PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 65 (79), 1999, 69–96

ASYMPTOTIC EXPANSIONS FOR DIRICHLET SERIES ASSOCIATED TO CUSP FORMS

Andreas Guthmann

Communicated by Aleksandar Ivić

Abstract. We prove an asymptotic expansion of Riemann-Siegel type for Dirichlet series associated to cusp forms. Its derivation starts from a new integral formula for the Dirichlet series and uses sharp asymptotic expansions for partial sums of the Fourier series of the cusp form.

0. Introduction

Since its publication in 1932 [**16**] the *Riemann-Siegel formula* has become a fundamental and indispensable tool in the theory of the zeta function. This formula yields an arbitrary sharp approximation for $\zeta(s)$ if s tends to infinity in a vertical strip. To be more precise, let $s = \sigma + it$, $\sigma_0 \le \sigma \le \sigma_1$, where σ_0, σ_1 are fixed, $t \ge t_0$ is sufficiently large, and $N = [(\frac{t}{2\pi})^{\frac{1}{2}}]$. Then

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + 2^{s-1}\pi^s\Gamma(s)^{-1}\sec(\tfrac{\pi s}{2})\sum_{n=1}^{N} n^{s-1} - (-1)^N(2\pi)^{\frac{s+1}{2}}\Gamma(s)^{-1}t^{\frac{s-1}{2}}e^{\pi is-\frac{it}{2}-\frac{\pi i}{8}}S \tag{0.1}$$

for $t \to +\infty$. The two sums of length $[(\frac{t}{2\pi})^{\frac{1}{2}}]$ are to be considered as the main approximation, while the third term is given as an asymptotic series of the shape

$$S = \sum_{k=0}^{\nu-1} a_k \sum_{0\le 2r\le k} b_{kr}F^{(k-2r)}(\delta) + O\big((3n/t)^{\frac{\nu}{6}}\big). \tag{0.2}$$

Here a_k, b_{kr} are certain complex numbers, $a_k = O(t^{-\frac{k}{6}})$, ν is a positive integer not exceeding $2\cdot 10^{-8}t$, $\delta = \sqrt{t} - (N+\frac{1}{2})\sqrt{2\pi}$, and F denotes the function $F(z) =$

AMS Subject Classification (1991): Primary 11M41

Keywords: Riemann-Siegel formula, cusp forms, Dirichlet series

$\cos(z^2+\frac{3\pi}{8})/\cos(\sqrt{2\pi}z)$. Despite its quite complicate structure, the Riemann-Siegel formula has found numerous applications, most notably by Levinson on zeros of the zeta function [**11**]. For further information the reader should consult the book of Ivić [**7**]. In view of these facts, it is quite natural to ask whether similar asymptotic expansions can be given for other types of Dirichlet series. Not surprisingly, this is the case for Dirichlet L functions, as shown by Siegel some years later [**17**] and by Deuring [**2**]. Afterwards, the entire subject fell into some kind of slumber, and it is only recently that Motohashi in his deep work found an analogue for $\zeta^2(s)$ [**12, 13**]. His argument depends on another version of the Riemann-Siegel formula and properties of the divisor function.

In the present paper we shall derive a formula of Riemann-Siegel type for a large class of Dirichlet series, namely for those associated to cusp forms for the modular group. Hitherto the only result in this direction is due to Jutila [**8**], who found the analogue of the approximate functional equation for $\zeta(s)$ (see also [**4**]). Our result allows arbitrary sharp approximations, like (0.1) and (0.2), but is less complicated. For example, a condition like $\nu \leq 2\cdot 10^{-8}t$ is not required. Apart from its theoretical value, the explicit form of the final result suggests applications to numerical purposes as well.

We fix some notation to be maintained throughout the paper. Let k be a positive even integer, $\mathcal{H}=\{z\in\mathbf{C}|\,\mathrm{Im}(z)>0\}$ the upper half plane, and denote by S_k the $\mathbf{C}$ vector space of cusp forms of weight k for the modular group $SL_2(\mathbf{Z})$. Thus $f\in S_k$ precisely if the following conditions are satisfied:

i) $f:\mathcal{H}\to\mathbf{C}$ is holomorphic.

ii) The function f satisfies $f(\frac{az+b}{cz+d})=(cz+d)^k f(z)$ for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbf{Z})$.

iii) f admits a Fourier expansion of the shape

$$f(z)=\sum_{n=1}^{\infty}a(n)e^{2\pi inz},\quad \mathrm{Im}(z)>0. \tag{0.3}$$

The most prominent example of such a cusp form is the discriminant

$$\Delta(z)=e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}$$

of weight 12. Its Fourier coefficients in $\Delta(z)=\sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}$ are given by Ramanujan's tau function.

The analytical foundations of the theory were mainly laid by Hecke in his classical works on Dirichlet series satisfying certain functional equations [**5, 6**]. Thus to each $f\in S_k$ with Fourier series (0.3) one associates a Dirichlet series

$$\varphi(s)=\sum_{n=1}^{\infty}a(n)n^{-s},\quad s=\sigma+it, \tag{0.4}$$

via its Mellin transform, i.e.

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = \int_0^\infty f(ix)x^{s-1}\mathrm{d}x. \tag{0.5}$$

From Hecke's theory it also follows that the Fourier coefficients do not grow too fast. The estimates

$$|a(n)| = O(n^{\frac{k}{2}}), \quad A(x) := \sum_{n\leq x} a(n) = O(x^{\frac{k}{2}}), \tag{0.6}$$

are classical and are easily proved from the properties of f. We shall also assume that f is an eigenfunction of the Hecke algebra. This is no restriction since S_k has a basis of such functions, and it has the advantage that we may employ Deligne's result $|a(n)| \leq d(n)n^{\frac{k-1}{2}}$ [**1**], $d(n)$ denoting the divisor function. Although it is not absolutely necessary to use this inequality, most of our proofs concerning convergence of series involving $a(n)$ are considerably simplified. As a matter of fact, Hecke's classical formulas are always sufficient, yielding the same results in a more roundabout way.

It now follows that the series in (0.4) converges absolutely for $\sigma > \frac{k+1}{2}$. Moreover, Property ii) above implies $f(-\frac{1}{z}) = z^k f(z)$, which in turn gives the functional equation

$$\varphi(s) = (2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\varphi(k-s).$$

We have thus a situation completely analogous to that of the Riemann zeta function. The "critical strip" is given by $\frac{k-1}{2} \leq \sigma \leq \frac{k+1}{2}$.

Despite of much research being done on the entire subject no analogue of the Riemann-Siegel formula was known before, apart form Jutila's approximate functional equation [**8, 4**]. The success of our approach depends on some new ideas. Starting from an integral representation for $\varphi(s)$ proved earlier [**3, 4**], we employ the usual saddle point method. It will be seen that certain types of "incomplete" cusp forms appear naturally in the analysis. The success of the method then depends on a remarkable asymptotic expansion of these partial sums of the Fourier series (0.3) (Theorem 2).

The contents of the paper are as follows. In Section 1 we derive some auxiliary results, mainly asymptotic expansions for various functions occurring later. The next section contains our main formulas for the incomplete cusp forms mentioned above which are applied in Section 3 for the derivation of the Riemann-Siegel formula. In the last section we finally state some special cases of our asymptotic formula and indicate various problems for further study.

1. Asymptotic Expansions of Integrals

In this section we collect some results that will be extensively used in the sequel. The formulas derived here give asymptotic expansions of certain integrals.

The application of the Voronoi summation formula in Section 2 leads in a natural way to functions $U(z,x)$ and $V(z,x)$ (see (2.4), (2.5)) whose behaviour in turn is determined by that of the integrals treated here. In our context $z \geq 1$ is a real number, x is complex subject to suitable restrictions like $\mathrm{Re}(x) > 0$ or $|\arg(x)| \leq \frac{\pi}{4}$.

We start with the fundamental

LEMMA 1. *Let $\nu \in \mathbf{N}$, h, k, τ_1, τ_2 be fixed real numbers satisfying the inequalities*

$$0 < h \leq 1, \ \tau_1 > 0, \ \tau_2 > 0, \ \max\{\tau_1, \tau_2\} < h. \tag{1.1}$$

Set $s_1 = i + \tau_1 e^{-\frac{\pi i}{4}}$, $s_2 = i - \tau_2 e^{-\frac{\pi i}{4}}$. For $z \geq 1$ real and complex x such that $|x - i| \geq h$ define

$$\tilde{I}_{k\nu}(z,x) = \int_{s_1}^{s_2} e^{z(s-\frac{1}{s})} s^{-k} \frac{\mathrm{d}s}{(x-s)^\nu}.$$

Then for each fixed $M \in \mathbf{N}_0$ we have

$$\tilde{I}_{k\nu}(z,x) = -e^{2iz} z^{-\frac{1}{2}} i^{-k} e^{-\frac{\pi i}{4}} (x-i)^{-\nu} \left\{ \sum_{m=0}^{M-1} \alpha_m(x) \Gamma(m+\tfrac{1}{2}) z^{-m} + O(z^{-M}) \right\}$$

uniformly in x. The coefficients $\alpha_m(x)$ are given by the formula

$$\alpha_m(x) = e^{\frac{\pi i m}{2}} \sum_{\mu=0}^{2m} \binom{-\nu}{\mu} \binom{m-k+\frac{1}{2}}{2m-\mu} i^{-\mu} (x-i)^{-\mu}.$$

They are rational functions of x having a pole of order $2m$ at $x = i$. Moreover, the inequality

$$|\alpha_m(x)| \leq h^{-2m} \sum_{\mu=0}^{2m} \left| \binom{-\nu}{\mu} \binom{m-k+\frac{1}{2}}{2m-\mu} \right|$$

is satisfied.

Proof. We proceed along standard lines using the saddle point method. We shall, however, be very careful because of the dependency of the integrand on x. First let $s = w + i$ and $\varepsilon = e^{\frac{\pi i}{4}}$. Then $s - s^{-1} = \frac{s^2-1}{s} = 2i - \frac{iw^2}{1-iw}$. Thus

$$\tilde{I}_{k\nu}(z,x) = e^{2iz} i^{-k} (x-i)^{-\nu} \int_{\varepsilon^{-1}\tau_1}^{-\varepsilon^{-1}\tau_2} e^{-izw^2/(1-iw)} (1-iw)^{-k} \frac{\mathrm{d}w}{(1-\frac{w}{x-i})^\nu}.$$

Here i^{-k} is defined by $i^{-k} = e^{-\frac{\pi i k}{2}}$. If $w = \varepsilon^{-1}\tau$, we now have

$$\tilde{I}_{k\nu}(z,x) = -e^{2iz} i^{-k} \varepsilon^{-1} (x-i)^{-\nu} \int_{-\tau_2}^{\tau_1} e^{-zf(\tau)} h(\tau) \mathrm{d}\tau, \tag{1.2}$$

where

$$f(\tau) = \frac{\tau^2}{1-\varepsilon\tau}, \quad H(\tau) = (1-\varepsilon\tau)^{-k}(1-\beta\varepsilon^{-1}\tau)^{-\nu}, \quad \beta = \frac{1}{x-i}. \tag{1.3}$$

Using (1.1) we take τ_0 fixed such that $\max\{\tau_1, \tau_2\} < \tau_0 < h$. This implies $|\tau| \le \tau_0 < 1$, as well as $|\beta\tau| \le h^{-1}\tau_0 < 1$ for $|\tau| \le \tau_0$. Consequently, the integrand in (1.2) is holomorphic in the domain $B := \{\tau \in \mathbf{C}; |\tau| < \tau_0\}$ which contains the path of integration.

