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ON THE NON-COMMUTATIVE NEUTRIX PRODUCT OF x_+^λ AND $x_+^{-\lambda-r}$

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Abstract. The non-commutative neutrix product of the distributions x_+^λ and $x_+^{-\lambda-r}$ is evaluated for $\lambda \neq 0, \pm 1, \pm 2 \ldots$

In the following, we let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$, (iii) $\rho(x) = \rho(-x)$,

(ii) $\rho(x) \geq 0$, (iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [**2**].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$

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is a locally summable function in $L^q(a,b)$ with $1/p+1/q=1$. Then the product $fg=gf$ of f and g is defined on the interval (a,b) by

$$fg=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[Fg^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [4] and generalizes Definition 1.

Definition 2. Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x)=(g*\delta_n)(x)$. We say that the neutrix product $f\circ g$ of f and g exists and is equal to the distribution h on the interval (a,b) if

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle f(x)g_n(x),\varphi(x)\rangle=\langle h(x),\varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a,b), where N is the neutrix, see van der Corput [**1**], having domain $N'=\{1,2,\ldots,n,\ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda\ln^{r-1}n,\quad \ln^r n:\qquad \lambda>0,\quad r=1,2,\ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It was proved in [**4**] that if the product fg exists by Definition 1, then the product $f\circ g$ exists by Definition 2 and the two are equal.

The following theorem was proved in [**5**].

Theorem 1. *The neutrix product $x_+^\lambda\circ x_+^{-\lambda-1}$ exists and*

$$x_+^\lambda\circ x_+^{-\lambda-1}=x_+^{-1}-[\gamma+\tfrac{1}{2}\,\psi(-\lambda)+\tfrac{1}{2}\,\psi(\lambda+1)+2c(\rho)]\delta(x)$$

for $\lambda\neq 0,\pm1,\pm2,\ldots$, where γ denotes Euler's constant and

$$\psi(\mu)=\frac{\Gamma'(\mu)}{\Gamma(\mu)}.$$

Before proving our main result we need the following definition of the Beta function given in [**6**].

Definition 3. The Beta function $B(\lambda,\mu)$ is defined for all λ,μ by

$$B(\lambda,\mu)=\mathrm{N}\!-\!\lim_{n\to\infty}\int_{1/n}^{1-1/n}t^{\lambda-1}(1-t)^{\mu-1}\,dt.$$

It was proved that if $\lambda, \mu \neq 0, -1, -2, \ldots$, then the above definition is in agreement with the standard definition of the Beta function.

In particular, it was proved in [**6**] that

$$B(-r, \lambda) = \frac{(-1)^r \Gamma(\lambda)}{r! \Gamma(\lambda - r)} [\phi(r) - \gamma - \psi(\lambda - r)]$$

for $r = 0, 1, 2, \ldots$ and $\lambda, \mu \neq 0, -1, -2, \ldots$, where

$$\phi(r) = \begin{cases} \sum\limits_{i=1}^{r} 1/i, & r = 1, 2, \ldots, \\ 0, & r = 0. \end{cases} \tag{1}$$

We now generalize theorem 1 in which the distribution x_+^{-r} is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^r$$

for $r = 1, 2, \ldots$ and not as in Gel'fand and Shilov [**7**].

THEOREM 2. *The neutrix product* $x_+^{\lambda} \circ x_+^{-\lambda - r}$ *exists and*

$$x_+^{\lambda} \circ x_+^{-\lambda - r} = x_+^{-r} + a_r(\lambda) \delta^{(r-1)}(x), \tag{2}$$

where

$$a_r(\lambda) = \frac{(-1)^r [\gamma + 2c(\rho) + \frac{1}{2} \psi(\lambda + 1) + \frac{1}{2} \psi(-\lambda - r + 1) - \phi(r - 1)]}{(r-1)!} + \\ + \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1},$$

for $r = 1, 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. We first of all suppose that $-1 < \lambda < 0$ and put

$$\begin{aligned} (x_+^{-\lambda - r})_n &= x_+^{-\lambda - r} * \delta_n(x) \\ &= \begin{cases} \dfrac{(-1)^{r-1} \Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \displaystyle\int_{-1/n}^{1/n} (x - t)^{-\lambda - 1} \delta_n^{(r-1)}(t) \, dt, & x > 1/n, \\ \dfrac{(-1)^{r-1} \Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \displaystyle\int_{-1/n}^{x} (x - t)^{-\lambda - 1} \delta_n^{(r-1)}(t) \, dt, & -1/n \leq x \leq 1/n, \\ 0, & x < -1/n. \end{cases} \end{aligned}$$

