cusp form, so Deligne's estimate is applicable since the square of a holomorphic cusp form is again a cusp form (with the weight doubled). On the other hand, the same sum for non-holomorphic cusp forms is more problematic, and no nontrivial estimate seems to be known at least in literature (though the classical method might work even in this context). Our main object is to fill this gap on showing that our variant of the circle method, in combination with Kuznetsov's trace formula, yields the expected estimate for the sum (1.4). In other words, the estimate will be of the same order as Motohashi's [M1] error term for the sum (1.2). The same method applies to the sums (1.3) as well, and the results coincide with those in [J2-3]. It is of methodical interest that the new approach dispenses with mean value estimates for inner products of the square of our (fixed) cusp form agains "variable" Maass forms; such estimates (see [G1], [J2-3]) played a crucial role in the previous work on such sums. Unfortunately our method does not apply to the classical sums (1.1) and (1.2) (the positivity of $d(n)$ is an obstacle), so a perfect methodical unification still remains unattained. However, as far as the *results* are concerned, the correspondence between binary additive divisor problems and their cusp form analogues appears to be now quite satisfactory.

We are going to confine ourselves to the sums (1.3) and (1.4) for non-holomorphic cusp forms. The latter sum will be discussed in more detail, and after that the necessary modifications for dealing with the former sum are outlined more briefly.

Turning to the formulation of our results, we first recall some basic properties of non-holomorphic cusp forms, or Maass (wave) forms, for the full modular group Γ. Such a form $u(z) = u(x+yi)$ is a Γ-invariant function in the upper half-plane $y > 0$ which is an eigenfunction (for the eigenvalue $1/4 + \kappa^2$, say) of the hyperbolic Laplacian $-y^2(\partial_x^2 + \partial_y^2)$ and square integrable with respect to the invariant measure $dx\,dy/y^2$ over a fundamental domain of Γ. Moreover, we may suppose that u is an eigenfunction of all Hecke operators T_n with respective eigenvalues $t(n)$, and that it is an even or odd function of x. Then $t(n)$ is a multiplicative function. Our form is represented by its Fourier series

$$u(x+yi) = \sqrt{y} \sum_{n \neq 0} \rho(n) K_{i\kappa}(2\pi |n| y) e(nx), \tag{1.5}$$

where $K_{i\kappa}(\cdots)$ stands for a modified Bessel function in the standard notation. Here the coefficients satisfy

$$\rho(n) = \rho(1) t(n), \quad \rho(-n) = \pm \rho(n) \qquad (n \geq 1), \tag{1.6}$$

where $\pm$ is the parity sign of the form (plus for even, minus for odd). Let u_j $(j = 1, 2, \dots)$ be an orthonormal system of Maass forms with respect to the Petersson inner product. The indexing of these forms together with the related coefficents $\rho_j(n)$ and $t_j(n)$ corresponds to the indexing of the respective parameters κ_j arranged into a non-decreasing sequence.

As to the order of the coefficients $t(n)$, let θ be a constant such that

$$t(n) \ll n^{\theta + \varepsilon} \text{ for all } n \geq 1, \tag{1.7}$$

where the implied constant depends only on $\varepsilon > 0$ (but not on the form). It is known [BDHI] that $\theta \leq 5/28$, and it is a famous conjecture that $\theta = 0$ is admissible.

The sums (1.3) and (1.4) for Maass forms now amount to

$$T(N;f)=\sum_{n=1}^{N}t(n)t(n+f), \tag{1.8}$$

$$T(N)=\sum_{n=1}^{N-1}t(n)t(N-n), \tag{1.9}$$

respectively. More generally, we are going to investigate sums

$$T_g(N;f)=\sum_{n=1}^{N}t(n)t(n+f)g\Big(\frac{n}{N}\Big), \tag{1.10}$$

$$T_g(N)=\sum_{n=1}^{N-1}t(n)t(N-n)g\Big(\frac{n}{N}\Big) \tag{1.11}$$

involving a smooth weight function g. We suppose that g is a C^∞ function in the interval (0,1) such that

$$|g^{(\nu)}(x)|\le c(\nu)(\min(x,1-x))^{-\nu},\quad \nu=0,1,\dots \tag{1.12}$$

for certain positive numbers $c(\nu)$.

Theorem. *For any fixed $\varepsilon>0$, we have*

$$T_g(N;f)\ll N^{2/3+\varepsilon}\qquad (1\le f\ll N^{2/3}), \tag{1.13}$$

$$T_g(N)\ll N^{1/2+\theta+\varepsilon}, \tag{1.14}$$

where the implied constants depend only on ε and $c(\nu)$ for $\nu\le\nu(\varepsilon)$.

Corollary. *We have*

$$T(N;f)\ll N^{2/3+\varepsilon}\qquad (1\le f\ll N^{2/3}) \tag{1.15}$$

$$T(N)\ll N^{1/2+\theta+\varepsilon}. \tag{1.16}$$

Remark 1. The estimate (1.15) is contained in the corollary of Theorem 3 in [J2]. The right hand side in (1.14) and (1.16) is of the same order as the error term in Motohashi's [M1] asymptotic formula for the sum (1.2).

Remark 2. The argument of the proof of our theorem applies to holomorphic cusp forms as well with obvious modifications. Let $a(n)$ run over the Fourier coefficients of a holomorphic cusp form of weight k, and let $\tilde{a}(n)=a(n)n^{-(k-1)/2}$ be the "normalized" coefficients (comparable with $d(n)$). Then the estimates (1.13)–(1.16) remains true if the coefficients $t(n)$ are replaced by $\tilde{a}(n)$. Returning to original coefficients, let $A(N)$ stand for a sum similar to $T(N)$ with $t(n)$ replaced by $a(n)$ in (1.9). Then, as an analogue of (1.16), we have

$$A(N) \ll N^{k-1/2+\theta+\varepsilon}.$$

However, since the sums $A(N)$ are actually Fourier coefficients of a holomorphic cusp forms of weight $2k$, Deligne's theorem implies the above estimate with $\theta = 0$! This means that as to the order of $A(N)$, the hypothesis $\theta = 0$ is of the same consequence as Deligne's theorem. Nevertheless, though our bound for $A(N)$ is thus weaker than what is known otherwise, the analogue of (1.14) for holomorphic cusp forms seems to be new.

The scheme of the proof of the theorem is as follows. In Sec. 2, we apply our variant of the circle method to the sum $T_g(N)$, and single out for that a certain explicit expression involving exponential sums. These are transformed in Sec. 3 by use of the Γ-invariance of Maass forms. At this stage, Kloosterman sums emerge. Next, in Sec. 4, Kuznetsov's trace formula is applied to sums of Kloosterman sums, and the resulting expression is estimated by Iwaniec's spectral large sieve to complete the proof of (1.14). Finally, in Sec. 6, the preceding argument is adapted to the sum $T_g(N; f)$.