Consider the function

$$g(\tau) = \tau(1-\varepsilon\tau)^{-\frac{1}{2}}, \quad |\tau| < 1. \tag{1.4}$$

Here the principal value of the square root is taken. Obviously, g is holomorphic in the interior of the unit circle and $g^2 = f$ from (1.3). Moreover it is easy to show that g is conformal and *schlicht* there. In particular, g is *schlicht* in $\overline{B} = \{\tau \in \mathbf{C}; |\tau| \le \tau_0\}$. Let $\overline{C} = g(\overline{B})$ be the image of $\overline{B}$ under g. Hence g maps $\overline{B}$ bijectively onto $\overline{C}$. Therefore there exists the inverse map $p : \overline{C} \to \overline{B}$, which is also bijective and holomorphic in the interior C of $\overline{C}$. Explicitly, we get from (1.4)

$$p : \overline{C} \to \overline{B}, \quad p(u) = u\left(1 + \frac{i}{4}u^2\right)^{\frac{1}{2}} - \frac{\varepsilon}{2}u^2. \tag{1.5}$$

The integral (1.2) will now be transformed by the substitution $\tau = p(u)$. We get ($u = g(\tau)$)

$$\tilde{I}_{k\nu}(z,x) = -e^{2iz}i^{-k}\varepsilon^{-1}(x-i)^{-\nu}\int_{u_2}^{u_1} e^{-zu^2}H\big(p(u)\big)p'(u)\mathrm{d}u, \tag{1.6}$$

where $u_2 = g(-\tau_2)$, $u_1 = g(\tau_1)$. Here we have $\mathrm{Re}(u_2) < 0$, $\mathrm{Re}(u_1) > 0$, as follows immediatley from (1.4). Explicitly

$$u_2 = -\tau_2(1-\varepsilon\tau_2)^{-\frac{1}{2}}, \quad u_1 = \tau_1(1-\varepsilon\tau_1)^{-\frac{1}{2}}. \tag{1.7}$$

The path of integration from u_2 to u_1 lies entirely in $\overline{C}$, even in C, since $\tau_1, -\tau_2 \in B$ are no boundary points of B. The function $H\big(p(u)\big)p'(u)$ is holomorphic in C (since $|p(u)| < \tau_0 < h$) so that we may write

$$H\big(p(u)\big)p'(u) = \sum_{m=0}^{\infty} a_m u^m, \quad u \in C. \tag{1.8}$$

For $M \in \mathbf{N}_0$ define

$$H\big(p(u)\big)p'(u) = \sum_{m=0}^{M-1} a_m u^m + u^M R_M(u). \tag{1.9}$$

For each fixed M the remainder R_M is bounded on $\overline{C}$, since R_M is holomorphic. Thus $|R_M(u)| \leq c_1$ for each $u \in \overline{C}$. The value of the constant c_1 depends on M, τ_0, h but not on z or x. Inserting (1.9) with $2M+1$ instead of M into (1.6) yields

$$\tilde{I}_{k\nu}(z,x) = -e^{2iz} i^{-k} \varepsilon^{-1}(x-i)^{-\nu} \int_{u_2}^{u_1} e^{-zu^2} \left[\sum_{m=0}^{2M} a_m u^m + u^{2M+1} R_{2M+1}(u) \right] \mathrm{d}u. \tag{1.10}$$

Using $|R_{2M+1}(u)| \leq c_2$ for $u \in \overline{C}$ and a suitable constant c_2, we see that

$$\begin{aligned} \int_{u_2}^{u_1} e^{-zu^2} u^{2M+1} R_{2M+1}(u) \mathrm{d}u &= z^{-\frac{1}{2}} \int_{u_2\sqrt{z}}^{u_1\sqrt{z}} e^{-t^2} \left(\frac{t}{\sqrt{z}} \right)^{2M+1} R_{2M+1} \left(\frac{t}{\sqrt{z}} \right) \mathrm{d}t \\ &= O(z^{-M-1}). \end{aligned} \tag{1.11}$$

Moreover

$$\begin{aligned} \int_{u_2}^{u_1} e^{-zu^2} u^m \mathrm{d}u &= z^{-\frac{m+1}{2}} \int_{u_2\sqrt{z}}^{u_1\sqrt{z}} e^{-t^2} t^m \mathrm{d}t \\ &= z^{-\frac{m+1}{2}} \left(\int_{-\infty}^{\infty} e^{-t^2} t^m \mathrm{d}t + O(e^{-c_3 z}) \right) \end{aligned} \tag{1.12}$$

with suitable $c_3 = c_3(M, \tau_0, h) > 0$, which is independent from z and x. The integrals vanish for m odd, while for m even they take the value

$$\int_{-\infty}^{\infty} e^{-t^2} t^m \mathrm{d}t = 2 \int_0^{\infty} e^{-t^2} t^m \mathrm{d}t = \Gamma\left(\frac{m+1}{2} \right). \tag{1.13}$$

Inserting (1.11), (1.12), (1.13) into (1.10) yields

$$\begin{aligned} \tilde{I}_{k\nu}(z,x) &= -e^{2iz} i^{-k} \varepsilon^{-1}(x-i)^{-\nu} \left\{ \sum_{\substack{m=0 \\ 2|m}}^{2M} a_m \Gamma(\tfrac{m+1}{2}) z^{-\frac{m+1}{2}} + O(z^{-M-1}) \right\} \\ &= -e^{-2iz} z^{-\frac{1}{2}} i^{-k} \varepsilon^{-1}(x-i)^{-\nu} \left\{ \sum_{m=0}^{M} a_{2m} \Gamma(m+\tfrac{1}{2}) z^{-m} + O(z^{-M-\frac{1}{2}}) \right\} \\ &= -e^{-2iz} z^{-\frac{1}{2}} i^{-k} \varepsilon^{-1}(x-i)^{-\nu} \left\{ \sum_{m=0}^{M-1} a_{2m} \Gamma(m+\tfrac{1}{2}) z^{-m} + O(z^{-M}) \right\}. \end{aligned} \tag{1.14}$$

This proves the existence of the asymptotic expansion and it remains to compute the coefficients a_m from (1.8). To this end let γ denote a simple closed curve encircling the origin in the positive direction and which lies entirely in C. Then by Cauchy's formula

$$a_m = \frac{1}{2\pi i} \int_\gamma \frac{H\big(p(u)\big) p'(u)}{u^{m+1}} \mathrm{d}u.$$

Here we substitute with the inverse function $p : C \to B$ (i.e. $p(u) = \tau$, $u = g(\tau)$). Then $\beta = p(\gamma)$ is a simple closed curve around the origin lying entirely in B. Furthermore

$$a_m = \frac{1}{2\pi i}\int_\beta \frac{H(\tau)p'(u)}{g(\tau)^{m+1}} g'(\tau)\mathrm{d}t = \frac{1}{2\pi i}\int_\beta \frac{H(\tau)}{g(\tau)^{m+1}}\mathrm{d}\tau. \tag{1.15}$$

By the residue theorem, the last integral equals the coefficient of τ^{-1} in the power series expansion of $H(\tau)g(\tau)^{-m-1}$ (note that $g(\tau) \neq 0$ if $\tau \neq 0$). From (1.3), (1.4) we obtain

$$H(\tau)g(\tau)^{-m-1} = (1-\varepsilon\tau)^{-k+\frac{m+1}{2}}(1-\beta\varepsilon^{-1}\tau)^{-\nu}\tau^{-m-1}.$$

It follows that a_m is given as the coefficient of τ^m of $(1-\varepsilon\tau)^{-k+\frac{m+1}{2}}(1-\beta\varepsilon^{-1}\tau)^{-\nu}$. One readily computes

$$a_m = (-1)^m\varepsilon^m \sum_{\mu=0}^{m} \binom{-\nu}{\mu}\binom{\frac{m+1}{2}-k}{m-\mu}\varepsilon^{-2\mu}\beta^\mu.$$

Thus

$$\alpha_m(x) = a_{2m} = e^{\frac{\pi i m}{2}} \sum_{\mu=0}^{2m} \binom{-\nu}{\mu}\binom{m-k+\frac{1}{2}}{2m-\mu} i^{-\mu}\beta^\mu, \quad \beta = (x-i)^{-1}.$$

This completes the proof of the theorem as the remaining assertions are obvious.

In applications it should be noted that the parameters ν, h, k, τ_1, and τ_2 must be fixed in advance. The asymptotic expansion then will hold for $z \to \infty$ uniformly in x, as stated.

To derive our next result, we now define a sequence of polynomials p_l, q_l, by

$$i^{-l}(t-i)^{-l} = (1+it)^{-l} = \frac{p_l(t)+iq_l(t)}{(t^2+1)^l}, \quad l \geq 0,\ t \in \mathbf{R}. \tag{1.16}$$

The first few polynomials are $p_0(t) = 1$, $q_0(t) = 0$, $p_1(t) = 1$, $q_1(t) = -t$, $p_2(t) = 1 - t^2$, and $q_2(t) = -2t$. Clearly, if $\mu \geq 1$ then p_μ is even and q_μ is odd. Moreover, the following recursions hold:

$$p_{l+1}(t) = p_l(t) + tq_l(t), \quad q_{l+1}(t) = q_l(t) - tp_l(t), \quad l \geq 0.$$

Let z be real, $x \in \mathbf{C}$ with $|x-i| \geq h > 0$. From

$$e^{iz}(1+ix)^{-l} = \big[p_l(x)\cos z - q_l(x)\sin z + ip_l(x)\sin z + q_l(x)\cos z\big](x^2+1)^{-l}$$

we get the following inequalities:

$$\left|\frac{p_l(x)\cos z - q_l(x)\sin z}{(x^2+1)^l}\right| = \left|\mathrm{Re}\big(e^{iz}(1+ix)^{-l}\big)\right| \le h^{-l}, \tag{1.17}$$

$$\left|\frac{p_l(x)\sin z + q_l(x)\cos z}{(x^2+1)^l}\right| = \left|\mathrm{Im}\big(e^{iz}(1+ix)^{-l}\big)\right| \le h^{-l}. \tag{1.18}$$

Replacing x with $-x$, adding and subtracting, respectively, we similarly obtain

$$\left|\frac{p_l(x)}{(x^2+1)^l}\right| \le 2h^{-l}, \quad \left|\frac{q_l(x)}{(x^2+1)^l}\right| \le 2h^{-l}, \quad |x-i|\ge h,\ |x+i|\ge h. \tag{1.19}$$

The next formula is an easy corollary to Lemma 1. We use it in Section 2 to derive the asymptotic expansion of the functions U and V as defined above.

Lemma 2. *Let $k>0$ be an even integer, $h>0$ real. Assume $z\ge 1$, and let x be complex, such that $|x-i|\ge h$ and $|\overline{x}-i|\ge h$. Then*

$$\tilde{I}_{k+1,1}(z,x) - \overline{\tilde{I}_{k+1,1}(z,\overline{x})} = -(-1)^{\frac{k}{2}} 2iz^{-\frac12} \sum_{m=0}^{M-1} \beta_m(z,x) z^{-m} + O(z^{-M-\frac12})$$

for each $M\ge 0$, uniformly in x. The coefficients β_m are given by

$$\beta_m(z,x) =$$
$$\Gamma(m+\tfrac12)\sum_{\mu=0}^{2m}(-1)^\mu \binom{m-k-\frac12}{2m-\mu} \frac{p_{\mu+1}(x)\sin(2z+c_m)+q_{\mu+1}(x)\cos(2z+c_m)}{(x^2+1)^{\mu+1}},$$

where $c_m = \frac{\pi m}{2}-\frac{\pi}{4}$, and the polynomials $p_{\mu+1}, q_{\mu+1}$ are defined by (1.16)*. Moreover, $\beta_m(z,x)$ is uniformly bounded for the values of z and x permitted.*

Proof. The first two formulas follow at once from Lemma 1 with $k+1$ instead of k and $\nu=1$, since

$$\tilde{I}_{k+1,1}(z,x) = -z^{-\frac12}\sum_{m=0}^{M-1}\Gamma(m+\tfrac12)e^{2iz-\frac{\pi ik}{2}+\frac{\pi im}{2}-\frac{\pi i}{4}}z^{-m}$$
$$\times \sum_{\mu=0}^{2m}(-1)^\mu\binom{m-k-\frac12}{2m-\mu}\frac{p_{\mu+1}(x)+iq_{\mu+1}(x)}{(x^2+1)^{\mu+1}}.$$

The remaining assertions are obvious from the properties of $\alpha_m(x)$, $p_l(x)$, and $q_l(x)$ stated above.

2. Incomplete Cusp Forms

It will be seen later that a major role in the derivation of the Riemann-Siegel formula is played by truncations of the Fourier series of the underlying cusp form. Let $f \in S_k$ with Fourier series (0.3). We shall find it convenient in the sequel to work in the right half plane, and hence we define

$$\psi(x) = f(ix) = \sum_{n=1}^{\infty} a(n)e^{-2\pi nx}, \quad \mathrm{Re}(x) > 0. \tag{2.1}$$

Then $\psi(x) = (-1)^{\frac{k}{2}} x^{-k} \psi(\frac{1}{x})$, which shows that $\psi(x)$ decays rapidly at $x = 0$. This fact will be used below.