Then

$$
\begin{aligned}
\frac{(-1)^{r-1}\Gamma(\lambda+r)}{\Gamma(\lambda+1)} & \int_{-1}^{1} x_+^\lambda (x_+^{-\lambda-r})_n x^i\,dx \\
&= \int_0^{1/n} x^{\lambda+i} \int_{-1/n}^{x} (x-t)^{-\lambda-1}\delta_n^{(r-1)}(t)\,dt\,dx \\
&\quad + \int_{1/n}^{1} x^{\lambda+i} \int_{-1/n}^{1/n} (x-t)^{-\lambda-1}\delta_n^{(r-1)}(t)\,dt\,dx \\
&\quad + \int_0^{1/n} \delta_n^{(r-1)}(t) \int_t^1 x^{\lambda+i}(x-t)^{-\lambda-1}\,dx\,dt \\
&\quad + \int_{-1/n}^{0} \delta_n^{(r-1)}(t) \int_0^1 x^{\lambda+i}(x-t)^{-\lambda-1}\,dx\,dt \\
&= n^{r-i-1} \int_0^1 \rho^{(r-1)}(v) \int_v^n u^{\lambda+i}(u-v)^{-\lambda-1}\,du\,dv \\
&\quad - (-1)^r n^{r-i-1} \int_0^1 \rho^{(r-1)}(v) \int_0^n u^{\lambda+i}(u+v)^{-\lambda-1}\,du\,dv,
\end{aligned} \tag{3}
$$

where the substitutions $nt = v$ and $nx = u$ have been made in the first integral and $nt = -v$ and $nx = u$ in the second integral.

We have

$$
\begin{aligned}
&\int_v^n u^{\lambda+i}(u-v)^{-\lambda-1}\,du - (-1)^r \int_0^n u^{\lambda+i}(u+v)^{-\lambda-1}\,du = \\
&= \int_v^n u^{\lambda+i}[(u-v)^{-\lambda-1} - (-1)^r(u+v)^{-\lambda-1}]\,du - (-1)^r \int_0^v u^{\lambda+i}(u+v)^{-\lambda-1}\,du
\end{aligned}
$$

and it follows for the cases $i = 0, 1, \ldots, r-2$ that

$$
\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} n^{r-i-1} & \Big[\int_v^n u^{\lambda+i}(u-v)^{-\lambda-1}\,du - (-1)^r \int_0^n u^{\lambda+i}(u+v)^{-\lambda-1}\,du\Big] = \\
&= \operatorname*{N-lim}_{n\to\infty} n^{r-i-1} \int_v^n u^{\lambda+i}[(u-v)^{-\lambda-1} - (-1)^r(u+v)^{-\lambda-1}]\,du \\
&= \operatorname*{N-lim}_{n\to\infty} n^{r-i-1} \sum_{j=0}^{\infty} \binom{-\lambda-1}{j} [(-1)^j - (-1)^r] v^j \int_v^n u^{i-j-1}\,du \\
&= \frac{2(-1)^r}{r-i-1}\binom{-\lambda-1}{r-1} v^{r-1} = -\frac{2\Gamma(\lambda+r)}{(r-i-1)(r-1)!\,\Gamma(\lambda+1)} v^{r-1}.
\end{aligned}
$$

It follows that

$$
\operatorname*{N-lim}_{n\to\infty} \int_{-1}^{1} x_+^\lambda (x_+^{-\lambda-r})_n x^i\,dx = -\frac{1}{r-i-1} \tag{4}
$$