Notation. The parameter κ will be fixed throughout, so the constants implied by the symbol $\ll$ and $O(\cdots)$ will depend possibly on κ, and also on ε, a small positive number, whenever it occurs in an inequality. The meaning of ε is not necessarily the same at each occurrence. Moreover, constants may occasionally depend on finitely many first constants $c(\nu)$ introduced in (1.12); the number of the relevant indices ν may depend on ε. The possible dependence of implied constants on other parameters is indicated by subscripts, say $\ll_\nu$. We write $A \asymp B$ to mean that A and B are positive numbers of the same order of magnitude, thus $A \ll B \ll A$.

We are going to encounter frequently "smooth" functions whose actual definition is irrelevant. Therefore, following [I1], we adopt a common notation $v(x_1, \ldots, x_r)$ for such functions. Here the variables x_i lie respectively in intervals of length $\asymp X_i$, and we suppose that for any r-tuple $(\nu_1, \ldots, \nu_r)$ of nonnegative integers and any $\varepsilon > 0$ it holds

$$\frac{\partial^{\nu_1+\cdots+\nu_r} v}{\partial x_1^{\nu_1} \cdots \partial x_r^{\nu_r}} \ll X_1^{-\nu_1} \cdots X_r^{-\nu_r} N^{\varepsilon}, \tag{1.17}$$

the implied constant depending on the r-tuple and ε.

2. The circle method with overlapping intervals

The basic idea of this method, explained in [J4], is approximating the characteristic function of the unit interval by a linear combination of characteristic functions of neighbourhoods related to a system of rationals (in their lowest terms)

$a/q \in [0,1]$ with q of a given order. These neighbourhoods overlap with a variable multiplicity, which turns out to be mostly not far from its expectation; this is a consequence of a certain well-distribution property of rational numbers [J1]. There is no distinction between "major arcs" and "minor arcs" and no "levelling problem", because the denominators are all of the same order and the subintervals are chosen to be of the same length. The following lemma (corollary to Lemma 1 in [J4]) estimates the approximation error in the mean square.

Lemma 1. *Let $\chi_{a/q}(x)$ denote the characteristic function of the interval $[a/q-\delta, a/q+\delta]$, let $w(q) \in [0,1]$ for $q \in [Q,2Q]$, write $\lambda = 2\delta L$ with*

$$L = \sum_{Q \le q \le 2Q} w(q)\varphi(q),$$

and define

$$\tilde{\chi}(\alpha) = \lambda^{-1} \sum_{Q \le q \le 2Q} w(q) \sum_{a=1}^{q}{}^{*} \chi_{a/q}(\alpha),$$

where the asterisk indicates the coprimality condition $(a,q)=1$. Let $\chi(\alpha)$ be the characteristic function of the interval $[0,1]$. Then, if $L \gg Q^2$ and $Q^{-2} \ll \delta \ll Q^{-1}$, we have

$$\int (\chi(\alpha) - \tilde{\chi}(\alpha))^2 \, d\alpha \ll (\delta Q^2)^{-1} Q^{\varepsilon}.$$

In the sequel, frequent use will be made of the following estimates involving the Hecke eigenvalues $t(n)$ for a form related to the parameter κ:

$$\sum_{x \le n \le x+y} t^2(n) \ll_{\varepsilon} (y + x^{3/5}) x^{\varepsilon} \qquad (0 < y \le x), \tag{2.1}$$

$$\sum_{n \le x} t^2(n) \ll_{\varepsilon} x\kappa^{\varepsilon}, \tag{2.2}$$

$$\sum_{n \le x} t(n) e(n\alpha) \ll \kappa x^{1/2} \log(2x). \tag{2.3}$$

The short-interval result (2.1) is a consequence of an asymptotic sum formula of the Rankin type, the spectrally uniform estimate (2.2) is due to Iwaniec [I3], and (2.3) (see [I4, Theorem 8.1]) is an analogue of the corresponding classical estimate of Wilton for holomorphic cusp forms.

Consider now the sum $T_g(N)$ defined in (1.11). To begin with, we split it up into subsums, in each of which n is restricted to an interval of the form $[M, 2M]$. It is easy to see that there are $\ll \log N$ nonnegative C^{∞} functions φ_i satisfying the smoothness condition (1.12), having support in an interval $[\alpha_i, 2\alpha_i]$ with $0 < \alpha_i \le 1/3$, and giving a "decomposition of unity" in an interval relevant for the sum $T_g(N)$:

$$\sum_i (\varphi_i(x) + \varphi_i(1-x)) = 1 \text{ for } 1/N \le x \le 1 - 1/N.$$

The functions φ_i may be chosen in such a way that the graphs of φ_i and φ_{i+1} differ only by the scale in the direction of the x-axis. Therefore we may suppose that (1.12) for φ_i holds uniformly in i for any given ν. Consequently, the sum $T_g(N)$ can be decomposed into a sum of $\ll \log N$ analogous sums with $g(x)$ replaced by the smooth function $g(x)\varphi_i(x)$ or $g(x)\varphi_i(1-x)$, and the constants $c(\nu)$ in (1.12) for these functions are independent of i. By symmetry, it suffices to consider the sum related to the first mentioned function, which we denote again by g for simplicity. Then the function $g(x/N)$ vanishes outside a certain interval $[M, 2M]$ with $M \le N/3$.

We may suppose that

$$N^{1/2+\varepsilon} \le M \le N/3. \tag{2.4}$$

For if $M < N^{1/2+\varepsilon}$, then

$$T_g(N) \ll N^{1/2+\theta+\varepsilon}$$

by (1.7), Cauchy's inequality, and (2.2).

For the sake of formal symmetry, we prefer to consider sums of the shape

$$T = T_{g_1,g_2}(N) = \sum_{n=1}^{N-1} t(n)t(N-n)g_1\left(\frac{n}{N}\right) g_2\left(\frac{N-n}{N}\right)$$

in place of $T_g(N)$. If the functions g_i satisfy the smoothness condition (1.12), this sum is just a special case of $T_g(N)$. Suppose that the supports of the functions g_1 and g_2 lie, respectively, in the intervals $[M/N, 2M/N]$ and $[1-(5/2)(M/N), 1-(1/2)(M/N)]$ with M as in (2.4). Then, if we specify $g_1 = g$ and choose g_2 so that $g_2((N-x)/N) = 1$ for $M \le x \le 2M$, then the above sum T amounts to $T_g(N)$.