The partial sums of $\psi(x)$ to be considered here depend on two parameters, $\eta > 0$ real, x complex, and are defined by

$$\psi_1(\eta, x) = \sum_{n>\eta} a(n)e^{-2\pi nx}, \quad \mathrm{Re}(x) > 0, \tag{2.2}$$

and

$$\psi_{-1}(\eta, x) = \sum_{n\leq\eta} a(n)e^{2\pi nx}. \tag{2.3}$$

Obviously, $\psi_{-1}(\eta, \cdot)$ is an entire function, while $\psi_1(\eta, \cdot)$ has the line $\mathrm{Re}(x) = 0$ as a natural boundary. For our purpose we need asymptotic expansions of these functions for large η and x restricted to the sector $|\arg(x)| \leq \frac{\pi}{4}$. In this respect, the behaviour for $x \to 0$ is important. A first approximation is clearly given by

$$\psi_{-1}(\eta, x) \sim \sum_{n\leq\eta} a(n), \quad x \to 0,$$

and by

$$\psi_1(\eta, x) = \sum_{n>\eta} a(n)e^{-2\pi nx} = \psi(x) - \sum_{n\leq\eta} a(n)e^{-2\pi nx} \sim -\sum_{n\leq\eta} a(n),$$
$$x \to 0,\ \mathrm{Re}(x) > 0.$$

Here it has been used that $\psi(x)$ vanishes exponentially for $x \to 0$. For our purposes, however, these formulas are much too crude, and we are going to replace them by much sharper ones. Our final goal is to obtain the remarkable approximations furnished by Theorem 2, which appear to be of independent interest and may have other applications.

To investigate the functions ψ_1 and ψ_{-1} further, we use a variant of the well known Voronoi summation formula [**9**]. In a natural way we are thus led to the functions

$$U(z, x) = e^{zx} \int_z^{\infty} J_k(\sqrt{t})t^{\frac{k}{2}} e^{-xt}\mathrm{d}t, \quad \mathrm{Re}(x) > 0, \tag{2.4}$$

$$V(z, x) = e^{-zx} \int_0^{z} J_k(\sqrt{t})t^{\frac{k}{2}} e^{xt}\mathrm{d}t. \tag{2.5}$$

Employing the results of Section 1, it is an easy task to derive sharp asymptotic expansion for $U(z,x)$, $V(z,x)$, when z is large and x is allowed to vary in the sector $|\arg(x)| \leq \frac{\pi}{4}$ (see Theorem 1). First we transform $\psi_{\pm 1}$ using explicit formulas of Voronoi type. Thus let

$$A(x) = \sum_{n \leq x} a(n), \quad A_1(x) = \int_0^x A(t)\mathrm{d}t = \sum_{n \leq x} a(n)(x-n). \tag{2.6}$$

For the latter sum we only need the explicit formula

$$A_1(x) = (-1)^{\frac{k}{2}} \frac{1}{2\pi} x^{\frac{k+1}{2}} \sum_{n=1}^{\infty} a(n) n^{-\frac{k+1}{2}} J_{k+1}(4\pi\sqrt{nx}), \tag{2.7}$$

which can be proved by elementary means [9]. The series is absolutely convergent, as follows from Deligne's estimate $|a(n)| \leq n^{\frac{k-1}{2}} d(n)$ [**1**] and well known properties of the Bessel function J_{k+1}.

Lemma 3. *Let $\eta > 0$, x complex. For the functions ψ_1 and ψ_{-1} defined by* (2.2) *and* (2.3) *we have*

$$e^{\eta x}\psi_1\left(\eta, \frac{x}{2\pi}\right) = -A(\eta) + \Phi_1(\eta, x), \quad \mathrm{Re}(x) > 0,$$
$$e^{-\eta x}\psi_{-1}\left(\eta, \frac{x}{2\pi}\right) = \quad A(\eta) - \Phi_{-1}(\eta, x),$$

where

$$\Phi_1(\eta, x) = (-1)^{\frac{k}{2}} (4\pi)^{-k-2} x \sum_{n=1}^{\infty} a(n) n^{-k-1} U(\eta_n, x_n),$$
$$\Phi_{-1}(\eta, x) = (-1)^{\frac{k}{2}} (4\pi)^{-k-2} x \sum_{n=1}^{\infty} a(n) n^{-k-1} V(\eta_n, x_n),$$

and $\eta_n = 16\pi^2 n\eta$, $x_n = \frac{x}{16\pi^2 n}$.

Proof. Assume $\eta > 0$, $\mathrm{Re}(x) > 0$. By partial summation

$$\psi_1\left(\eta, \frac{x}{2\pi}\right) = \sum_{n > \eta} a(n) e^{-nx} = -A(\eta)e^{-\eta x} + x\int_\eta^\infty A(t)e^{-xt}\mathrm{d}t.$$

The integral equals

$$\int_\eta^\infty e^{-xt}\mathrm{d}A_1(t) = -A_1(\eta)e^{-\eta x} + x\int_\eta^\infty A_1(t)e^{-xt}\mathrm{d}t.$$

Inserting the expression (2.7) for $A_1(t)$ and interchanging the order of integration and summation then yields

$$x\int_\eta^\infty A_1(t)e^{-xt}\mathrm{d}t =$$
$$(-1)^{\frac{k}{2}} \frac{x}{2\pi} \sum_{n=1}^{\infty} a(n) n^{-\frac{k+1}{2}} (16\pi^2 n)^{-1-\frac{k+1}{2}} \int_{\eta_n}^\infty u^{\frac{k+1}{2}} J_{k+1}(\sqrt{u}) e^{-x_n u}\mathrm{d}u.$$

From the familiar differential equation of the Bessel function we deduce $\frac{\mathrm{d}}{\mathrm{d}u}[u^{\frac{k}{2}}J_k(\sqrt{u})] = \frac{1}{2}u^{\frac{k-1}{2}}J_{k-1}(\sqrt{u})$. Thus integration by parts shows that

$$\begin{aligned} x\int_\eta^\infty A_1(t)e^{-xt}\mathrm{d}t &= (-1)^{\frac{k}{2}}\frac{x}{2\pi}\sum_{n=1}^\infty a(n)n^{-\frac{k+1}{2}}(16\pi^2 n)^{-1-\frac{k+1}{2}}\times \\ &\qquad \times\left\{\eta_n^{\frac{k+1}{2}}J_{k+1}(\sqrt{\eta_n})\frac{e^{-x_n\eta_n}}{x_n}+\frac{1}{2x_n}\int_{\eta_n}^\infty u^{\frac{k}{2}}J_k(\sqrt{u})e^{-x_n u}\mathrm{d}u\right\} \\ &= (-1)^{\frac{k}{2}}\frac{x}{2\pi}\eta^{\frac{k+1}{2}}\frac{e^{-\eta x}}{x}\sum_{n=1}^\infty a(n)n^{-\frac{k+1}{2}}J_{k+1}(4\pi\sqrt{n\eta})+ \\ &\quad +(-1)^{\frac{k}{2}}\frac{1}{4\pi}\sum_{n=1}^\infty a(n)n^{-\frac{k+1}{2}}(16\pi^2 n)^{-\frac{k+1}{2}}\int_{\eta_n}^\infty u^{\frac{k}{2}}J_k(\sqrt{u})e^{-x_n u}\mathrm{d}u \\ &= e^{-\eta x}A_1(\eta)+(-1)^{\frac{k}{2}}(4\pi)^{-k-2}e^{-\eta x}\sum_{n=1}^\infty a(n)n^{-k-1}U(\eta_n,x_n). \end{aligned}$$

Altogether we have

$$x\int_\eta^\infty A(t)e^{-xt}\mathrm{d}t = (-1)^{\frac{k}{2}}x(4\pi)^{-k-2}e^{-\eta x}\sum_{n=1}^\infty a(n)n^{-k-1}U(\eta_n,x_n).$$

This immediately implies the assertion for ψ_1. Similarly we proceed in the case of ψ_{-1}. Summation by parts and then integration by parts yields

$$\psi_{-1}\left(\eta,\frac{x}{2\pi}\right) = A(\eta)e^{\eta x}-A_1(\eta)xe^{\eta x}+x^2\int_0^\eta A_1(t)e^{xt}\mathrm{d}t.$$

Here

$$\int_0^\eta A_1(t)e^{xt}\mathrm{d}t = (-1)^{\frac{k}{2}}\frac{1}{2\pi}\sum_{n=1}^\infty a(n)n^{-\frac{k+1}{2}}(16\pi^2 n)^{-1-\frac{k+1}{2}}\int_0^{\eta_n}u^{\frac{k+1}{2}}J_{k+1}(\sqrt{u})e^{x_n u}\mathrm{d}u.$$

Integrating by parts, we see

$$\int_0^{\eta_n}u^{\frac{k+1}{2}}J_{k+1}(\sqrt{u})e^{x_n u}\mathrm{d}u = \eta_n^{\frac{k+1}{2}}J_{k+1}(\sqrt{\eta_n})\frac{e^{\eta_n x_n}}{x_n}-\frac{1}{2x_n}\int_0^{\eta_n}u^{\frac{k}{2}}J_k(\sqrt{u})e^{x_n u}\mathrm{d}u.$$

Consequently,

$$x^2\int_0^\eta A_1(t)e^{xt}\mathrm{d}t = xe^{\eta x}A_1(\eta)-(-1)^{\frac{k}{2}}\frac{x}{4\pi}\sum_{n=1}^\infty a(n)n^{-\frac{k+1}{2}}(16\pi^2 n)^{-\frac{k+1}{2}}e^{\eta x}V(\eta_n,x_n),$$

and this completes the proof of the lemma.

After this preliminary transformation, showing the appearance of $U(z,x)$ and $V(z,x)$, we are now prepared to find asymptotic expansions for these functions, which are defined by (2.4) and (2.5), respectively. In fact, the defining integrals can be expressed in terms of the integral $\tilde{I}_{k+1,1}$ introduced in Section 1. To show this, we use the familiar formulas $J_k(t) = \frac{1}{2}\big[H_k^{(1)}(t) + H_k^{(2)}(t)\big]$ and $H_k^{(2)}(t) = \overline{H_k^{(1)}(t)}$ [**19**, p. 74]. Then

$$U(z,x) = \frac{1}{2}\left[e^{xz}\int_z^\infty t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{-xt}\mathrm{d}t + \overline{e^{\overline{x}z}\int_z^\infty t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{-\overline{x}t}\mathrm{d}t}\right], \tag{2.8}$$

and

$$V(z,x) = \frac{1}{2}\left[e^{-xz}\int_0^z t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{xt}\mathrm{d}t + \overline{e^{-\overline{x}z}\int_0^z t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{\overline{x}t}\mathrm{d}t}\right]. \tag{2.9}$$

The two relevant integrals occurring here are treated in the next two lemmata.

Lemma 4. *Let $z \geq 1$, $|\arg(x)| \leq \frac{\pi}{4}$. Then*

$$e^{xz}\int_z^\infty t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{-xt}\mathrm{d}t = z^{\frac{k+1}{2}}\frac{2}{\pi i}\tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, 2x\sqrt{z}\right) + O(e^{-\sqrt{z}/10})$$

uniformly in x. Moreover, the parameters in $\tilde{I}_{k+1,1}$ can be chosen according to $h = \frac{1}{\sqrt{2}}$, $\tau_1 = \frac{\sqrt{2}}{1+\tan\delta_1}$, $\tau_2 = \frac{\sqrt{2}}{\cot\delta_2 - 1}$, where $\delta_1 = \frac{3\pi}{8}$, $\delta_2 = \frac{\pi}{12}$.