for $i = 0, 1, \ldots, r-2$, since it is easily proved by induction that

$$\int_0^1 v^r \rho^{(r)}(v)\,dv = \tfrac{1}{2}(-1)^r r!.$$

When $i = r-1$, we have on making the substitution $u = v/y$

$$\begin{aligned}
\int_v^n u^{\lambda+r-1}(u-v)^{-\lambda-1}\,du &= v^{r-1}\int_{v/n}^1 y^{-r}(1-y)^{-\lambda-1}\,dy \\
&= v^{r-1}\int_{v/n}^1 y^{-r}\Big[(1-y)^{-\lambda-1} - \sum_{j=0}^{r-1}\binom{-\lambda-1}{j}(-y)^j\Big]\,dy \\
&\quad + v^{r-1}\sum_{j=0}^{r-1}(-1)^j\binom{-\lambda-1}{j}\int_{v/n}^1 y^{j-r}\,dy \\
&= v^{r-1}\int_{v/n}^1 y^{-r}\Big[(1-y)^{-\lambda-1} - \sum_{j=0}^{r-1}\binom{-\lambda-1}{j}(-y)^j\Big]\,dy \\
&\quad + v^{r-1}\sum_{j=0}^{r-2}(-1)^j\binom{-\lambda-1}{j}\frac{1-(n/v)^{r-j-1}}{j-r+1} \\
&\quad - \binom{-\lambda-1}{r-1}(-v)^{r-1}(\ln v - \ln n).
\end{aligned}$$

It follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_v^n u^{\lambda+r-1}(u-v)^{-\lambda-1}\,du = v^{r-1}B(-r+1,-\lambda) - \binom{-\lambda-1}{r-1}(-v)^{r-1}\ln v$$

and so

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty}\int_0^1 \rho^{(r-1)}(v)\int_v^n u^{\lambda+r-1}(u-v)^{-\lambda-1}\,du\,dv = \qquad (5)\\
&\quad = \tfrac{1}{2}(-1)^{r-1}(r-1)!B(-r+1,-\lambda) - \binom{-\lambda-1}{r-1}(r-1)![\tfrac{1}{2}\,\phi(r-1) + c(\rho)] \\
&\quad = \tfrac{1}{2}(-1)^{r-1}(r-1)!B(-r+1,-\lambda) + \frac{(-1)^r\Gamma(\lambda+r)}{\Gamma(\lambda+1)}[\tfrac{1}{2}\,\phi(r-1) + c(\rho)]
\end{aligned}$$

since it is easily proved by induction that

$$\int_0^1 v^r \ln v\,\rho^{(r)}(v)\,dv = \tfrac{1}{2}(-1)^r r!\phi(r) + (-1)^r r!c(\rho).$$

Further, making the substitution $u = v(y^{-1}-1)$, we have

$$\begin{aligned}
\int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1}\,du &= v^{r-1}\int_{v/(n+v)}^1 y^{-r}(1-y)^{\lambda+r-1}\,dy \\
&= v^{r-1}\int_{v/(n+v)}^1 y^{-r}\Big[(1-y)^{\lambda+r-1} - \sum_{j=0}^{r-1}\binom{\lambda+r-1}{j}(-y)^j\Big]\,dy+ \\
&\quad + v^{r-1}\sum_{j=0}^{r-1}(-1)^j\binom{\lambda+r-1}{j}\int_{v/(n+v)}^1 y^{j-r}\,dy \\
&= v^{r-1}\int_{v/(n+v)}^1 y^{-r}\Big[(1-y)^{\lambda+r-1} - \sum_{j=0}^{r-1}\binom{\lambda+r-1}{j}(-y)^j\Big]\,dy+ \\
&\quad + v^{r-1}\sum_{j=0}^{r-2}(-1)^j\binom{\lambda+r-1}{j}\frac{1-(n/v+1)^{r-j-1}}{j-r+1}+
\end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty}\int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1}\,du &= v^{r-1}B(-r+1,\lambda+r)+ \\
&\quad + v^{r-1}\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j-1} + \binom{\lambda+r-1}{r-1}(-1)^r v^{r-1}\ln v
\end{aligned}$$

and so

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty}\int_0^1 \rho^{(r-1)}(v)\int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1}\,du\,dv \\
&= \tfrac{1}{2}(-1)^{r-1}(r-1)!B(-r+1,\lambda+r) \\
&\quad + \tfrac{1}{2}(-1)^{r-1}(r-1)!\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j-1} \\
&\quad - \binom{\lambda+r-1}{r-1}(r-1)![\tfrac{1}{2}\,\phi(r-1)+c(\rho)] \\
&= \tfrac{1}{2}(-1)^{r-1}(r-1)!B(-r+1,\lambda+r) \\
&\quad + \tfrac{1}{2}(-1)^{r-1}(r-1)!\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j-1} \\
&\quad - \frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)}[\tfrac{1}{2}\,\phi(r-1)+c(\rho)]. \qquad (6)
\end{aligned}$$