Define now the exponential sums

$$S_j(\alpha) = \sum_n t(n)g_j(n/N)e(n\alpha), \quad j = 1, 2. \tag{2.5}$$

Then

$$T = \int_0^1 S_1(\alpha)S_2(\alpha)e(-N\alpha)\,d\alpha.$$

By periodicity, we may replace α here by $\alpha + \mu$ for any real μ, so

$$T = \int_0^1 S_1(\alpha+\mu)S_2(\alpha+\mu)e(-N(\alpha+\mu))\,d\alpha; \tag{2.6}$$

this shifting device will be motivated in the next section.

Let $\chi(\alpha)$ and $\tilde{\chi}(\alpha)$ be as in Lemma 1 with

$$Q = MN^{-\varepsilon}, \qquad \delta = 1/Q. \tag{2.7}$$

Further, let $w(x)$ be a smooth function of support in $[Q, 2Q]$ such that $0 \le w(x) \le 1$ and

$$\sum_q w(q)\varphi(q) \gg Q^2.$$

Then, by Lemma 1, we have

$$\int (\chi(\alpha) - \tilde{\chi}(\alpha))^2 \, d\alpha \ll Q^{-1+\varepsilon}. \tag{2.8}$$

The integral (2.6) is now decomposed as follows:

$$\begin{aligned} T = &\int \tilde{\chi}(\alpha) S_1(\alpha+\mu) S_2(\alpha+\mu) e(-N(\alpha+\mu)) \, d\alpha \\ &+ \int (\chi(\alpha) - \tilde{\chi}(\alpha)) S_1(\alpha+\mu) S_2(\alpha+\mu) e(-N(\alpha+\mu)) \, d\alpha = T_1 + T_2. \end{aligned} \tag{2.9}$$

To estimate the "error term" T_2, note that $S_1(\alpha) \ll M^{1/2+\varepsilon}$ by (2.3) and partial summation, and similarly $S_2(\alpha) \ll N^{1/2+\varepsilon}$. Thus, by Cauchy's inequality, (2.8), and (2.7), we obtain

$$T_2 \ll N^{1/2+\varepsilon}. \tag{2.10}$$

This is of admissible order, so it remains to estimate the term T_1 in (2.9).

By definition,

$$T_1 = \lambda^{-1} \sum_q w(q) \sideset{}{^*}\sum_{a=1}^{q} \int_{\mu-\delta}^{\mu+\delta} S_1\left(\frac{a}{q}+\eta\right) S_2\left(\frac{a}{q}+\eta\right) e\left(-N\left(\frac{a}{q}+\eta\right)\right) \, d\eta,$$

where $\lambda \asymp Q$. Therefore

$$T_1 \ll M^{-2+\varepsilon} \max_{|\eta-\mu| \le \delta} |\tau(\eta)|, \tag{2.11}$$

where

$$\tau(\eta) = \sum_q w(q) \sideset{}{^*}\sum_{a=1}^{q} S_1\left(\frac{a}{q}+\eta\right) S_2\left(\frac{a}{q}+\eta\right) e\left(-\frac{Na}{q}\right). \tag{2.12}$$

Finally, we specify

$$\mu = 3/Q = 3N^{\varepsilon}/M. \tag{2.13}$$

The rest of the proof will consist in the estimation of $\tau(\eta)$. The variable η will be fixed during the following discussion with the understanding that all the estimations will be uniform in η.

3. Transformation of exponential sums

As the first step in our analysis of the sum $\tau(\eta)$ in (2.12), we are going to transform the exponential sums $S_j(\alpha)$ by use of the automorphy of the form u. Let $(a,q)=1$, $a\bar{a}\equiv 1 \pmod q$, and apply the Möbius transform with the matrix $\begin{pmatrix} \bar{a} & (1-a\bar{a})/q \\ -q & a \end{pmatrix}$; then

$$u\left(\frac{a}{q}+\xi\right)=u\left(-\frac{\bar{a}}{q}-\frac{1}{q^2\xi}\right) \text{ for } \operatorname{Im}\xi>0. \tag{3.1}$$

The sums S_j can be written in terms of $u(z)$ by a simple Fourier analysis of the Fourier series (1.5). Recall some properties of the K-Bessel functions occurring in that series. By definition,

$$K_{i\kappa}(x)=\frac{\pi}{2}\frac{I_{-i\kappa}(x)-I_{i\kappa}(x)}{i\sinh\pi\kappa} \text{ for } x>0$$

with

$$I_\nu(x)=\sum_{n=0}^{\infty}\frac{(x/2)^{2n+\nu}}{n!\Gamma(n+1+\nu)}.$$

This is an appropriate representation if x is bounded, and otherwise a suitable integral representation, say (see [L, Eq. (5.10.23)])

$$K_{i\kappa}(x)=\int_0^\infty e^{-x\cosh t}\cos(\kappa t)\,dt$$

shows that $K_{i\kappa}(x)$ decays exponentially as x increases; in fact, $K_{i\kappa}(x)\sim\left(\frac{\pi}{2x}\right)^{1/2}e^{-x}$ as $x\to\infty$.

Let now $y_1=b/M$, $y_2=b/N$, where b is a sufficiently large constant (depending on κ). Then $K_{i\kappa}(2\pi n y_j)$ is bounded away from zero if n lies in the support of $g_j(x/N)$. Hence the Fourier transforms

$$B_j(\beta)=\int g_j(\lambda/N)K_{i\kappa}(2\pi\lambda y_j)^{-1}e(-\beta\lambda)\,d\lambda \quad (j=1,2) \tag{3.2}$$

are smooth functions, and

$$S_j(\alpha)=y_j^{-1/2}\rho(1)^{-1}\int_{-\infty}^{\infty}u(\alpha+\beta+y_j i)B_j(\beta)\,d\beta \tag{3.3}$$

by (1.5), (1.6), (2.5), and the Fourier inversion.

Repeated integration by parts in (3.2) shows that

$$B_j(\beta)\ll_\nu M^{1-\nu}|\beta|^{-\nu}, \quad \nu=0,1,\dots.$$

Hence, letting $v(\beta)$ be a suitable smooth bounded weight function of support in the interval $[-\beta_0, \beta_0]$ with

$$\beta_0 = 1/Q = M^{-1}N^{\varepsilon}, \tag{3.4}$$

we may truncate the integral (3.3) as follows:

$$S_j(\alpha) = y_j^{-1/2}\rho(1)^{-1}\int u(\alpha+\beta+y_j i)v(\beta)B_j(\beta)\,d\beta + O_A(N^{-A}) \tag{3.5}$$

for any fixed $A > 0$. The main term on the right is substituted into (2.12), and the error term can be ignored.