Proof. Assume first $\mathrm{Re}(x) \geq 1$. We use Schläfli's integral for the Hankel function $H_\nu^{(1)}$ in the form [**19**, p. 179]

$$H_\nu^{(1)}(w) = (2w)^{-\nu}\frac{1}{\pi i}\int_0^{i\infty} e^{w^2 s - \frac{1}{4s}} s^{-\nu-1}\mathrm{d}s, \quad w > 0,\ \nu > 0.$$

The path of integration is chosen so as to run from 0 to $\frac{1}{2}i$ along the half-circle $s = \frac{1}{4}(i + e^{i\psi})$, $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$, and then from $\frac{1}{2}i$ to $i\infty$ on the positive imaginary axis. The integral is absolutely convergent at both limits of integration. Note that $\mathrm{Re}(s) \leq \frac{1}{4}$ on the path and hence $\mathrm{Re}(x-s) \geq \frac{3}{4}$. Writing the above expression for $H_k^{(1)}$ in the form

$$t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t}) = 2^{-k}\frac{1}{\pi i}\int_0^{i\infty} e^{ts - \frac{1}{4s}} s^{-k-1}\mathrm{d}s, \quad t \geq 0,\ k > 0, \tag{2.10}$$

we get

$$\begin{aligned} e^{xz}\int_z^\infty t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{-xt}\mathrm{d}t &= 2^{-k}e^{xz}\frac{1}{\pi i}\int_0^{i\infty} e^{-\frac{1}{4s}} s^{-k-1}\int_z^\infty e^{-t(x-s)}\mathrm{d}t\mathrm{d}s \\ &= 2^{-k}\frac{1}{\pi i}\int_0^{i\infty} e^{zs - \frac{1}{4s}} s^{-k-1}\frac{\mathrm{d}s}{x-s}. \end{aligned}$$

The interchange of the order of integration is permitted by absolute convergence. Substituting $\frac{s}{2\sqrt{z}}$ for s yields

$$e^{xz}\int_z^\infty t^{\frac{k}{2}}H_k^{(1)}(\sqrt{t})e^{-xt}\mathrm{d}t = z^{\frac{k+1}{2}}\frac{2}{\pi i}\int_0^{i\infty} e^{\frac{1}{2}\sqrt{z}(s-\frac{1}{s})}s^{-k-1}\frac{\mathrm{d}s}{2x\sqrt{z}-s}. \tag{2.11}$$

Our initial restriction $\mathrm{Re}(x) \geq 1$ can now be relaxed to $|\arg(x)| \leq \frac{\pi}{4}$ by analytic continuation. Let $s_1 = i + \tau_1 e^{-\frac{\pi i}{4}}$, $s_2 = i - \tau_2 e^{-\frac{\pi i}{4}}$. Then s_1, s_2 are the points of intersection of the line $s = i + \tau e^{-\frac{\pi i}{4}}$ (τ real) with the rays $\rho e^{i\delta_1}$ and $\rho e^{i(\frac{\pi}{2}+\delta_2)}$ ($\rho \geq 0$), respectively. Moreover,

$$\rho_1 := |s_1| = \frac{1}{\sin\delta_1 + \cos\delta_1}, \quad \rho_2 := |s_2| = \frac{1}{\cos\delta_2 - \sin\delta_2} = \sqrt{2}.$$

Numerical values, rounded to three decimal places, are $\tau_1 = 0.414$, $\tau_2 = 0.518$, $\rho_1 = 0.765$, $\rho_2 = 1.414$. Therefore the conditions of Lemma 1 are satisfied. With these parameters we define a path $P = P_0 \cup P_1 \cup P_2$ consisting of straight line segments P_0, P_1, P_2, connecting the points 0, s_1, s_2, and $\infty e^{i(\frac{\pi}{2}+\delta_2)}$. By Cauchy's theorem, the path of integration in (2.11) can be replaced by P. Hence

$$e^{xz}\int_z^\infty t^{\frac{k}{2}}H_k^{(1)}(\sqrt{t})e^{-xt}\mathrm{d}t = z^{\frac{k+1}{2}}\frac{2}{\pi i}\int_P e^{\frac{1}{2}\sqrt{z}(s-\frac{1}{s})}s^{-k-1}\frac{\mathrm{d}s}{2x\sqrt{z}-s}, \quad |\arg(x)| \leq \frac{\pi}{4}. \tag{2.12}$$

Thus the lemma is proved if we show that the integrals along P_0 and P_2 are sufficiently small. Consider first

$$\int_{P_0} e^{z(s-\frac{1}{s})}s^{-\nu}\frac{\mathrm{d}s}{x-s} = \int_0^{s_1} e^{z(s-\frac{1}{s})}s^{-\nu}\frac{\mathrm{d}s}{x-s}, \quad z \geq A > 0,\ \nu > 0,\ |\arg(x)| \leq \frac{\pi}{4}, \tag{2.13}$$

where A is a fixed positive number. Let $s = \rho e^{i\delta_1}$, $0 \leq \rho \leq \rho_1$. If u, ψ are real, $u \geq 0$, $|\psi| \leq \pi$, then $|1 - ue^{i\psi}| \geq |\sin\psi|$. Thus for $x = re^{i\psi}$, $r \geq 0$, $|\psi| \leq \frac{\pi}{4}$,

$$|x-s| = |re^{i\psi} - \rho e^{i\delta_1}| = \rho\left|1 - \frac{r}{\rho}e^{i(\psi-\delta_1)}\right| \geq \rho|\sin(\psi-\delta_1)| \geq \rho\sin\frac{\pi}{8}.$$

We also have $\mathrm{Re}(s - s^{-1}) = (\rho - \rho^{-1})\cos\delta_1$. Therefore,

$$\begin{aligned}\left|\int_{P_0} e^{z(s-\frac{1}{s})}s^{-\nu}\frac{\mathrm{d}s}{x-s}\right| &\leq \csc\frac{\pi}{8}\int_0^{\rho_1}\exp\bigl[z(\rho-\rho^{-1})\cos\delta_1\bigr]\rho^{-\nu-1}\mathrm{d}\rho \\ &\ll \int_{\rho_1^{-1}}^\infty \exp\bigl[-z(u-u^{-1})\cos\delta_1\bigr]u^{\nu-1}\mathrm{d}u \\ &\leq \int_{\rho_1^{-1}}^\infty \exp\bigl[-zu(1-\rho_1^2)\cos\delta_1\bigr]u^{\nu-1}\mathrm{d}u \\ &\ll \exp\bigl[-z(\rho_1^{-1}-\rho_1)\cos\delta_1\bigr]\int_0^\infty \exp\bigl[-zu(1-\rho_1^2)\cos\delta_1\bigr]u^{\nu-1}\mathrm{d}u \\ &\ll e^{-c_1 z}, \quad c_1 = (\rho_1^{-1}-\rho_1)\cos\delta_1,\end{aligned} \tag{2.14}$$

since $z \geq A > 0$. Finally we consider

$$\int_{P_2} e^{z(s-\frac{1}{s})} s^{-\nu} \frac{\mathrm{d}s}{x-s} = \int_{s_2}^{\infty e^{i\psi}} e^{z(s-\frac{1}{s})} s^{-\nu} \frac{\mathrm{d}s}{x-s}, \quad z \geq A > 0,\ |\arg(x)| \leq \frac{\pi}{4}. \tag{2.15}$$

Here we have set $\psi = \frac{\pi}{2} + \delta_2 = \arg(s_2)$ for brevity. We have now $|x-s| \geq |\operatorname{Re}(s)| \geq |\operatorname{Re}(s_2)| = |\rho_2 \cos\psi| = (\cos\delta_2 - \sin\delta_2)^{-1} \sin\delta_2$. Using the parametrization $s = \rho e^{i\psi}$, $\rho \geq \rho_2$, we have $\operatorname{Re}(s - s^{-1}) = -(\rho - \rho^{-1}) \sin\delta_2$ and $|s| > 1$. Thus

$$\begin{aligned} \left| \int_{P_2} e^{z(s-\frac{1}{s})} s^{-\nu} \frac{\mathrm{d}s}{x-s} \right| &\leq (\cos\delta_2 - \sin\delta_2) \csc\delta_2 \int_{\rho_2}^{\infty} \exp\left[-z(\rho - \rho^{-1}) \sin\delta_2\right] \mathrm{d}\rho \\ &\leq (\cot\delta_2 - 1) \int_{\rho_2}^{\infty} \exp\left[-z\rho(1 - \rho_2^{-2}) \sin\delta_2\right] \mathrm{d}\rho \\ &\ll e^{-c_2 z}, \quad c_2 = \rho_2 - \rho_2^{-1}. \end{aligned} \tag{2.16}$$

From the values given above we compute $c_1 = 0.207\ldots$, $c_2 = 2^{-\frac{1}{2}}$. By (2.14) and (2.16) we thus have proved

$$\int_{P_0 \cup P_2} e^{z(s-\frac{1}{s})} \frac{\mathrm{d}s}{x-s} = O(e^{-0.207z}), \quad z \geq A > 0,\ |\arg(x)| \leq \frac{\pi}{4},$$

uniformly in x, and this completes the proof of the lemma.

Lemma 5. *Let $z \geq 1$, $|\arg(x)| \leq \frac{\pi}{4}$. Then*

$$\begin{aligned} e^{-xz} \int_0^z t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t}) e^{xt} \mathrm{d}t = -z^{\frac{k+1}{2}} \frac{2}{\pi i} \tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, -2x\sqrt{z}\right) \\ + \frac{2i}{\pi} e^{-xz} \int_0^{\infty} u^{\frac{k}{2}} K_k(\sqrt{u}) e^{-xu} \mathrm{d}u + O(e^{-\sqrt{z}/10}) \end{aligned}$$

uniformly in x. Moreover, the parameters in $\tilde{I}_{k+1,1}$ can be chosen according to $h = \frac{1}{\sqrt{2}}$, $\tau_1 = \frac{\sqrt{2}}{1+\tan\delta_1}$, $\tau_2 = \frac{\sqrt{2}}{\cot\delta_2 - 1}$, where $\delta_1 = \frac{3\pi}{8}$, $\delta_2 = \frac{\pi}{12}$.

Proof. First let $\operatorname{Re}(x) \geq 1$. Using the asymptotic expansion of the Hankel function, namely [**10, 19**]

$$H_k^{(1)}(w) = \left(\frac{2}{\pi w}\right)^{\frac{1}{2}} e^{iw - \frac{\pi i k}{2} - \frac{\pi i}{4}} \left[1 + O(w^{-1})\right], \quad |\arg(w)| \leq \pi - \delta < \pi,$$

we can write

$$\int_0^z t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t}) e^{xt} \mathrm{d}t = \left(\int_0^{\infty e^{\pi i}} - \int_z^{\infty e^{\pi i}}\right) t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t}) e^{xt} \mathrm{d}t.$$

Here the paths of integration remain entirely in the upper half plane. Let $t = e^{\pi i}u$. Since $H_k^{(1)}(we^{\frac{\pi i}{2}}) = \frac{2}{\pi i}e^{-\frac{\pi i k}{2}}K_k(w)$ [**10**, p. 109], we obtain

$$\int_0^{\infty e^{\pi i}} t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{xt}\mathrm{d}t = \frac{2i}{\pi}\int_0^{\infty} u^{\frac{k}{2}} K_k(\sqrt{u})e^{-xu}\mathrm{d}u.$$

To treat the second integral from z to $\infty e^{\pi i}$ we again use Schläfli's representation (2.10). From $\mathrm{Re}(x+s) \geq 1$ we get

$$\begin{aligned}\int_z^{\infty e^{\pi i}} t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{xt}\mathrm{d}t &= 2^{-k}\frac{1}{\pi i}\int_0^{i\infty} e^{-\frac{1}{4s}}s^{-k-1}\int_z^{\infty e^{\pi i}} e^{t(x+s)}\mathrm{d}t\mathrm{d}s \\ &= -2^{-k}e^{xz}\frac{1}{\pi i}\int_0^{i\infty} e^{sz-\frac{1}{4s}}s^{-k-1}\frac{\mathrm{d}s}{x+s}.\end{aligned}$$

Hence

$$\begin{aligned}&e^{-xz}\int_0^z t^{\frac{k}{2}} H_k^{(1)}(\sqrt{t})e^{xt}\mathrm{d}t = \\ &\quad z^{\frac{k+1}{2}}\frac{2}{\pi i}\int_0^{i\infty} e^{\frac{1}{2}\sqrt{z}(s-\frac{1}{s})}s^{-k-1}\frac{\mathrm{d}s}{2x\sqrt{z}+s} + \frac{2i}{\pi}e^{-xz}\int_0^{\infty} u^{\frac{k}{2}}K_k(\sqrt{u})e^{-xu}\mathrm{d}u.\end{aligned}$$

The first integral is now taken along the path P, as defined in the proof of the previous lemma. By analytic continuation, the formula then holds for $|\arg(x)| \leq \frac{\pi}{4}$. The contribution over the paths P_0 and P_2 is estimated as before, the result being

$$\int_{P_0\cup P_2} e^{\frac{1}{2}\sqrt{z}(s-\frac{1}{s})}s^{-k-1}\frac{\mathrm{d}s}{2x\sqrt{z}+s} = O(e^{-\sqrt{z}/10})$$

uniformly in x. But since

$$\int_{P_1} e^{\frac{1}{2}\sqrt{z}(s-\frac{1}{s})}s^{-k-1}\frac{\mathrm{d}s}{2x\sqrt{z}+s} = -\tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, -2x\sqrt{z}\right)$$

the assertion follows.