It follows from equation (1) that

$$B(-r+1,\lambda+r)=\frac{(-1)^{r-1}\Gamma(\lambda+r)}{(r-1)!\Gamma(\lambda+1)}[\phi(r-1)-\gamma-\psi(\lambda+1)],$$

$$\begin{aligned}B(-r+1,-\lambda)&=\frac{(-1)^{r-1}\Gamma(-\lambda)}{(r-1)!\Gamma(-\lambda-r+1)}[\phi(r-1)-\gamma-\psi(-\lambda-r+1)]\\&=\frac{\Gamma(\lambda+r)}{(r-1)!\Gamma(\lambda+1)}[\phi(r-1)-\gamma-\psi(-\lambda-r+1)]\end{aligned}$$

and so

$$B(-r+1,-\lambda)-(-1)^rB(-r+1,\lambda+r)=$$
$$\frac{\Gamma(\lambda+r)}{(r-1)!\Gamma(\lambda+1)}[2\phi(r-1)-2\gamma-\psi(\lambda+1)-\psi(-\lambda-r+1)]. \quad (7)$$

It now follows from equations (3), (5), (6) and (7) that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)}\int_{-1}^{1}x_+^{\lambda}(x_+^{-\lambda-r})_n x^{r-1}\,dx&=\\&=\tfrac{1}{2}(r-1)![B(-r+1,-\lambda)-(-1)^rB(-r+1,\lambda+r)]\\&\quad-\tfrac{1}{2}(r-1)!\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^{r-j}}{r-j-1}-\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)}[\phi(r-1)+2c(\rho)]\\&=\frac{(-1)^{r-1}(r-1)!\Gamma(\lambda+r)}{\Gamma(\lambda+1)}b_r(\lambda),\end{aligned} \quad (8)$$

where

$$b_r(\lambda)=\frac{(-1)^r[\gamma+2c(\rho)+\frac{1}{2}\psi(\lambda+1)+\frac{1}{2}\psi(-\lambda-r+1)]}{(r-1)!}+$$
$$+\frac{\Gamma(\lambda+1)}{2\Gamma(\lambda+r)}\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j-1}.$$

It was proved in [**3**] that

$$\langle x_+^{-r},\varphi(x)\rangle=$$
$$\int_0^{\infty}x^{-r}\Big[\varphi(x)-\sum_{i=0}^{r-2}\frac{\varphi^{(i)}(0)}{i!}x^i-\frac{\varphi^{(r-1)}(0)}{(r-1)!}x^{r-1}H(1-x)\Big]dx+-\frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0),$$

for all φ in $\mathcal{D}$. In particular, if the support of φ is contained in the interval $[-1,1]$, we have

$$\langle x_+^{-r},\varphi(x)\rangle= \quad (9)$$
$$=\int_0^1 x^{-r}\Big[\varphi(x)-\sum_{i=0}^{r-1}\frac{\varphi^{(i)}(0)}{i!}x^i\Big]dx-\sum_{i=0}^{r-2}\frac{\varphi^{(i)}(0)}{i!}\int_1^{\infty}x^{-r+i}\,dx-\frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0)$$
$$=\int_0^1 x^{-r}\Big[\varphi(x)-\sum_{i=0}^{r-1}\frac{\varphi^{(i)}(0)}{i!}x^i\Big]dx-\sum_{i=0}^{r-2}\frac{\varphi^{(i)}(0)}{i!(r-i-1)}-\frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0).$$

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1,1]$. By the mean value theorem

$$\varphi(x) = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(r)}(\xi x)}{r!} x^r,$$

where $0 < \xi < 1$ and so

$$\begin{aligned}\langle x_+^\lambda (x_+^{-\lambda-r})_n, \varphi(x)\rangle &= \int_0^1 x^\lambda (x_+^{-\lambda-r})_n \varphi(x)\,dx \\ &= \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_0^1 x^{\lambda+i}(x_+^{-\lambda-r})_n\,dx + \frac{1}{r!}\int_0^1 x^\lambda [x^r (x_+^{-\lambda-r})_n] \varphi^{(r)}(\xi x)\,dx.\end{aligned}$$