The Maass form in (3.5) for $\alpha = a/q + \eta$ is next transformed by (3.1) and (1.5). Write $\zeta = \eta + \beta$ noting that $\zeta \asymp Q^{-1}$ for $|\beta| \le \beta_0$ (we specified the parameter μ as in (2.13) in order to stabilize the order of ζ). Then

$$\begin{aligned} u\left(\frac{a}{q} + \zeta + y_j i\right) &= u\left(\frac{\bar a}{q} - \frac{1}{q^2(\zeta + y_j i)}\right) \\ &= \frac{y_j^{1/2}}{q(\zeta^2 + y_j^2)^{1/2}} \sum_{n \ne 0} \rho(n) K_{i\kappa}\left(\frac{2\pi |n| y_j}{q^2(\zeta^2 + y_j^2)}\right) e\left(-\frac{n\bar a}{q}\right) e\left(-\frac{n\zeta}{q^2(\zeta^2 + y_j^2)}\right). \end{aligned}$$

The last oscillating factor here is approximately $e(-nq^{-2}\zeta^{-1})$, so we write

$$u\left(\frac{a}{q} + \zeta + y_j i\right) = y_j^{1/2} \sum_{n \ne 0} \rho(n) e\left(-\frac{n\bar a}{q}\right) e(-nq^{-2}\zeta^{-1})\varphi_j(n, q, \zeta), \tag{3.6}$$

where

$$\varphi_j(n, q, \zeta) = q^{-1}(\zeta^2 + y_j^2)^{-1/2} K_{i\kappa}\left(\frac{2\pi |n| y_j}{q^2(\zeta^2 + y_j^2)}\right) e\left(\frac{n y_j^2}{q^2(\zeta^2 + y_j^2)\zeta}\right). \tag{3.7}$$

Equipped with a smooth weight function of n with support in the range $n \asymp y_j^{-1}$, this becomes a smooth function of the type $v(n, q, \zeta)$ in the sense of (1.17). We show next that such a truncation of the n-sum is indeed admissible.

Substitute (3.6) into (3.5), where $\alpha = a/q + \eta$ and $B_j(\beta)$ is written according to its definition (3.2). Then the integral over β amounts to the integrals

$$\int_{\eta-\beta_0}^{\eta+\beta_0} v(\zeta - \eta)\varphi(n, q, \zeta) e(-\zeta\lambda - nq^{-2}\zeta^{-1})\,d\zeta \tag{3.8}$$

for different values of n. This exponential integral may have a saddle point only if $n \asymp \lambda$, that is for $n \asymp M$ if $j = 1$, and for $n \asymp N$ if $j = 2$. These conditions mean that $n \asymp y_j^{-1}$. On the other hand, if $n > cy_j^{-1}$ or $n < (cy_j)^{-1}$ for a sufficiently large positive constant c, then there is no saddle point, and repeated integration by parts shows that the integral (3.8) is very small. Moreover, the function $\varphi_j(n,q,\zeta)$ decays exponentially as n exceeds y_j^{-1}. Thus we are left with a sum over the critical range $n \asymp y_j^{-1}$, which may be equipped by a smooth weight.

To summarize, we may write (3.5) as follows:

$$S_j\left(\frac{a}{q}+\eta\right) = \int \sum_{n\asymp 1/y_j} t(n)e\left(-\frac{n\bar{a}}{q}\right)\int_{\eta-\beta_0}^{\eta+\beta_0} v(n,q,\zeta,\lambda)e(-\zeta\lambda - nq^{-2}\zeta^{-1})\,d\zeta\,d\lambda \\ + O_A(N^{-A}),$$

where the range for λ is the support of the function $g_j(\lambda/N)$; thus it runs over an interval of length $\asymp M$ in a neighbourhood of M or N.

We treat the integral over ζ following [DI2], Sec. 7. In the variable

$$\xi = \xi(\zeta) = \sqrt{\zeta\lambda} - \frac{\sqrt{n}}{q\sqrt{\zeta}},$$

this integral can be written as

$$Q^{-1/2}y_j^{1/2}e\left(-2\frac{\sqrt{n\lambda}}{q}\right)\int v(n,q,\xi,\lambda)e(-\xi^2)\,d\xi.$$

Here the range for ξ is $\xi \ll (Qy_j)^{-1/2}$, but the integral converges rapidly for $|\xi|>1$ owing to the oscillatory nature of the function $e(-\xi^2)$, so it defines a function of the type $v(n,q,\lambda)$.

The resulting transformation formula for S_j can be written into the form

$$S_j\left(\frac{a}{q}+\eta\right) = Q^{-1/2}y_j^{1/2}\sum_{n\asymp 1/y_j} t(n)e\left(-\frac{n\bar{a}}{q}\right)\int e\left(-2\frac{\sqrt{n\lambda}}{q}\right)v(n,q,\lambda)\,d\lambda \\ + O_A(N^{-A}),$$

where $\lambda \asymp M$ for $j=1$ and $N-\lambda \asymp M$ for $j=2$.

Note that the factor $e(-2\sqrt{n\lambda}/q)$ can be included into the v-function for $j=1$, while for $j=2$ we may replace the same factor by $e(-2\sqrt{nN}/q)$, the approximation error being again absorbed into the respective v-function. Then, integrating over λ, we obtain

$$S_1\left(\frac{a}{q}+\eta\right) = \sum_{m\asymp M} t(m)e\left(-\frac{m\bar{a}}{q}\right)v(m,q) + O_A(N^{-A}),$$

(3.9)

$$S_2\left(\frac{a}{q}+\eta\right) = (M/N)^{1/2}\sum_{n\asymp N} t(n)e\left(-\frac{n\bar{a}}{q}\right)e\left(-2\frac{\sqrt{nN}}{q}\right)v(n,q) + O_A(N^{-A}).$$

When the above expressions for S_1 and S_2 are substituted into (2.12), the sum over a produces Kloosterman sums $S(m+n,N;q)$, and we end up with the formula

$$\tau(\eta)=\tau_1(\eta)+O_A(N^{-A}),$$

where

$$\tau_1(\eta)=(M/N)^{1/2}\sum_{q\asymp Q}\sum_{m\asymp M}\sum_{n\asymp N}t(m)t(n)S(m+n,N;q) \times e\left(-2\frac{\sqrt{(m+n)N}}{q}\right)v(m,n,q);$$

here we replaced n by $m+n$ under the root sign and included the error into the v-factor.