It is now an easy task to derive asymptotic expansions for $U(z,x)$, $V(z,x)$, and certain related functions. We summarize the formulas in the following

THEOREM 1. *Let $z \geq 4$, $|\arg(x)| \leq \frac{\pi}{4}$. Then*

$$\begin{aligned}U(z,x) &= -(-1)^{\frac{k}{2}}2^{\frac{3}{2}}\pi^{-1}z^{\frac{k}{2}+\frac{1}{4}}\sum_{m=0}^{M-1}\beta_m\left(\frac{1}{2}\sqrt{z}, 2x\sqrt{z}\right)\left(\frac{z}{4}\right)^{-\frac{m}{2}} + O(z^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}), \\ V(z,x) &= (-1)^{\frac{k}{2}}2^{\frac{3}{2}}\pi^{-1}z^{\frac{k}{2}+\frac{1}{4}}\sum_{m=0}^{M-1}\beta_m\left(\frac{1}{2}\sqrt{z}, -2x\sqrt{z}\right)\left(\frac{z}{4}\right)^{-\frac{m}{2}} + O(z^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}),\end{aligned}$$

for each fixed $M \geq 0$, uniformly in x. The coefficients β_m are given by

$$\beta_m(z,x) =$$
$$\Gamma(m+\tfrac{1}{2}) \sum_{\mu=0}^{2m} (-1)^\mu \binom{m-k-\frac{1}{2}}{2m-\mu} \frac{p_{\mu+1}(x)\sin(2z+c_m) + q_{\mu+1}(x)\cos(2z+c_m)}{(x^2+1)^{\mu+1}},$$

where $c_m = \frac{\pi m}{2} - \frac{\pi}{4}$ and the polynomials p_l, q_l are given by (1.16). *Moreover, $\beta_m(z,x)$ is uniformly bounded for $z \geq 1$ and complex x with $|\arg(x)| \leq \frac{\pi}{4}$.*

Proof. To derive the result for $U(z,x)$, use (2.8) and Lemma 4. Accordingly

$$U(z,x) = \frac{z^{\frac{k+1}{2}}}{\pi i}\left[\tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, 2x\sqrt{z}\right) - \overline{\tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, 2\overline{x}\sqrt{z}\right)}\right] + O(e^{-\sqrt{z}/10}),$$

uniformly in x. Then the formula stated follows from Lemma 2. Similarly, (2.9) and Lemma 5 show that

$$V(z,x) = -\frac{z^{\frac{k+1}{2}}}{\pi i}\left[\tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, -2x\sqrt{z}\right) - \overline{\tilde{I}_{k+1,1}\left(\frac{1}{2}\sqrt{z}, -2\overline{x}\sqrt{z}\right)}\right] + O(e^{-\sqrt{z}/10}),$$

since the two infinite integrals coming from Lemma 5 cancel. This completes the proof of our assertion.

We apply the previous result to Lemma 3. With $\eta_n = 16\pi^2 n\eta$, $x_n = \frac{x}{16\pi^2 n}$, we get

$$\begin{aligned} U(\eta_n, x_n) &= -(-1)^{\frac{k}{2}} 2^{\frac{5}{2}} \pi^{-\frac{1}{2}} (4\pi)^k (n\eta)^{\frac{k}{2}+\frac{1}{4}} \sum_{m=0}^{M-1} \beta_m\Big(2\pi\sqrt{n\eta}, \frac{x}{2\pi}\sqrt{\eta/n}\Big)(4\pi^2 n\eta)^{-\frac{m}{2}} \\ &\qquad + O\big((n\eta)^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}\big), \\ V(\eta_n, x_n) &= (-1)^{\frac{k}{2}} 2^{\frac{5}{2}} \pi^{-\frac{1}{2}} (4\pi)^k (n\eta)^{\frac{k}{2}+\frac{1}{4}} \sum_{m=0}^{M-1} \beta_m\Big(2\pi\sqrt{n\eta}, -\frac{x}{2\pi}\sqrt{\eta/n}\Big)(4\pi^2 n\eta)^{-\frac{m}{2}} \\ &\qquad + O\big((n\eta)^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}\big). \end{aligned}$$

Here $\beta_m(z,\cdot)$ are the rational functions defined in Lemma 2. Their only poles occur at $x = \pm i$. Inserting these formulas into those of Lemma 3 we get the main result of the present section.

Theorem 2. *Let $\eta \geq 1$, x complex such that $|\arg(x)| \leq \frac{\pi}{4}$. Define the functions Φ_1 and Φ_{-1} as in Lemma 3. Then*

$$\Phi_1(\eta, x) =$$
$$-2^{-\frac{3}{2}}\pi^{-\frac{5}{2}} x \eta^{\frac{k}{2}+\frac{1}{4}} \sum_{m=0}^{M-1} (4\pi^2\eta)^{-\frac{m}{2}} \sum_{n=1}^{\infty} a(n) n^{-\frac{k}{2}-\frac{3}{4}-\frac{m}{2}} \beta_m\left(2\pi\sqrt{n\eta}, \frac{x}{2\pi}\sqrt{\eta/n}\right)$$
$$+ O(|x|\eta^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}),$$

$$\Phi_{-1}(\eta,x)=$$
$$2^{-\frac{3}{2}}\pi^{-\frac{5}{2}}x\eta^{\frac{k}{2}+\frac{1}{4}}\sum_{m=0}^{M-1}(4\pi^2\eta)^{-\frac{m}{2}}\sum_{n=1}^{\infty}a(n)n^{-\frac{k}{2}-\frac{3}{4}-\frac{m}{2}}\beta_m\left(2\pi\sqrt{n\eta},\frac{x}{2\pi}\sqrt{\eta/n}\right)$$
$$+O(|x|\eta^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}),$$

for each integer $M\geq 0$, uniformly in x. The infinite series are absolutely convergent and uniformly bounded for the permissible values of η and x. The coefficients β_m are as in Theorem 1.

Proof. It is only necessary to insert the above formulas for $U(\eta_n,x_n)$ and $V(\eta_n,x_n)$ into Lemma 3. Since $|a(n)|\leq d(n)n^{\frac{k-1}{2}}$, absolute convergence is ensured by the boundedness of the coefficients β_m. This proves the theorem.

As a special case we note

$$\Phi_1(\eta,x)=O(|x|\eta^{\frac{k}{2}+\frac{1}{4}}),\quad \Phi_{-1}(\eta,x)=O(|x|\eta^{\frac{k}{2}+\frac{1}{4}}),\tag{2.17}$$

subject to $\eta\geq 1$ and $|\arg(x)|\leq\frac{\pi}{4}$.

In the next section we also need the two functions $\Phi^{(+)}$, $\Phi^{(-)}$, defined by

$$\Phi^{(\pm)}(\eta,x)=\Phi_1(\eta,x)\pm\Phi_{-1}(\eta,x),\quad \eta>0,\ \mathrm{Re}(x)>0.\tag{2.18}$$

Using the previous result it is a trivial matter to derive formulas of the above type for $\Phi^{(\pm)}$. With $\beta_m(z,x)$ as in Lemma 2, let $\beta_m^{(\pm)}(z,x)=\frac{1}{2}\left[\beta_m(z,x)\pm\beta_m(z,-x)\right]$. Using the fact that p_l is even, q_l is odd, we thus get

$$\beta_m^{(+)}(z,x)=\Gamma(m+\tfrac{1}{2})\sin(2z+\tfrac{\pi m}{2}-\tfrac{\pi}{4})\sum_{\mu=0}^{2m}(-1)^{\mu}\binom{m-k-\frac{1}{2}}{2m-\mu}\frac{p_{\mu+1}(x)}{(x^2+1)^{1+\mu}},\tag{2.19}$$

$$\beta_m^{(-)}(z,x)=\Gamma(m+\tfrac{1}{2})\cos(2z+\tfrac{\pi m}{2}-\tfrac{\pi}{4})\sum_{\mu=0}^{2m}(-1)^{\mu}\binom{m-k-\frac{1}{2}}{2m-\mu}\frac{q_{\mu+1}(x)}{(x^2+1)^{1+\mu}}.\tag{2.20}$$

We then have

Theorem 3. *Let $\eta\geq 1$, x complex such that $|\arg(x)|\leq\frac{\pi}{4}$. Define the functions $\Phi^{(+)}$ and $\Phi^{(-)}$ by $\Phi^{(\pm)}(\eta,x)=\Phi_1(\eta,x)\pm\Phi_{-1}(\eta,x)$, where $\Phi_{\pm 1}$ are defined in Lemma 3. Then*

$$\Phi^{(\pm)}(\eta,x)=$$
$$-2^{-\frac{1}{2}}\pi^{-\frac{5}{2}}x\eta^{\frac{k}{2}+\frac{1}{4}}\sum_{m=0}^{M-1}(4\pi^2\eta)^{-\frac{m}{2}}\sum_{n=1}^{\infty}a(n)n^{-\frac{k}{2}-\frac{3}{4}-\frac{m}{2}}\beta_m^{(\mp)}\left(2\pi\sqrt{n\eta},\frac{x}{2\pi}\sqrt{\eta/n}\right)$$
$$+O(|x|\eta^{\frac{k}{2}+\frac{1}{4}-\frac{M}{2}}),$$

for each integer $M \geq 0$, uniformly in x. The infinite series are absolutely convergent and uniformly bounded for the permissible values of η and x.

3. The Riemann-Siegel Formula

We are now ready to derive our main results on the asymptotic expansion of $\varphi(s)$, i.e. the analogue of the Riemann-Siegel formula. We take $f \in S_k$ and define $\psi(x) = f(ix)$ as in (2.1)

Let $s = \sigma + it$. We shall assume in the sequel $0 < \sigma_0 \leq \sigma \leq \sigma_1$, where σ_0, σ_1 are fixed, $t \geq 2\pi$, and we define $\eta = \frac{t}{2\pi}$. As we have shown elsewhere [**3, 4**], $\varphi(s)$ is determined by

$$T(s) = (2\pi)^s \Gamma(s)^{-1} \int_0^\infty \psi(x)(i+x)^{s-1}\mathrm{d}x, \tag{3.1}$$

in virtue of the formula

$$\varphi(s) = T(s) + (-1)^{\frac{k}{2}} X(s)\overline{T(k-\overline{s})}, \quad X(s) = (2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}. \tag{3.2}$$

Hence we seek an asymptotic expansion for $T(s)$ and our main result is Theorem 4 below. We first require some preliminary work transforming $T(s)$.

Write $\psi(x) = \psi_{-1}(\eta, -x) + \psi_1(\eta, x)$, where $\psi_{\pm 1}$ are the incomplete cusp forms, as defined in Section 2. The integral involving ψ_{-1} is transformed according to

$$\begin{aligned}
\int_0^\infty \psi_{-1}(\eta, -x)(i+x)^{s-1}\mathrm{d}x &= \int_i^{i+\infty} \psi_{-1}(\eta, -x)x^{s-1}\mathrm{d}x \\
&= \left(\int_0^i + \int_i^{i+\infty}\right)\psi_{-1}(\eta, -x)x^{s-1}\mathrm{d}x \\
&= \sum_{n\leq\eta} a(n)\int_0^\infty e^{-2\pi n x}x^{s-1}\mathrm{d}x - \int_0^i \psi_{-1}(\eta, w-i)(i-w)^{s-1}\mathrm{d}w \\
&= (2\pi)^{-s}\Gamma(s)\sum_{n\leq\eta} a(n)n^{-s} - \int_0^i \psi_{-1}(\eta, x)(i-x)^{s-1}\mathrm{d}x.
\end{aligned}$$

We therefore have

$$\begin{aligned}
&T(s) = \\
&\sum_{n\leq\eta} a(n)n^{-s} + (2\pi)^s\Gamma(s)^{-1}\left[\int_0^\infty \psi_1(\eta, x)(i+x)^{s-1}\mathrm{d}x - \int_0^i \psi_{-1}(\eta, x)(i-x)^{s-1}\mathrm{d}x\right].
\end{aligned} \tag{3.3}$$

Next, the functions $\Phi_{\pm 1}$ from Lemma 3 are used, namely

$$\begin{aligned}
\psi_1(\eta, x) &= e^{-2\pi\eta x}\big[-A(\eta) + \Phi_1(\eta, 2\pi x)\big], \\
\psi_{-1}(\eta, x) &= e^{2\pi\eta x}\big[A(\eta) - \Phi_{-1}(\eta, 2\pi x)\big].
\end{aligned} \tag{3.4}$$

Hence

$$\int_0^\infty \psi_1(\eta,x)(i+x)^{s-1}\mathrm{d}x - \int_0^i \psi_{-1}(\eta,x)(i-x)^{s-1}\mathrm{d}x$$
$$= -A(\eta)\left[\int_0^\infty e^{-2\pi\eta x}(i+x)^{s-1}\mathrm{d}x + \int_0^i e^{2\pi\eta x}(i-x)^{s-1}\mathrm{d}x\right]$$
$$+ \int_0^\infty e^{-2\pi\eta x}(i+x)^{s-1}\Phi_1(\eta,2\pi x)\mathrm{d}x + \int_0^i e^{2\pi\eta x}(i-x)^{s-1}\Phi_{-1}(\eta,2\pi x)\mathrm{d}x.$$