Since the sequence of continuous functions $\{x^r(x_+^{-\lambda-r})_n\}$ converges uniformly to the continuous function $x^{-\lambda}$ on the closed interval $[0,1]$, it follows on using equations (4), (8) and (9) that

$$\begin{aligned}&\mathrm{N}\text{-}\lim_{n\to\infty} \langle x_+^\lambda (x_+^{-\lambda-r})_n, \varphi(x)\rangle = \\ &= \mathrm{N}\text{-}\lim_{n\to\infty} \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-1}^1 x_+^{\lambda+i} (x_+^{-\lambda-r})_n\,dx + \lim_{n\to\infty} \frac{1}{r!} \int_{-1}^1 x_+^\lambda [x^r (x_+^{-\lambda-r})_n] \varphi^{(r)}(\xi x)\,dx \\ &= \frac{1}{r!}\int_0^1 \varphi^{(r)}(\xi x)\,dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + (-1)^{r-1} b_r(\lambda) \varphi^{(r-1)}(0) \\ &= \int_0^1 x^{-r}\Big[\varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i\Big]\,dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0) \\ &\qquad + \frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0) + (-1)^{r-1} b_r(\lambda)\varphi^{(r-1)}(0) \\ &= \langle x_+^{-r}, \varphi(x)\rangle + (-1)^{r-1} a_r(\lambda) \varphi^{(r-1)}(0),\end{aligned}$$

giving equation (1) on the interval $[-1,1]$ when $-1 < \lambda < 0$. However, since $x_+^\lambda . x_+^{-\lambda-r} = x_+^{-r}$ on any closed interval not containing the origin, equation (1) holds on the real line when $-1 < \lambda < 0$.

Now suppose that equation (2) holds for some r and $\lambda \neq 0, \pm 1, \pm 2, \ldots$. This is true for $r = 1$. It is also true for $r+1$ when $-1 < \lambda < 0$. Assume it is also true for $r+1$ when $-k < \lambda < -k+1$. Then if $-k < \lambda < -k+1$, equation (2) holds by our assumption and differentiating this equation, we get

$$\lambda x_+^{\lambda-1} \circ x_+^{-\lambda-r} - (\lambda+r) x_+^\lambda \circ x_+^{-\lambda-r-1} = -r x_+^{-r-1} + a_r(\lambda)\delta^{(r)}(x).$$

It follows from our assumptions that

$$\lambda x_+^{\lambda-1} \circ x_+^{-\lambda-r} = \lambda x_+^{-r-1} + [(\lambda+r) a_{r+1}(\lambda) + a_r(\lambda)]\delta^{(r)}(x).$$

We have

$$
\begin{aligned}
(\lambda+r)a_{r+1}(\lambda)+a_r(\lambda) = {} & \frac{(-1)^{r-1}(\lambda+r)}{r!}[\gamma+\tfrac{1}{2}\,\psi(-\lambda-r)+\tfrac{1}{2}\,\psi(\lambda+1)+2c(\rho)] \\
& + \frac{\Gamma(\lambda+1)}{2\Gamma(\lambda+r)}\sum_{j=0}^{r-1}\binom{\lambda+r}{j}\frac{(-1)^j}{r-j}+\frac{(-1)^r(\lambda+r)\phi(r)}{r!} \\
& + \frac{(-1)^r}{(r-1)!}[\gamma+\tfrac{1}{2}\,\psi(-\lambda-r+1)+\tfrac{1}{2}\,\psi(\lambda+1)+2c(\rho)] \\
& + \frac{\Gamma(\lambda+1)}{2\Gamma(\lambda+r)}\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j-1}-\frac{(-1)^r\phi(r-1)}{(r-1)!}.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
(\lambda+r)\psi(-\lambda-r) &= 1+(\lambda+r)\psi(-\lambda-r+1), \\
\lambda\psi(\lambda+1) &= 1+\lambda\psi(\lambda), \\
\sum_{j=0}^{r-1}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j} &= \sum_{j=0}^{r-1}\binom{\lambda+r}{j}\frac{(-1)^j}{r-j}+\sum_{j=0}^{r-2}\binom{\lambda+r-1}{j}\frac{(-1)^j}{r-j-1},
\end{aligned}
$$

it follows that

$$(\lambda+r)a_{r+1}(\lambda)+a_r(\lambda) = \lambda a_{r+1}(\lambda-1)$$

and we see that equation (2) holds when $-k-1<\lambda<-k$.

Equation (2) therefore holds by induction for negative $\lambda \neq -1,-2,\ldots$ and $r=1,2,\ldots$. A similar argument shows that equation (2) holds for positive $\lambda \neq 1,2\ldots$. This completes the proof of the theorem.

Corollary 2.1 *The neutrix product $x_-^{\lambda} \circ x_-^{-\lambda-r}$ exists and*

$$x_-^{\lambda} \circ x_-^{-\lambda-r} = x_-^{-r} - (-1)^r a_r(\lambda)\delta^{(r-1)}(x) \tag{10}$$

for $\lambda \neq 0,\pm1,\pm2,\ldots$.