4. Application of Kuznetsov's trace formula

The sum over Kloosterman sums in (3.10) is now translated into the language of the spectral theory by means of the following identity due to Kuznetsov [K1] (for a neat proof, see [M2, Theorem 2.3]). Let the functions

$$\sum_{n=1}^{\infty}\rho_{j,k}(n)n^{(k-1)/2}e(nz)\quad(1\le j\le\vartheta(k))$$

constitute an orthonormal basis for the holomorphic cusp forms of weight k, and put $a_k=2^{2-2k}\pi^{-k-1}(k-1)!$. As usual, write $\sigma_\alpha(n)=\sum_{d|n}d^\alpha$.

Lemma 2. *Let $\psi(x)\in C^3(0,\infty)$, and suppose that for $\nu=0,1,2,3$*

$$\psi^{(\nu)}(x)\ll\begin{cases}x^{1/2-\nu+\varepsilon} & \text{as } x\to+0,\\ x^{-1-\nu-\varepsilon} & \text{as } x\to\infty,\end{cases}$$

where ε is an arbitrarily small positive number. Then, for any integers $m,n\ge1$, we have

$$\begin{aligned}(4.1)\quad &\sum_{q=1}^{\infty}q^{-1}S(m,n;q)\psi\left(\frac{4\pi\sqrt{mn}}{q}\right)\\ &=\sum_{j=1}^{\infty}\frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh\pi\kappa_j}\hat\psi(\kappa_j)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}\hat\psi(r)\,dr\\ &\quad+\sum_{k=1}^{\infty}a_k\sum_{j=1}^{\vartheta(k)}\overline{\rho_{j,k}(m)}\rho_{j,k}(n)\hat\psi((1-k)i/2),\end{aligned}$$

where

$$\hat{\psi}(r) = \frac{\pi i}{2\sinh \pi r} \int_0^\infty (J_{2ir}(x) - J_{-2ir}(x))\psi(x)\frac{dx}{x}. \tag{4.2}$$

In the case of the sum (3.10), the function $\psi(x)$ involves the oscillating factor e^{-ix} and a smooth factor. In the next lemma, we estimate the corresponding transform $\hat{\psi}(r)$. Actually, with a proof of the estimate (1.13) also in mind, we take the oscillating factor in the more general form $e^{\pm iax}$, where $a > 0$ is a parameter. To avoid repetition, we state the result only in the case where r is real; the values $\hat{\psi}((1-k)i/2)$ occurring in (4.1) can be estimated completely analogously.

Lemma 3. *Let*

$$\psi(x) = e^{\pm iax} C(x),$$

where $a > 0$ and $C(x)$ is a function supported in the interval $[X, cX]$, where $c > 1$ is a constant and X is a positive parameter such that aX is large. Suppose that

$$C^{(\nu)}(x) \ll_\nu X^{-\nu}, \quad \nu = 0, 1, \ldots \tag{4.3}$$

Then, for all real r, we have

$$\hat{\psi}(r) \ll \min\left(X^{-1/2}, X^{-1}|a^2-1|^{-1/2}\right)(aX)^\varepsilon. \tag{4.4}$$

Moreover, for any fixed positive A, we have

$$\hat{\psi}(r) \ll (|r| + aX)^{-A} \tag{4.5}$$

if

$$|r| \gg \max\left(X^{1/2}, X\sqrt{|a^2-1|}\right)(aX)^\varepsilon, \tag{4.6}$$

and also if

$$a \le 1 - X^{\varepsilon - 1}. \tag{4.7}$$

Proof. We modify the argument of the proof of Lemma 7.1 in [DI1]. The J-Bessel functions occurring in (4.2) may be given by the integral representation (see [L, p. 139])

$$\frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh \pi r} = \frac{4}{\pi i}\int_{-\infty}^{\infty} \cos(x\cosh\xi)\cos(2r\xi)\,d\xi. \qquad (r \ne 0, x > 0). \tag{4.8}$$

Thus

$$\hat{\psi}(r) = 2\int_{-\infty}^{\infty}\int C(x)x^{-1}e^{\pm iax}\cos(x\cosh\xi)\cos(2r\xi)\,dx\,d\xi.$$

Here the x-integral is small as a smooth oscillatory integral unless $\cosh\xi$ lies near a. This cannot happen in the case (4.7). Then (4.5) is easily verified by repeated integration by parts over x, and also over ξ if $|r| \gg X$. Otherwise we extract those values of ξ with

$$\cosh\xi = a + O(X^{-1}(aX)^{\varepsilon}),$$

and the extracted part is understood as a weighted integral.

The remaining part is small, namely of the order of the right hand side of (4.5). Indeed, in view of (4.3), repeated integration by parts sufficiently times with respect to x saves as high a power of aX as we wish. Moreover, if $|r|$ exceeds aX, repeated integration by parts over ξ saves any power of r.

It remains to estimate the contribution of the critical range of ξ extracted above. First, to prove (4.4), we observe that the length of this range is of the order of the right hand side in (4.4), and estimate both integrals trivially. Alternatively, if r satisfies (4.6), we first integrate by parts with respect to the factor $\cos(2r\xi)$ sufficiently many times getting again the estimate (4.5).

For the convenience of reference, we state separately the case $a = 1$ of Lemma 3.

Corollary. *Let $\psi(x) = e^{\pm ix}C(x)$, where $C(x)$ satisfies the conditions of Lemma 3 for a large parameter X. Then, for all real r, we have*

$$\hat{\psi}(r) \ll X^{-1/2+\varepsilon},$$

and also

$$\hat{\psi}(r) \ll |r|^{-A} \text{ for } |r| \gg X^{1/2+\varepsilon}.$$

Remark 1. To deal with the transforms $\hat{\psi}((1-k)i/2)$, we may use the integral representation (see [L, Eq. (5.10.8)])

$$J_{k-1}(x) = \frac{1}{\pi}\int_0^{\pi} \cos((k-1)\xi - x\sin\xi)\, d\xi,$$

which implies that

$$\frac{\pi i}{2\sinh((1-k)\pi i/2)}(J_{k-1}(x) - J_{-(k-1)}(x)) = \int_{-\pi/2}^{\pi/2} \sin((k-1)\xi - x\cos\xi)\, d\xi.$$

This is analogous to (4.8), and the resulting estimates for $\hat{\psi}((1-k)i/2)$ are similar to those in lemma 3, with k playing the same role as r.