Since

$$\int_0^i e^{2\pi\eta x}(i-x)^{s-1}\mathrm{d}x = e^{2\pi i\eta}\int_0^i e^{-2\pi\eta x}x^{s-1}\mathrm{d}x,$$
$$\int_0^\infty e^{-2\pi\eta x}(i+x)^{s-1}\mathrm{d}x = e^{2\pi i\eta}\int_i^{i+\infty} e^{-2\pi\eta x}x^{s-1}\mathrm{d}x,$$

we see that the bracketed term in (3.3) equals $-A(\eta)e^{2\pi i\eta}(2\pi\eta)^{-s}\Gamma(s)$. Inserting this into (3.3), and observing $2\pi\eta = t$, we obtain

$$T(s) = \sum_{n\le\eta} a(n)n^{-s} - A(\eta)e^{it}\eta^{-s} + $$
$$+(2\pi)^s\Gamma(s)^{-1}\left[\int_0^\infty e^{-tx}(i+x)^{s-1}\Phi_1(\eta,2\pi x)\mathrm{d}x\right.$$
$$\left. + \int_0^i e^{tx}(i-x)^{s-1}\Phi_{-1}(\eta,2\pi x)\mathrm{d}x\right]. \quad (3.5)$$

We proceed to show that the main contribution to the last two integrals comes from the part where $|x| \le \frac{1}{2}$. Consider first the finite integral. Let $x_1 = \frac{1}{2}e^{\frac{\pi i}{4}} = \frac{\sqrt{2}}{4}(1+i)$. Then

$$\int_{x_1}^i e^{tx}(i-x)^{s-1}\Phi_{-1}(\eta,2\pi x)\mathrm{d}x = e^{it}\int_0^{i-x_1} e^{-tx}x^{s-1}\left[A(\eta) - e^{-it+tx}\psi_{-1}(\eta,-x)\right]\mathrm{d}x,$$

on using (3.4). Let $\delta = \arg(i-x_1) = \pi - \arctan\frac{4-\sqrt{2}}{\sqrt{2}}$, and $x = \rho e^{i\delta}$, $0 \le \rho \le |i-x_1| = \frac{1}{2}(5-2\sqrt{2})^{\frac{1}{2}}$. Hence

$$\int_0^{i-x_1} e^{-tx}x^{s-1}\mathrm{d}x = e^{i\delta s}\int_0^{|i-x_1|}\exp\left[-t\rho\left(\cos\delta + i\sin\delta\right)\right]\rho^{s-1}\mathrm{d}\rho,$$

and consequently

$$\left|\int_0^{i-x_1} e^{-tx}x^{s-1}\mathrm{d}x\right| \le e^{-\delta t}\int_0^{|i-x_1|}\exp\left(-t\rho\cos\delta\right)\rho^{\sigma-1}\mathrm{d}\rho$$
$$\le \exp\left(-\delta t - t|i-x_1|\cos\delta\right)\int_0^{|i-x_1|}\rho^{\sigma-1}\mathrm{d}\rho$$
$$\le |i-x_1|^\sigma\sigma^{-1}\exp\left(-\delta t - t|i-x_1|\cos\delta\right).$$

The same inequality holds for t replaced by $2\pi n$, since $2\pi n \le 2\pi\eta \le t$ in the sum for ψ_{-1}. Noting that $|i-x_1|\cos\delta = \mathrm{Re}(i-x_1) = -\frac{\sqrt{2}}{4}$, we thus find

$$\begin{aligned}\left|\int_{x_1}^{i} e^{tx}(i-x)^{s-1}\Phi_{-1}(\eta,2\pi x)\mathrm{d}x\right| \\ &\le \Big(|A(\eta)| + \sum_{n\le\eta}|a(n)|\Big)|i-x_1|^{\sigma}\sigma^{-1}\exp\big(-\delta t - t|i-x_1|\cos\delta\big) \\ &\ll t^{C}e^{-\frac{\pi t}{2}-c_1 t}, \quad c_1 = \delta - \tfrac{\sqrt{2}}{4} - \tfrac{\pi}{2} = 0.1469\ldots \end{aligned} \tag{3.6}$$

In the last equation C denotes a suitable positive constant. Next we consider the infinite integral in (3.5). We shall show that the contribution of the part where $|x| \ge \frac{1}{2}$ is negligible. Again let $x_1 = \frac{1}{2}e^{\frac{\pi i}{4}}$. Turning the line of integration appropriately, we have

$$\int_{x_1}^{\infty} e^{-tx}(i+x)^{s-1}\Phi_1(\eta,2\pi x)\mathrm{d}x = e^{it}\int_{i+x_1}^{\infty e^{i\delta}} e^{-tx}x^{s-1}\Phi_1\big(\eta,2\pi(x-i)\big)\mathrm{d}x,$$

where $\delta = \arg(i+x_1) = \arctan\frac{4+\sqrt{2}}{\sqrt{2}}$. The new integral can be parametrized by $x = e^{i\delta}u$, with $u \ge |i+x_1| = \frac{1}{2}(5+2\sqrt{2})^{\frac{1}{2}}$. Using (3.4)

$$\int_{x_1}^{\infty} e^{-tx}(i+x)^{s-1}\Phi_1(\eta,2\pi x)\mathrm{d}x = e^{it}\int_{i+x_1}^{\infty e^{i\delta}} e^{-tx}x^{s-1}\big[A(\eta)+e^{tx-it}\psi_1(\eta,x)\big]\mathrm{d}x. \tag{3.7}$$

Now let $c > 0$ be arbitrary. Then

$$\begin{aligned}\left|\int_{i+x_1}^{\infty e^{i\delta}} e^{-cx}x^{s-1}\mathrm{d}x\right| &= \left|e^{i\delta s}\int_{|i+x_1|}^{\infty}\exp(-ce^{i\delta}u)u^{s-1}\mathrm{d}u\right| \le e^{-\delta t}\int_{|i+x_1|}^{\infty} e^{-cu\cos\delta}u^{\sigma-1}\mathrm{d}u \\ &= e^{-\delta t}(c\cos\delta)^{-\sigma}\Gamma(\sigma,|i+x_1|c\cos\delta),\end{aligned}$$

where $\Gamma(\sigma,z) = \int_z^{\infty} e^{-u}u^{\sigma-1}\mathrm{d}u$ denotes the incomplete gamma function. If σ is fixed it is well known [**14**] that $\Gamma(\sigma,z) = e^{-z}z^{\sigma-1}[1+O(z^{-1})]$ for $z \ge 1$. Since $|i+x_1|\cos\delta = \mathrm{Re}(i+x_1) = \frac{\sqrt{2}}{4}$, we find thus

$$\left|\int_{i+x_1}^{\infty e^{i\delta}} e^{-cx}x^{s-1}\mathrm{d}x\right| \ll c^{-1}e^{-\delta t - c\sqrt{2}/4} \ll e^{-\delta t - c\sqrt{2}/4}, \quad c \ge 2\sqrt{2}.$$

We employ this in (3.7) with $c = t \ge 2\pi$, and $c = 2\pi n \ge 2\pi$, to get

$$\begin{aligned}\left|\int_{x_1}^{\infty} e^{-tx}(i+x)^{s-1}\Phi_1(\eta,2\pi x)\mathrm{d}x\right| &\ll |A(\eta)|e^{-\delta t - t\sqrt{2}/4} + \sum_{n>\eta}|a(n)|e^{-\delta t - \pi n\sqrt{2}/2} \\ &\ll t^{C}e^{-\frac{\pi t}{2}-c_2 t}, \quad c_2 = \delta + \tfrac{\sqrt{2}}{4} - \tfrac{\pi}{2} = 0.09805\ldots \end{aligned} \tag{3.8}$$

for a suitable $C > 0$. Using estimates (3.6) and (3.7) together with Stirling's formula in the form $|\Gamma(s)|^{-1} \sim \sqrt{2\pi} e^{\frac{\pi t}{2}} t^{\frac{1}{2}-\sigma}$, $t \to \infty$, we obtain from (3.5)

$$\begin{aligned} T(s) &= \sum_{n \le \eta} a(n) n^{-s} - A(\eta) e^{it} \eta^{-s} + (2\pi)^s e^{\frac{\pi i}{2}(s-1)} \Gamma(s)^{-1} \times \\ &\times \left[\int_0^{x_1} e^{-tx} (1-ix)^{s-1} \Phi_1(\eta, 2\pi x) \mathrm{d}x + \int_0^{x_1} e^{tx} (1+ix)^{s-1} \Phi_{-1}(\eta, 2\pi x) \mathrm{d}x \right] + O(e^{-t/11}). \end{aligned} \tag{3.9}$$

After these preparations we now arrived at the central problem, viz. the asymptotic expansion of the integrals in (3.9). With $f(x) = x - i\log(1-ix)$ they can both be written as

$$\int_0^{x_1} e^{-tf(\pm x)} (1 \mp ix)^{\sigma-1} \Phi_{\pm 1}(\eta, 2\pi x) \mathrm{d}x. \tag{3.10}$$

Since $f(x) = -\frac{i}{2}x^2$ for $x \to 0$, it is reasonable to put $f(x) = -\frac{i}{2}x^2 + g(x)$, i.e. $g(x) = x + \frac{i}{2}x^2 - i\log(1-ix)$, and hence

$$\int_0^{x_1} e^{-tf(\pm x)} (1 \mp ix)^{\sigma-1} \Phi_{\pm 1}(\eta, 2\pi x) \mathrm{d}x = \int_0^{x_1} e^{\frac{it}{2}x^2} G(\pm x) \Phi_{\pm 1}(\eta, 2\pi x) \mathrm{d}x, \tag{3.11}$$

where $G(x) = e^{-tg(x)}(1-ix)^{\sigma-1}$. Formally, we may proceed as follows. Let $G(x) = \sum_{l=0}^{\infty} \gamma_l(\sigma, t) x^l$ be the Taylor series of G around 0. We then expect

$$\begin{aligned} \int_0^{x_1} e^{\frac{it}{2}x^2} G(\pm x) \Phi_{\pm 1}(\eta, 2\pi x) \mathrm{d}x &\sim \sum_{l=0}^{\infty} (\pm 1)^l \gamma_l(\sigma, t) \int_0^{\infty e^{\pi i/4}} e^{\frac{it}{2}x^2} x^l \Phi_{\pm 1}(\eta, 2\pi x) \mathrm{d}x \\ &= \sum_{l=0}^{\infty} (\pm 1)^l \gamma_l(\sigma, t) \left(\frac{2i}{t} \right)^{\frac{l+1}{2}} \int_0^{\infty} e^{-u^2} u^l \Phi_{\pm 1}(\eta, 2u\sqrt{\pi i/\eta}) \mathrm{d}u \end{aligned}$$

to be the correct asymptotic expansion of the integral (3.10). Finally, using the results from Section 2 we complete our task by deriving explicit formulas for the infinite integrals involving $\Phi_{\pm 1}$. In order to validate this procedure, the function G has to be investigated more closely, and in particular its dependency on σ and t. We introduce the remainder R_L through

$$G(x) = \sum_{l=0}^{L-1} \gamma_l(\sigma, t) x^l + x^L R_L(x), \quad |x| < 1, \ L \ge 0.$$

We then have

Lemma 6. *Assume $0 < \sigma_0 \le \sigma \le \sigma_1$, where σ_0, σ_1 are fixed, and $t \ge 2\pi$. For x complex, $|x| < 1$, define*

$$g(x) = x + \tfrac{i}{2}x^2 - i\log(1-ix), \quad G(x) = e^{-tg(x)}(1-ix)^{\sigma-1},$$

and let R_L be given as above. If $|x| \le \frac{1}{2}$, then $R_L(x) = O(t^{\frac{L+1}{3}} e^{\frac{2}{5}t|x|^2})$ uniformly in σ, t, and x.