Proof. Equation (10) follows immediately on replacing x by $-x$ in equation (2).

In the next corollary, the distribution $(x+i0)^{\lambda}$ is defined by

$$(x+i0)^{\lambda} = x_+^{\lambda} + e^{i\lambda\pi}x_-^{\lambda}$$

for $\lambda \neq 0,\pm1,\pm2,\ldots$ and

$$(x+i0)^{-r} = x^{-r} + \frac{(-1)^r i\pi}{(r-1)!}\delta^{(r-1)}(x), \tag{11}$$

for $r=1,2,\ldots$, see [**7**].

Corollary 2.2 *The neutrix product* $(x+i0)^\lambda \circ (x+i0)^{-\lambda-r}$ *exists and*

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-r} = (x+i0)^{-r} \tag{12}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. The neutrix product is distributive with respect to addition and so

$$\begin{aligned}(x+i0)^\lambda \circ (x+i0)^{-\lambda-r} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13)\\ = x_+^\lambda \circ x_+^{-\lambda-r} + (-1)^r x_-^\lambda \circ x_-^{-\lambda-r} + (-1)^r e^{-i\lambda\pi} x_+^\lambda \circ x_-^{-\lambda-r} + e^{i\lambda\pi} x_-^\lambda \circ x_+^{-\lambda-r}.\end{aligned}$$

Further, it was proved in [**4**] that

$$x_+^\lambda \circ x_-^{-\lambda-r} = (-1)^{r-1} x_-^\lambda \circ x_+^{-\lambda-r} = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}(x) \tag{14}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$. It follows from equations (2), (10), (11), (13) and (14) that

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-r} = x^{-r} + \frac{(-1)^r i\pi}{(r-1)!}\delta^{(r-1)}(x) = (x+i0)^{-r},$$

proving equation (12).

We finally note that the following results can be proved similarly.

$$\begin{aligned}
|x|^\lambda \circ (\operatorname{sgn} x|x|^{-\lambda-2r+1}) &= x^{-2r+1},\\
|x|^\lambda \circ (\operatorname{sgn} x|x|^{-\lambda-2r}) &= \operatorname{sgn} x|x|^{-2r} + \Big[2a_{2r}(\lambda) + \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-1)!}\Big]\delta^{(2r-1)}(x),\\
(\operatorname{sgn} x|x|^\lambda) \circ |x|^{-\lambda-2r+1} &= x^{-2r+1},\\
(\operatorname{sgn} x|x|^\lambda) \circ |x|^{-\lambda-2r} &= \operatorname{sgn} x|x|^{-2r} + \Big[2a_{2r}(\lambda) - \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-1)!}\Big]\delta^{(2r-1)}(x),\\
|x|^\lambda \circ |x|^{-\lambda-2r+1} &= |x|^{-2r+1} + \Big[2a_{2r-1}(\lambda) - \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-2)!}\Big]\delta^{(2r-2)}(x),\\
|x|^\lambda \circ |x|^{-\lambda-2r} &= x^{-2r},\\
(\operatorname{sgn} x|x|^\lambda) \circ (\operatorname{sgn} x|x|^{-\lambda-2r+1}) &= |x|^{-2r+1} + \Big[2a_{2r-1}(\lambda) + \frac{\pi \operatorname{cosec}(\pi\lambda)}{(2r-2)!}\Big]\delta^{(2r-2)}(x),\\
(\operatorname{sgn} x|x|^\lambda) \circ (\operatorname{sgn} x|x|^{-\lambda-2r}) &= x^{-2r}
\end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

References

1. J.G. van der Corput, *Introduction to the neutrix calculus*, J. Analyse Math. **7** (1959-60), 291–398.
2. B. Fisher, *The product of distributions*, Quart. J. Math. Oxford (2), **22** (1971), 291–298.
3. B. Fisher, *Some notes on distributions*, Math. Student **48**(3) (1980), 269–281.
4. B. Fisher, *A non-commutative neutrix product of distributions*, Math. Nachr. **108** (1982), 117–127.
5. B. Fisher and F. Al-Sirehy, *On the non-commutative neutrix product of x_+^λ and $x_+^{-\lambda-1}$*, Submitted for publication.
6. B. Fisher and Y. Kuribayashi, *Neutrices and the Beta function*, Rostock. Math. Kolloq. **32** (1987), 56–66.
7. I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.

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