Remark 2. The "opposite-sign" case of Kuznetsov's trace formula reads

$$\sum_{q=1}^{\infty} q^{-1}S(m,-n;q)\psi\left(\frac{4\pi\sqrt{mn}}{q}\right)$$
$$= \sum_{j=1}^{\infty} \frac{\overline{\rho_j(m)}\rho_j(-n)}{\cosh\pi\kappa_j}\psi^{-}(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}\psi^{-}(r)\, dr$$

where

$$\psi^-(r) = 2\cosh(\pi r)\int_0^\infty K_{2ir}(x)\psi(x)\frac{dx}{x}$$

(see [M2, Theorem 2.5]). The assumptions on the function ψ are as in Lemma 2. The appropriate integral representation for the K-Bessel function here is ([W], p. 183)

$$\cosh(\pi r)K_{2ir}(x) = \int_0^\infty \cos(x\sinh\xi)\cos(2r\xi)\,d\xi.$$

The estimate corresponding to (4.4) is now

$$\psi^-(r) \ll ((a+1)X)^{-1+\varepsilon},$$

and the stronger estimate of the type (4.5) holds for $|r| \gg ((a+1)X)^{1+\varepsilon}$.

5. Estimation of the sum $T_g(N)$

Returning to the sum (3.10), we rewrite it as follows:

(5.1)
$$\tau_1(\eta) = (M/N)^{1/2}Q\sum_{m\asymp M}\sum_{n\asymp N} t(m)t(n)\sum_{q\asymp Q} q^{-1}S(m+n,N;q)e^{-ix}v(x,m,n),$$

where $x = 4\pi\sqrt{(m+n)N}/q$. Then $x \asymp N/Q \asymp N^{1+\varepsilon}/M$, so $X \gg N^\varepsilon$ in the Corollary of Lemma 3.

The sum over q in (5.1) is now transformed by Lemma 2, and $\tau(\eta)$ is decomposed accordingly:

$$\tau_1(\eta) = \tau_1^{(d)}(\eta) + \tau_1^{(c)}(\eta) + \tau_1^{(h)}(\eta). \tag{5.2}$$

The treatment of these ingredients will be analogous, so let us consider $\tau_1^{(d)}(\eta)$, the contribution of the discrete spectrum, as an example. The Corollary of Lemma 3 shows that the relevant range for κ_j is

$$\kappa_j \le K = (N/M)^{1/2}N^\varepsilon, \tag{5.3}$$

and then $\hat{\psi}(\kappa_j) \ll (M/N)^{1/2}N^\varepsilon$. This transform is again a smooth function of m and n. Therefore, putting $\alpha_j = |\rho_j(1)|^2/\cosh\pi\kappa_j$, we have

$$\tau_1^{(d)}(\eta) = M^2N^{-1}\sum_{m\asymp M}\sum_{n\asymp N} t(m)t(n)\sum_{\kappa_j\le K}\alpha_j t_j(m+n)t_j(N)v_j(m,n)+O(1), \tag{5.4}$$

where the v_j are smooth functions in the same sense as before.

The spectral sum in (5.4), like analogous expressions related to the continuous spectrum or holomorphic cusp forms, are now estimated by use of the spectral large sieve due to H. Iwaniec [I1-2] (see also [M2, Sec. 3.5], for a simplified approach to this topic).

Lemma 4. *Let $K \geq 1$, $N \geq 1/2$, $\varepsilon > 0$, and let b_n for $N < n \leq 2N$ be any complex numbers. Then the expressions*

$$\sum_{k \leq K} a_k \sum_{j=1}^{\vartheta(k)} \Big| \sum_{N<n\leq 2N} b_n \rho_{j,k}(n) \Big|^2,$$

$$\sum_{\kappa_j \leq K} \alpha_j \Big| \sum_{N<n\leq 2N} b_n t_j(n) \Big|^2,$$

$$\int_{-K}^{K} \Big| \sum_{N<n\leq 2N} b_n \sigma_{2ir}(n) n^{-ir} \Big|^2 \, dr$$

are all majorized by

$$\ll (K^2 + N^{1+\varepsilon}) \sum_{N<n\leq 2N} |b_n|^2.$$

Before applying this to the sum (5.4), it is convenient to separate the variables m and n in the functions $v_j(m,n)$ by partial summation. In practice this means elimination of these functions, so we end up with the sum

$$M^2 N^{-1} \sum_{\kappa_j \leq K} |t_j(N)| \Big| \sum_{p \asymp N} b_p t_j(p) \Big|, \tag{5.5}$$

where

$$b_p = \sum_{m+n=p} t(m) t(n) \tag{5.6}$$

with m and n running over similar intervals as before.

We still need an estimate for b_p in mean square.

Lemma 5. *Let $M, N \geq 1$, and let b_p be as in (5.6) with $m \asymp M$, $n \asymp N$. Then*

$$\sum_p |b_p|^2 \ll (MN)^{1+\varepsilon}.$$

This is "dual" to Lemma 3 in [J4], and the proof, based on the estimate (2.3) for exponential sums, is closely analogous.

We are now in a position to complete the estimation of the sum (5.5). Since there are $O(K^2)$ numbers $\kappa_j \leq K$, this sum is by Cauchy's inequality and (1.7) at most

$$\ll M^2 N^{-1+\theta+\varepsilon} K \sqrt{\sum_{\kappa_j \leq K} \Big| \sum_{p \asymp N} b_p t_j(p) \Big|^2}.$$

By Lemmas 4 and 5, this is $\ll K M^{5/2} N^{\theta+\varepsilon}$, or $\ll M^2 N^{1/2+\theta+\varepsilon}$ by (5.3), which is our estimate for $\tau_1^{(d)}(\eta)$. The same holds for the other terms in (5.2), too, thus for $\tau_1(\eta)$ (and $\tau(\eta)$) as well.

Returning to (2.12), we observe by the preceding estimation that $T_1 \ll N^{1/2+\theta+\varepsilon}$. Together with (2.10) and (2.11), this entails the same estimate for our sum T, and the proof of the estimate (1.14) is complete.

6. Estimation of the sum $T_g(N;f)$

Let $S_1(\alpha)$ and $S_2(\alpha)$ be defined as in (2.5), and consider the sums

(6.1)
$$\sum_{n=1}^{N} t(n)t(n+f)g_j\left(\frac{n}{N}\right)g_j\left(\frac{n+f}{N}\right) = \int_0^1 S_j(\alpha+\mu)\overline{S}_j(\alpha+\mu)e(-f(\alpha+\mu))\,d\alpha.$$

Since $f \ll N^{2/3}$ by assumption, it is easy to see, by (2.1), that the sum $T_g(N;f)$ can be written as a sum of $\ll \log N$ sums of the type (6.1) with an error $\ll N^{2/3+\varepsilon}$. The parameter M may be restricted to the interval

$$N^{2/3} \le M \le N/3. \tag{6.2}$$

Let us consider the sum (6.1) for $j = 2$, the case $j = 1$ being similar but easier. Indeed, in the latter case, the argument leads to sums of Kloosterman sums with a smooth stationary weight, and following [J4], we end up with the estimate $\ll N^{1/2+\varepsilon}f^{\theta}$. This is $\ll N^{2/3+\varepsilon}$ with the choice $\theta = 1/4$.