Proof. By Cauchy's formula

$$R_L(x) = \frac{1}{2\pi i}\int_{C(\rho)} \frac{G(w)}{w-x} w^{-L} \mathrm{d}w, \quad |x| < \rho < 1,$$

where $C(\rho)$ denotes a circle of radius ρ around the origin. Hence

$$|R_L(x)| \le \rho^{-L}\big(\rho - |x|\big)^{-1} M_\rho, \quad M_\rho = \max\big\{|G(w)|; |w| = \rho\big\}. \tag{3.12}$$

Since $|G(w)| = |e^{-tg(w)}(1-iw)^{\sigma-1}| \ll e^{t|g(w)|}$ if $|w| \le \frac{9}{10}$, we need an estimate for

$$g(w) = w + \tfrac{i}{2}w^2 - i\log(1-iw) = i\sum_{\nu=3}^{\infty} \frac{(iw)^\nu}{\nu}.$$

If $\rho = |w|$, then clearly $|g(w)| \le \rho^2 H(\rho)$, where

$$H(\rho) = \rho\sum_{\nu=0}^{\infty} \frac{\rho^\nu}{\nu+3} = \frac{1}{\rho^2}\big[-\log(1-\rho) - \rho - \tfrac{1}{2}\rho^2\big]. \tag{3.13}$$

Thus $|G(w)| \ll e^{t\rho^2 H(\rho)}$ and $M_\rho \ll e^{t\rho^2 H(\rho)}$, provided $\rho = |w| \le \frac{9}{10}$. To prove the lemma, assume first $\frac{1}{2}t^{-\frac{1}{3}} \le |x| \le \frac{1}{2}$. Choose $\rho = \frac{11}{10}|x|$, so that $\rho \le \frac{9}{10}$ holds. Since $H'(\rho) > 0$ (see (3.13)), H is monotonically increasing and we get from (3.12)

$$\big|R_L(x)\big| \ll |x|^{-L-1} e^{121t|x|^2 H(11/20)/100} \ll t^{\frac{L+1}{3}} e^{\frac{2}{5}t|x|^2},$$

using $H(\frac{11}{20}) = 0.321$ (to three decimal places). For the remaining values of $|x|$, i.e. $0 \le |x| < \frac{1}{2}t^{-\frac{1}{3}}$, we simply take $\rho = t^{-\frac{1}{3}}$. Then

$$|g(w)| \le \sum_{\nu=3}^{\infty} \frac{\rho^\nu}{\nu} < \tfrac{1}{3}\rho^3 \sum_{\nu=0}^{\infty} \rho^\nu = \tfrac{1}{3}\frac{\rho^3}{1-\rho} = \tfrac{1}{3}\frac{t^{-1}}{1-\frac{1}{2}t^{-1/3}} \le \tfrac{2}{3}t^{-1}.$$

This yields $|G(w)| \ll e^{t|g(w)|} \ll 1$ and hence $M_\rho \ll 1$. From (3.12) we conclude

$$\big|R_L(x)\big| \ll t^{\frac{L}{3}}\big(t^{-\frac{1}{3}} - \tfrac{1}{2}t^{-\frac{1}{3}}\big)^{-1} \ll t^{\frac{L+1}{3}} \ll t^{\frac{L+1}{3}} e^{\frac{2}{5}t|x|^2}.$$

This completes the proof of the lemma.

Lemma 7. *Assume $t \ge 2\pi$, and let $\eta = \frac{t}{2\pi}$, $x_1 = \frac{1}{2}e^{\frac{\pi i}{4}}$. Then for fixed, non negative integers L, l*

$$\int_0^{x_1} e^{\frac{it}{2}x^2} x^L R_L(x)\Phi_{\pm1}(\eta, 2\pi x)\mathrm{d}x = O\big(t^{\frac{k}{2}-\frac{L}{6}-\frac{5}{12}}\big),$$

$$\int_0^{x_1} e^{\frac{it}{2}x^2} x^l \Phi_{\pm1}(\eta, 2\pi x)\mathrm{d}x = O\big(t^{\frac{k}{2}-\frac{l}{2}-\frac{3}{4}}\big).$$

Proof. Write $x = e^{\frac{\pi i}{4}}u$, $0 \le u \le \frac{1}{2}$. By the previous lemma and (2.17)

$$\begin{aligned}\left|\int_0^{x_1} e^{\frac{it}{2}x^2} x^L R_L(x)\Phi_{\pm 1}(\eta, 2\pi x)\mathrm{d}x\right| &= \left|\int_0^{\frac{1}{2}} e^{-\frac{t}{2}u^2} u^L R_L\big(e^{\frac{\pi i}{4}}u\big)\Phi_{\pm 1}\big(\eta, 2\pi e^{\frac{\pi i}{4}}u\big)\mathrm{d}u\right| \\ &\ll t^{\frac{L+1}{3}+\frac{k}{2}+\frac{1}{4}} \int_0^{\frac{1}{2}} e^{-\frac{t}{2}u^2+\frac{2t}{5}u^2} u^{L+1}\mathrm{d}u \\ &\le t^{\frac{L+1}{3}+\frac{k}{2}+\frac{1}{4}} \int_0^{\infty} e^{-\frac{t}{10}u^2} u^{L+1}\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{10}{t}\right)^{1+\frac{L}{2}} t^{\frac{L+1}{3}+\frac{k}{2}+\frac{1}{4}}\Gamma\big(\tfrac{L}{2}+1\big) \ll t^{\frac{k}{2}-\frac{L}{6}-\frac{5}{12}}.\end{aligned}$$

This proves the first formula. For the second we similarly find

$$\left|\int_0^{x_1} e^{\frac{it}{2}x^2} x^l \Phi_{\pm 1}(\eta, 2\pi x)\mathrm{d}x\right| \ll t^{\frac{k}{2}+\frac{1}{4}} \int_0^{\frac{1}{2}} e^{-\frac{t}{2}u^2} u^{l+1}\mathrm{d}u \ll t^{\frac{k}{2}-\frac{l}{2}-\frac{3}{4}}.$$

This finishes the proof of the assertion.

From the last result we now deduce

$$\begin{aligned}&\int_0^{x_1} e^{\frac{it}{2}x^2} G(\pm x)\Phi_{\pm 1}(\eta, 2\pi x)\mathrm{d}x = \\ &\quad = \sum_{l=0}^{L-1} (\pm 1)^l \gamma_l(\sigma, t) \int_0^{x_1} e^{\frac{it}{2}x^2} x^l \Phi_{\pm 1}(\eta, 2\pi x)\mathrm{d}x + O\big(t^{\frac{k}{2}-\frac{L}{6}-\frac{5}{12}}\big) \\ &\quad = \sum_{l=0}^{L-1} (\pm 1)^l \gamma_l(\sigma, t)\left(\frac{2i}{t}\right)^{\frac{l+1}{2}} \int_0^{\infty} e^{-u^2} u^l \Phi_{\pm 1}\big(\eta, 2\pi u\sqrt{i/\pi\eta}\big)\mathrm{d}u + O\big(t^{\frac{k}{2}-\frac{L}{6}-\frac{5}{12}}\big).\end{aligned}$$

Thus we can write, using the definition (2.18) of $\Phi^{(\pm)}(\eta, x)$

$$\begin{aligned}&\int_0^{x_1} e^{\frac{it}{2}x^2} G(x)\Phi_1(\eta, 2\pi x)\mathrm{d}x + \int_0^{x_1} e^{\frac{it}{2}x^2} G(-x)\Phi_{-1}(\eta, 2\pi x)\mathrm{d}x = \\ &\quad = \sum_{l=0}^{L-1} \gamma_l(\sigma, t)\left(\frac{2i}{t}\right)^{\frac{l+1}{2}} \int_0^{\infty} e^{-u^2} u^l \Phi^{(\pm)}(\eta, 2\varepsilon u\sqrt{\pi/\eta})\mathrm{d}u + O(t^{\frac{k}{2}-\frac{L}{6}-\frac{5}{12}}),\end{aligned} \tag{3.14}$$

where here, and in what follows, the upper sign applies for l even, and the lower sign for l odd. To complete our task we now show how to get explicit expressions for the last type of integrals. First note that Theorem 3 gives

$$\begin{aligned}&\Phi^{(\pm)}(\eta, 2\varepsilon u\sqrt{\pi/\eta}) = \\ &\quad - 2^{\frac{1}{2}}\pi^{-2}\varepsilon u \eta^{\frac{k}{2}-\frac{1}{4}} \sum_{m=0}^{M-1} (4\pi^2\eta)^{-\frac{m}{2}} \sum_{n=1}^{\infty} a(n) n^{-\frac{k}{2}-\frac{m}{2}-\frac{3}{4}} \beta_m^{(\mp)}\left(2\pi\sqrt{n\eta}, \frac{\varepsilon u}{\sqrt{\pi n}}\right) \\ &\qquad + O\big(|u|\eta^{\frac{k}{2}-\frac{1}{4}-\frac{M}{2}}\big)\end{aligned}$$

for fixed $M \geq 0$, uniformly in u. Hence on integrating (note our sign convention)

$$\int_0^\infty e^{-u^2} u^l \Phi^{(\pm)}(\eta, 2\varepsilon u\sqrt{\pi/\eta})\mathrm{d}u =$$
$$-2^{\frac{1}{2}}\pi^{-2}\varepsilon\eta^{\frac{k}{2}-\frac{1}{4}}\sum_{m=0}^{M-1}(4\pi^2\eta)^{-\frac{m}{2}}\sum_{n=1}^{\infty} a(n)n^{-\frac{k}{2}-\frac{m}{2}-\frac{3}{4}}\int_0^\infty e^{-u^2}u^{l+1}\beta_m^{(\mp)}\left(2\pi\sqrt{n\eta}, \frac{\varepsilon u}{\sqrt{\pi n}}\right)\mathrm{d}u$$
$$+ O\big(\eta^{\frac{k}{2}-\frac{1}{4}-\frac{M}{2}}\big).$$

These formulas suggest the following definitions. Let

$$W_{lm}(n) = \sum_{\mu=0}^{2m}(-1)^\mu\binom{m-k-\frac{1}{2}}{2m-\mu}\int_0^\infty e^{-u^2}u^{l+1}\frac{q_{\mu+1}(\varepsilon u/\sqrt{\pi n})}{(1+iu^2/\pi n)^{\mu+1}}\mathrm{d}u, \quad l \text{ even}, \tag{3.15}$$
$$W_{lm}(n) = \sum_{\mu=0}^{2m}(-1)^\mu\binom{m-k-\frac{1}{2}}{2m-\mu}\int_0^\infty e^{-u^2}u^{l+1}\frac{p_{\mu+1}(\varepsilon u/\sqrt{\pi n})}{(1+iu^2/\pi n)^{\mu+1}}\mathrm{d}u, \quad l \text{ odd}. \tag{3.16}$$

From (1.19) we see that

$$|W_{lm}(n)| \leq 2\sum_{\mu=0}^{2m}\left|\binom{m-k-\frac{1}{2}}{2m-\mu}\right| 2^{\frac{\mu+1}{2}}\int_0^\infty e^{-u^2}u^{l+1}\mathrm{d}u, \tag{3.17}$$

which is less than a constant depending only on l and m (and k, of course). With these coefficients let then

$$\Xi_{lm}(\eta) = \sum_{n=1}^{\infty} a(n)n^{-\frac{k}{2}-\frac{m}{2}-\frac{3}{4}}W_{lm}(n)\cos(4\pi\sqrt{n\eta}+\tfrac{\pi m}{2}-\tfrac{\pi}{4}), \quad l \text{ even}, \tag{3.18}$$
$$\Xi_{lm}(\eta) = \sum_{n=1}^{\infty} a(n)n^{-\frac{k}{2}-\frac{m}{2}-\frac{3}{4}}W_{lm}(n)\sin(4\pi\sqrt{n\eta}+\tfrac{\pi m}{2}-\tfrac{\pi}{4}), \quad l \text{ odd}. \tag{3.19}$$

Furthermore, if we define $D_l(\eta)$ by the equation

$$\int_0^\infty e^{-u^2}u^l\Phi^{(\pm)}(\eta, 2\varepsilon u\sqrt{\pi/\eta})\mathrm{d}u = -2^{\frac{1}{2}}\pi^{-2}\varepsilon\eta^{\frac{k}{2}-\frac{1}{4}}D_l(\eta), \tag{3.20}$$

then $D_l(\eta)$ has an asymptotic expansion of the form

$$D_l(\eta) = \sum_{m=0}^{M-1}\Gamma(m+\tfrac{1}{2})(4\pi^2\eta)^{-\frac{m}{2}}\Xi_{lm}(\eta) + O(\eta^{-\frac{M}{2}}). \tag{3.21}$$

In particular, we deduce $D_l(\eta) = O(1)$ for $\eta \geq 1$. It is plain that (3.14) and (3.20) then yield

$$\int_0^{x_1} e^{\frac{it}{2}x^2}G(x)\Phi_1(\eta, 2\pi x)\mathrm{d}x + \int_0^{x_1} e^{\frac{it}{2}x^2}G(-x)\Phi_{-1}(\eta, 2\pi x)\mathrm{d}x =$$
$$= -2i\pi^{-2}t^{-\frac{1}{2}}\eta^{\frac{k}{2}-\frac{1}{4}}\sum_{l=0}^{L-1}\gamma_l(\sigma,t)\left(\frac{2i}{t}\right)^{\frac{l}{2}}D_l(\eta) + O(t^{\frac{k}{2}-\frac{L}{6}-\frac{5}{12}}). \tag{3.22}$$

With these definitions we finally arrive at the following result, giving the asymptotic expansion for the fundamental function $T(s)$ from (3.1).

THEOREM 4. *Let* $t \geq 2\pi$, $\eta = \frac{t}{2\pi}$. *Then*

$$T(s) = \sum_{n \leq \eta} a(n)n^{-s} - A(\eta)e^{it}\eta^{-s} -$$

$$- 2\pi^{-2}(2\pi)^{s+\frac{1}{2}}e^{\frac{\pi is}{2}}\Gamma(s)^{-1}\eta^{\frac{k}{2}-\frac{3}{4}} \sum_{l=0}^{L-1} \gamma_l(\sigma, t) \left(\frac{2i}{t}\right)^{\frac{l}{2}} D_l(\eta) + O(t^{\frac{k}{2}-\sigma+\frac{1}{12}-\frac{L}{6}})$$

for each fixed integer L. Moreover, $D_l(\eta)$ *can be asymptotically approximated by* (3.21) *above.*

Proof. By (3.9)

$$T(s) = \sum_{n \leq \eta} a(n)n^{-s} - A(\eta)e^{it}\eta^{-s}$$

$$+(2\pi)^s e^{\frac{\pi i}{2}(s-1)}\Gamma(s)^{-1}\left[\int_0^{x_1} e^{\frac{it}{2}x^2}G(x)\Phi_1(\eta, 2\pi x)\mathrm{d}x + \int_0^{x_1} e^{\frac{it}{2}x^2}G(-x)\Phi_{-1}(\eta, 2\pi x)\mathrm{d}x\right]$$
$$+ O(e^{-t/11}).$$

Replacing the bracketed factor by (3.22) and using $|e^{\frac{\pi is}{2}}\Gamma(s)^{-1}| = O(t^{\frac{1}{2}-\sigma})$ then yields the assertion, q.e.d.

This result may be considered as our main theorem. In fact, it yields the asymptotic expansion of the function $T(s)$ occurring in (3.2), and it is our analogue of the Riemann-Siegel formula (0.1), (0.2). By reordering the terms involving $\gamma_l(\sigma, t)$ and $t^{-\frac{l}{2}}$ one gets an asymptotic series consisting of powers of $t^{-\frac{1}{2}}$. The coefficients of each term $t^{-\frac{m}{2}}$ are linear combinations of the $D_l(\eta)$ from (3.21) which are uniformly bounded for $\eta \geq 1$. In the final section we shall write up some special cases in order to show the result more explicitly.

4. Some Special Cases and Further Problems

In this final section we shall consider some special cases of Theorem 4. For the present purpose we first need a bit more information on the Taylor coefficients $\gamma_l(\sigma, t)$ of the function $G(x)$ in Lemma 6. From the differential equation

$$G'(x)(1 - ix) = -G(x)\big[tx^2 + i(\sigma - 1)\big]$$

we get the recursion formula

$$(l + 1)\gamma_{l+1} = -t\gamma_{l-2} + i(l + 1 - \sigma)\gamma_l, \quad l \geq 2, \tag{4.1}$$

with the starting values $\gamma_0 = 1$, $\gamma_1 = i(1-\sigma)$, and $\gamma_2 = -\frac{1}{2}(1-\sigma)(2-\sigma)$. It then follows easily by induction that $\gamma_l(\sigma,t)$ is a polynomial in σ and t, having t degree at most $[\frac{l}{3}]$. Hence

$$\gamma_l(\sigma,t)t^{-\frac{l}{2}} = O(t^{[\frac{l}{3}]-\frac{l}{2}}) = O(t^{-\frac{l}{6}}) \tag{4.2}$$

uniformly in σ and t.

We now derive the asymptotic expansion of $T(s)$, including all terms up to order $t^{\frac{k}{2}-\sigma-\frac{3}{4}}$. Further approximations are obviously possible in the same way. Take $L = 6$ in Theorem 4. Since $D_l(\eta) = O(1)$, we find

$$\begin{aligned}\sum_{l=0}^{5} \gamma_l(\sigma,t)\Big(\frac{2i}{t}\Big)^{\frac{l}{2}} &\\ &= \gamma_0(\sigma,t)D_0(\eta) + \gamma_1(\sigma,t)\left(\frac{2i}{t}\right)^{\frac{1}{2}} D_1(\eta) + \gamma_3(\sigma,t)\left(\frac{2i}{t}\right)^{\frac{3}{2}} D_3(\eta) + O(t^{-1})\\ &= D_0(\eta) + i(2i)^{\frac{1}{2}}(1-\sigma)t^{-\frac{1}{2}}D_1(\eta) - \tfrac{1}{3}(2i)^{\frac{3}{2}}t^{-\frac{1}{2}}D_3(\eta) + O(t^{-1}),\end{aligned}$$

where (4.2) has been employed. Using (3.21) with $M = 2$ for $D_0(\eta)$ and $M = 1$ for $D_1(\eta), D_3(\eta)$, we obtain

$$\sum_{l=0}^{5} \gamma_l(\sigma,t)\left(\frac{2i}{t}\right)^{\frac{l}{2}} =$$

$$\pi^{\frac{1}{2}}\Xi_{00}(\eta) + \big[2^{-\frac{3}{2}}\Xi_{01}(\eta) + i(2\pi i)^{\frac{1}{2}}(1-\sigma)\Xi_{10}(\eta) - \tfrac{1}{3}(2i)^{\frac{3}{2}}\pi^{\frac{1}{2}}\Xi_{30}(\eta)\big]t^{-\frac{1}{2}} + O(t^{-1}).$$

We further need the asymptotic expansion of $\Gamma(s)^{-1}$ for $t \to +\infty$. This can be accomplished using Stirling's formula [**14**], and the final result may be written in the form

$$(2\pi)^s e^{\frac{\pi i s}{2}}\Gamma(s)^{-1} \sim e^{it+\frac{\pi i}{4}}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}-s}\sum_{l=0}^{\infty}\delta_l(\sigma)t^{-l}, \tag{4.3}$$

with coefficients $\delta_l(\sigma)$ being independent of t. In particular, $\delta_0(\sigma) = 1$. Inserting these formulas into Theorem 4, we find

$$T(s) = \sum_{n\le\eta} a(n)n^{-s} - A(\eta)e^{it}\eta^{-s} - \left(\frac{2}{\pi}\right)^{\frac{3}{2}} e^{it+\frac{\pi i}{4}}\eta^{\frac{k}{2}-s-\frac{1}{4}}S, \tag{4.4}$$

with

$$S = \pi^{\frac{1}{2}}\Xi_{00}(\eta) + \big[2^{-\frac{3}{2}}\Xi_{01}(\eta) + i(2\pi i)^{\frac{1}{2}}(1-\sigma)\Xi_{10}(\eta) - \tfrac{1}{3}(2i)^{\frac{3}{2}}\pi^{\frac{1}{2}}\Xi_{30}(\eta)\big]t^{-\frac{1}{2}} + O(t^{-\frac{11}{12}}). \tag{4.5}$$

Omitting the term involving S we have the special case

$$T(s) = \sum_{n\le\eta} a(n)n^{-s} - A(\eta)e^{it}\eta^{-s} + O(t^{\frac{k}{2}-\sigma-\frac{1}{4}}).$$

If the Fourier coefficients $a(n)$ are real and $\sigma = \frac{k}{2}$ we thus get by (3.2)

$$\varphi(\tfrac{k}{2}+it) = \sum_{n\le\eta} a(n)n^{-\frac{k}{2}-it} + (-1)^{\frac{k}{2}} X(\tfrac{k}{2}+it) \sum_{n\le\eta} a(n)n^{-\frac{k}{2}+it}$$
$$- 2A(\eta)\eta^{-\frac{k}{2}} \cos(t - t\log\eta) + O(t^{-\frac{1}{4}}).$$

This formula shows that the error term in the approximate functional equation for $\varphi(\frac{k}{2}+it)$ is essentially determined by the behaviour of the sum function $A(\eta) = \sum_{n\le\eta} a(n)$. The well known results [**15, 18**]

$$A(\eta) = O(\eta^{\frac{k}{2}-\frac{1}{6}+\varepsilon}), \quad A(\eta) = \Omega(\eta^{\frac{k}{2}-\frac{1}{4}}),$$

then yield upper and lower bounds for this error term. Moreover, it is possible to derive mean value results in the usual way.

The series (3.18), (3.19) which occur in our formula are in a certain sense the analogue of the function $F(z) = \cos(z^2 + \frac{3\pi}{8})/\cos(\sqrt{2\pi}z)$ and its derivatives from the introduction. It might be of interest to find another representation for $\Xi_{lm}(\eta)$ or to estimate mean values like

$$\int_1^T \Xi_{lm}(u)\mathrm{d}u, \quad \int_1^T |\Xi_{lm}(u)|^j \mathrm{d}u, \quad j \in \mathbf{N}_0,\ T\to\infty.$$

For numerical purposes it is also important to have an effective method of computing $\Xi_{lm}(\eta)$.

In conclusion, we mention that our main result can be generalized twofold. First, instead of (3.1) a more general integral can be used, where i is replaced by $i\frac{p}{q}$ for positive, coprime integers p, q (see [**3**]). This will eventually yield an "unsymmetric" form of the approximate functional equation, where sums of length $\eta\frac{p}{q}$ and $\eta\frac{q}{p}$ occur. Secondly, the whole theory can be extended to include cusp forms for an arbitrary congruence subgroup of $SL_2(\mathbf{Z})$. In particular, L functions of certain elliptic curves can be treated in the same way. We shall return to these matters elsewhere.

References

[1] P. Deligne, La Conjecture de Weil I, Inst. Hautes Études Sci. Publ. Math. **43**, 273–307 (1974).

[2] Deuring, M., Asymptotische Entwicklungen der Dirichletschen L-Reihen, Math. Ann. **168**, 1–30(1967).

[3] Guthmann, A., The Riemann-Siegel Integral Formula for Dirichlet Series Associated to Cusp Forms, in: Analytic and Elementary Number Theory, Vienna 1996, 53–69.

[4] Guthmann, A., Die Riemann-Siegel-Integralformel für die Mellintransformation von Spitzenformen, Arch. Math. **69**, 391–402 (1997).

[5] Hecke, E.,Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung, Math. Ann. **112**, 664–699 (1936).

[6] Hecke, E., Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I, Math. Ann. **114**, 1–28 (1937), II, ibid. **114**, 316–351 (1937).

[7] Ivić, A., The Riemann Zeta Function, New York: Wiley 1985.

[8] Jutila, M., On the Approximate Functional Equation of $\zeta^2(s)$ and Other Dirichlet Series, Quarterly J. Math. Oxford Ser. **37**, 193–209 (1986).

[9] Jutila, M., Lectures on a Method in the Theory of Exponential Sums, Bombay: Tata Institute 1987.

[10] Lebedev, N.N., Special Functions And Their Applications, New York: Dover 1972.

[11] Levinson, N., More than One Third of Zeros of Riemann's Zeta-function are on $\sigma = 1/2$, Adv. in Math. **13**, 383–436 (1974).

[12] Motohashi, Y., A Note on the Approximate Functional Equation for $\zeta^2(s)$ I, Proc. Japan Acad. **59**A, 393–396 (1983), II, ibid. **59**A, 469–472 (1983), III, ibid. **62**A, 410–412 (1986).

[13] Motohashi, Y., An Asymptotic Expansion of the Square of the Riemann Zeta-Function, in: Sieve Methods, Exponential Sums, and their Applications in Number Theory, Proceedings Cardiff 1995, Cambridge University Press, Cambridge 1997, 293–307.

[14] Olver, F.W.J., Asymptotics and Special Functions, New York: Academic Press 1974.

[15] Rankin, R.A., Fourier Coefficients of Cusp Forms, Math. Proc. Camb. Phil. Soc. **100**, 5–29 (1986).

[16] Siegel, C.L., Über Riemanns Nachlaß zur analytischen Zahlentheorie, Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik **2**, 45–80 (1932).

[17] Siegel, C.L., Contribution to the Theory of the Dirichlet L Series and the Epstein Zeta-functions, Ann. of Math. **44**, 143–172 (1943).

[18] Walfisz, A., Über die Koeffizientensummen einiger Modulformen, Math. Ann. **108**, 75–90 (1933).

[19] Watson, G.N., A Treatise on the Theory of Bessel Functions, 2nd edition, Cambridge: Cambridge University Press 1944.

Fachbereich Mathematik
Universität Kaiserslautern
D-67663 Kaiserslautern
`guthmann@mathematik.uni-kl.de`

(Received 02 02 1998)