Denote the sum (6.1) for $j = 2$ by T again, and decompose it to the sum $T_1 + T_2$ as in Sec. 2. Then

$$T_2 \ll N^{1+\varepsilon}M^{-1/2} \ll N^{2/3+\varepsilon} \tag{6.3}$$

by previous arguments and (6.2).

The "main term" T_1 is estimated as in (2.11):

$$T_1 \ll M^{-2+\varepsilon} \max_{|\mu-\eta|\le\delta} |\tau(\eta)|, \tag{6.4}$$

where

$$\tau(\eta) = \sum_q w(q) \sideset{}{^*}\sum_{a=1}^{q} S_2\left(\frac{a}{q}+\eta\right)\overline{S}_2\left(\frac{a}{q}+\eta\right)e\left(-\frac{fa}{q}\right).$$

Next we substitute the expression (3.9) for S_2 transforming thus $\tau(\eta)$ into

(6.5)
$$\tau_1(\eta) = \frac{M}{N}\sum_{q\asymp Q}\sum_{m,n\asymp N} t(m)t(n)S(m-n,f;q)e\left(-2\frac{(\sqrt{m}-\sqrt{n})\sqrt{N}}{q}\right)v(m,n,q),$$

up to a negligible error. This is analogous to (3.10).

The diagonal terms involve Kloosterman (actually Ramanujan) sums $S(0,f;q)$ and the contribution of these terms to $\tau_1(\eta)$ is $\ll M^2N^{\varepsilon}$ by straightforward estimations.

The non-diagonal part of (6.5) contains two types of terms as to the sign of $m-n$. If $m-n>0$, then the q-sum can be transformed by Lemma 2. On the other hand, if $m-n<0$, then the formula given in Remark 2 in Sec. 4 is to be applied. Since the latter "opposite-sign" case does not present any new problems, we may confine ourselves to the "equal-sign" case.

We gather in (6.5) those terms with $m-n=p>0$ and restrict p for a moment to the range $p \asymp P$ for some $P \ll N$. The sum over p can be equipped with a smooth weight. Then the corresponding part of $\tau_1(\eta)$, say $\tau_2(\eta)$, can be written as

$$\tau_2(\eta)=\frac{MQ}{N}\sum_{q\asymp Q} q^{-1}\sum_{p\asymp P}\sum_{n\asymp N} t(n)t(n+p)S(p,f;q)e^{-ia(n/p)x}v(n,p,x) \tag{6.6}$$

with

$$x=4\pi\sqrt{pf}/q,$$
$$a(y)=\sqrt{N/f}(\sqrt{y+1}-\sqrt{y}).$$

We may suppose that

$$P \gg QN^{\varepsilon}, \tag{6.7}$$

for otherwise the factor $e^{-ixa(n/p)}$ in (6.6) is essentially stationary and can be absorbed into the function $v(n,p,x)$. Then, arguing as in [J4] and using its lemmas 3 and 4, we get the estimate $\tau_2(\eta) \ll M^2N^{1/2}f^{\theta}N^{\varepsilon}$ giving a contribution $\ll N^{2/3}$ to T_1.

Next we reformulate (6.6) by a shifting device on replacing n by $n+\ell$ with $\ell \asymp L$, where

$$L=N^{1-\varepsilon}Q/P. \tag{6.8}$$

Then, estimating partial derivatives, we find that $e^{-ixa((n+\ell)/p)}v(n+\ell,p,x)$ can be written as $e^{-ixa(n/p)}v_\ell(n,p,x)$ for a suitable new v-function v_ℓ. Moreover, v_ℓ is a stationary function of ℓ, that is its derivative with respect to ℓ is a function of the type $L^{-1}v_\ell$. Finally we average the resulting formula for $\tau_2(\eta)$ over the parameter ℓ. Since v_ℓ is stationary in ℓ, its dependence on ℓ can be eliminated by partial summation. In addition, we may separate the variables in $v(n,p,x)$ on expressing this as a Fourier integral in the variables n and p for given x. In practice this means that $v(n,p,x)$ can be written, with a negligible error, as the integral of $e(\beta_1 n+\beta_2 p)V(\beta_1,\beta_2;x)$ with V standing for the Fourier transform and the variables running over the ranges $\beta_1 \ll N^{-1+\varepsilon}$, $\beta_2 \ll P^{-1}N^{\varepsilon}$. Since the integration over the β's will be estimated trivially in the end, we may fix these henceforth. Then, in place of (6.6), we have to deal with an expression of the type

$$\tau_3(\eta)=\frac{MQ}{LN}\sum_{n\asymp N} e(\beta_1 n)\sum_{p\asymp P} e(\beta_2 p)\left(\sum_{\ell\asymp L} t(n+\ell)t(n+\ell+p)\right)\times$$
$$\times\sum_{q\asymp Q} q^{-1}S(p,f;q)e^{-ixa(n/p)}v(x).$$

By Lemma 2, this can be decomposed into $\tau_3^{(d)}(\eta) + \tau_3^{(c)}(\eta) + \tau_3^{(h)}(\eta)$ as in (5.2). Let us consider the most significant term $\tau_3^{(d)}(\eta)$ in more detail. We have

$$\tag{6.9} \begin{aligned} \tau_3^{(d)}(\eta) = \frac{MQ}{LN} \sum_{p \asymp P} e(\beta_2 p) \sum_{n \asymp N} e(\beta_1 n) \times \\ \times \left(\sum_{\ell \asymp L} t(n+\ell)t(n+\ell+p) \right) \sum_{j=1}^{\infty} \alpha_j t_j(p) t_j(f) \hat{\psi}(\kappa_j, n/p), \end{aligned}$$

where $\hat{\psi}(r,y)$ is the transform (in the sense (4.2)) of $\psi(x,y) = e^{-ia(y)x}v(x)$ as a function of x for given y.

Lemma 3 is now applicable to the last mentioned transform. Indeed, in the notation of that lemma, we have $X \asymp \sqrt{Pf}/Q$ and $a \asymp \sqrt{P/f}$, whence $aX \asymp P/Q$ is large by our assumption (6.7). Moreover, $a(y)$ is large by our assumptions on P and f (the ε in (6.7) should be taken bigger than that in (2.3)), so Lemma 3 gives

$$\tag{6.10} \hat{\psi}(r,y) \ll (Q/P)^{1-\varepsilon}$$

for all real r and the relevant values of y, while this transform is very small for $|r| \gg (P/Q)^{1+\varepsilon}$. Therefore the spectral sum in (6.9) can be truncated to $\kappa_j \le K$ with

$$\tag{6.11} K = (P/Q)^{1+\varepsilon}.$$

Following the argument of the preceding section, we next apply the spectral large sieve to the sum (6.9). However, it is a new complication that the transform $\hat{\psi}(\kappa_j, y)$ depends on y in an oscillatory way. Indeed, as an easy generalization of (6.10), we have

$$\frac{\partial^\nu \hat{\psi}(\kappa_j, y)}{\partial y^\nu} \ll_\nu \left(\frac{Q}{P}\right) \left(\frac{P^2}{NQ}\right)^\nu N^\varepsilon.$$

Therefore we may express $\hat{\psi}(\kappa_j, y)$ in any y-interval of length $\asymp N^{1-\varepsilon}Q/P^2$ around a given value $y_0 \asymp N/P$ by a Taylor polynomial of bounded degree in $y - y_0$ with a negligible error. If any one of the terms of this polynomial is taken into consideration, then the variables κ_j and y will be separated. The constant term dominates in the polynomial, so let us estimate its contribution.

Fixing an y-interval of the above type for a moment, we sum first over those pairs (p,n) in (6.9) for which $y = n/p$ lies in our interval. These pairs can be subdivided into subsets consisting of pairs $(p, n(p))$, where $n = n(p)$ is assigned to p somehow among $\ll N^{1-\varepsilon}Q/P = L$ possible choices. The contribution of one such subset is estimated by Cauchy's inequality and Lemma 4. Next we sum over the subsets just described, and finally sum over the y-intervals, again by Cauchy's inequality. To complete the estimations, we need two auxiliary results. The first of these is following spectral mean value estimate:

$$\sum_{\kappa_j \le K} \alpha_j t_j^2(f) \ll K^2 + f^{1/2+\varepsilon},$$

which is a weakened version of a theorem of N. V. Kuznetsov [K1]. Secondly, we need the following lemma for the purpose of estimating the ℓ-sums in (6.9) in mean. Its proof, based on the estimate (2.3) for exponential sums, is analogous to that of Lemma 3 in [J4].

Lemma 6. *For $N, P \geq 1$ and $1 \leq L \leq N$, we have*

$$\sum_{0 \leq p \leq P} \sum_{1 \leq n \leq N} \left| \sum_{1 \leq \ell \leq L} t(n+\ell)t(n+p+\ell) \right|^2 \ll (N+P)^{1+\varepsilon} NL.$$

We now put everything together, recalling (6.10) and estimating the fourfold sum in (6.9) as indicated above. In this way, we obtain

$$\tau_3^{(d)}(\eta) \ll (MQ/LN)(Q/P)L^{1/2}(N/L)^{1/2}(K^2+P)^{1/2}(N^{2+\varepsilon}L)^{1/2}(K^2+f^{1/2})^{1/2}.$$

In view of the choices (2.7), (6.8), and (6.11) of Q, K, and L, this is

$$\ll M^2(PM^{-1/2}+M^{1/2}f^{1/4})N^{\varepsilon} \ll M^2(NM^{-1/2}+M^{1/2}f^{1/4})N^{\varepsilon}.$$

The contribution of this to T_1 in (6.4) is $\ll N^{2/3+\varepsilon}$. Since the other ingredients of T_1 can be estimated in the same way, we have $T_1 \ll N^{2/3+\varepsilon}$. Finally, combining this with (6.3), we get the same estimate for our original sum T.

References

[BDHI] D. Bump, W. Duke, J. Hoffstein and H. Iwaniec, *An estimate for the Hecke eigenvalues of Maass forms*, Int. Math. Res. Notes No 4 (1994), 75–81.

[DI1] J.-M. Deshouillers and H. Iwaniec, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. **70** (1982), 219–288.

[DI2] J.-M. Deshouillers and H. Iwaniec, *An additive divisor problem*, J. London Math. Soc. (2) **26** (1982), 1–14.

[G1] A. Good, *Cusp Forms and Eigenfunctions of the Laplacian*, Math. Ann. **255** (1981), 523–548.

[G2] A. Good, *On Various Means Involving the Fourier Coefficients of Cusp Forms*, Math. Z. **183** (1983), 95–129.

[I1] H. Iwaniec, *Fourier coefficients of cusp forms and the Riemann zeta-function*, Séminaire de Théorie des Nombres, exposé No 18, Bordeaux 1979/80.

[I2] H. Iwaniec, *Mean values for Fourier coefficients of cusp forms and sums of Kloosterman sums*, Journées Arithmétiques de Exeter 1980, Cambridge Univ. Press, 1982, pp. 306–321.

[I3] H. Iwaniec, *The spectral growth of automorphic L-functions*, J. reine angew. Math. **428** (1992), 139–159.

[I4] H. Iwaniec, *Introduction to the Spectral Theory of Automorphic Functions*, Revista Matemática Iberoamericana, Madrid, 1995.

[J1] M. Jutila, *Transformations of exponential sums*, Proc. Amalfi Conf. on Analytic Number Theory 1989, Univ. di Salerno, Salerno, 1992, pp. 263–270.

[J2] M. Jutila, *The additive divisor problem and its analogs for Fourier coefficients of cusp forms. I*, Math. Z. **223** (1996), 435–461.

[J3] M. Jutila, *The additive divisor problem and its analogs for Fourier coefficients of cusp forms. II*, Math. Z. **225** (1997), 625–637.

[J4] M. Jutila, *A Variant of the Circle Method*, Sieve Methods, Exponential Sums, and their Applications in Number Theory, Cambridge Univ. Press, 1996, pp. 245–254.

[K1] N. V. Kuznetsov, *Petersson's hypothesis for parabolic forms of weight zero and Linnik's hypothesis. Sums of Kloosterman sums* (in Russian), Mat. Sb. **111** (1980), 334–383.

[K2] N. V. Kuznetsov, *Convolution of the Fourier coefficients of the Eisenstein-Maass series* (in Russian), Zap. Nauch. Sem. LOMI **129** (1983), 43–84.

[L] N. N. Lebedev, *Special Functions and their Applications*, Dover Publications, New York, 1972.

[M1] Y. Motohashi, *The Binary Additive Divisor Problem*, Ann. Sci. l'Ecole Norm. Sup. **27** (1994), 529–572.

[M2] Y. Motohashi, *Spectral Theory of the Riemann Zeta-Function*, Cambridge Univ. Press, 1997.

[TV] L. A. Tahtadjan and A. I. Vinogradov, *The zeta-function of the additive divisor problem and the spectral decomposition of the automorphic Laplacian* (in Russian), Zap. Nauch. Sem. LOMI **134** (1984), 84–116.

[W] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1966.

Department of Mathematics
University of Turku
FIN-20014 Turku
Finland
`matti.jutila@utu.fi`

(Received 23 02 1998)