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ON HANKEL TRANSFORMABLE DISTRIBUTION SPACES

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Abstract. We establish new properties of distribution spaces of slow growth and of exponential growth that are Hankel transformable. We obtain representations of those generalized functions as initial values of solutions of the Kepinski type equation. Also we analyze Hankel positive definite functions and generalized functions. Finally we obtain characterizations of Hankel transformable distributions having bounded above or bounded below support on $(0, \infty)$.

1. Introduction. In this paper we establish new properties of Hankel transformable distribution spaces of slow and of exponential growth.

The Hankel transformation is usually defined by

$$h_\mu(\phi)(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\phi(x)dx, \ \ y \in (0, \infty),$$

where J_μ represents the Bessel function of the first kind and order μ. Throughout this paper we will consider $\mu > -\frac{1}{2}$.

To study the Hankel transformation on distribution spaces A.H. Zemanian introduced in [**23**] the space H_μ that consists of all those complex valued and smooth functions ϕ on $(0, \infty)$ such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in(0,\infty)} (1 + x^2)^m \left| \left(\frac{1}{x}D\right)^k (x^{-\mu-\frac{1}{2}}\phi(x)) \right| < \infty,$$

for every $m, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. On H_μ we consider the topology generated by the family $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbb{N}_0}$ of seminorms. Thus H_μ is a Fréchet space. The Hankel transformation is an automorphism of H_μ [**23**, Lemma 8]. The dual space of H_μ is

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denoted by H'_μ and the elements of H'_μ are distributions of slow growth. The Hankel transformation h'_μ is defined on H'_μ as the transpose of the h_μ-transformation on H_μ. That is, if $T \in H'_\mu$ the Hankel transform $h'_\mu T$ of T is defined by

$$\langle h'_\mu T, \phi\rangle = \langle T, h_\mu \phi\rangle, \ \ \phi \in H_\mu.$$

Let $a > 0$. In [**24**] A.H. Zemanian defined the space $B_{\mu,a}$ constituted by all those functions $\phi \in H_\mu$ such that $\phi(x) = 0$, $x > a$. $B_{\mu,a}$ is a closed subspace of H_μ. The Hankel transform $h_\mu(B_{\mu,a})$ of $B_{\mu,a}$ was characterized in [**24**, Theorem 1]. It is clear that $B_{\mu,a}$ is continously contained in $B_{\mu,b}$, provided that $0 < a < b$. The space $B_\mu = \cup_{a>0} B_{\mu,a}$ is endowed with the inductive topology. As usual B'_μ will denote the dual space of B_μ.

Topological properties of the spaces H_μ, B_μ and their duals were established in [**2**] and [**3**].

J.J. Betancor and L. Rodríguez-Mesa [**8**] studied the Hankel transform of distributions of exponential growth. We introduced the space χ_μ that consists of all those complex valued and smooth functions ϕ defined on $(0,\infty)$ satisfying

$$\eta^\mu_{m,k}(\phi) = \sup_{x\in(0,\infty)} e^{mx}\Big|\Big(\frac{1}{x}D\Big)^k (x^{-\mu+\frac{1}{2}}\phi(x))\Big| < \infty,$$

for every $m, k \in \mathbb{N}_0$. χ_μ is a Fréchet space when we consider in χ_μ the topology generated by $\{\eta^\mu_{m,k}\}_{m,k\in\mathbb{N}_0}$. χ'_μ represents the dual space of χ_μ and the elements of χ'_μ are distributions of exponential growth.

By $\mathcal{Q}_\mu$ we denote the space of all those functions Φ verifying the following two conditions:

(i) $z^{-\mu-\frac{1}{2}}\Phi(z)$ is an even and entire function, and

(ii) for every $m, k \in \mathbb{N}_0$,

$$w^\mu_{m,k}(\Phi) \ = \sup_{|\operatorname{Im} z|\le k} (1+|z|^2)^m |z^{-\mu-\frac{1}{2}}\Phi(z)| < \infty.$$

The topology of $\mathcal{Q}_\mu$ is the one generated by $\{w^\mu_{m,k}\}_{m,k\in\mathbb{N}_0}$.

In [**8**, Theorem 2.1] it is proved that the Hankel transformation h_μ is an isomorphism from χ_μ onto $\mathcal{Q}_\mu$. The h_μ-transformation is defined on the dual spaces χ'_μ and $\mathcal{Q}'_\mu$ as the transpose of h_μ on $\mathcal{Q}_\mu$ and χ_μ, respectively.

F.M. Cholewinski [**10**], D.T. Haimo [**17**] and I.I. Hirschman Jr. [**18**] have investigated a convolution operation for a version of the Hankel transfomation closely connected to h_μ. After doing a single change of variable by taking into account the results in [**18**] we can define a convolution of the Hankel transformation h_μ. In particular if f and g are in $L_1(x^{\mu+\frac{1}{2}}dx, (0,\infty))$ the Hankel convolution $f\#g$ of f and g is defined by

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_x g)(y)dy, \ \ x \in (0,\infty),$$

where the Hankel translation τ_x, $x \in (0,\infty)$, is given by

$$(\tau_x g)(y) = \int_0^\infty D_\mu(x,y,z)g(z)dz, \;\; x,y \in (0,\infty),$$

and

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-\frac{1}{2}}(xt)^{\frac{1}{2}}J_\mu(xt)(yt)^{\frac{1}{2}}J_\mu(yt)(zt)^{\frac{1}{2}}J_\mu(zt)dt, \;\; x,y,z \in (0,\infty).$$

J. de Sousa-Pinto [**21**] started the investigation about the Hankel convolution in generalized functions. He defined the Hankel convolution of order $\mu = 0$ on distributions of compact support on $(0,\infty)$. More recently, J.J. Betancor and I. Marrero ([**4**], [**5**], [**6**], [**7**] and [**19**]), J.J. Betancor and B.J. González [**1**] and J.J. Betancor and L. Rodríguez-Mesa ([**8**] and [**9**]) have studied the Hankel convolution on distribution spaces of slow growth and of exponential growth.

In [**19**, Proposition 2.1, (i)] it is established that the Hankel translation τ_x, $x \in (0,\infty)$, defines a continuous mapping from H_μ into itself. The Hankel convolution $T\#\phi$ of $T \in H'_\mu$ and $\phi \in H_\mu$ is defined by

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle, \;\; x \in (0,\infty). \tag{1.1}$$

In [**19**, Proposition 4.3] and [**5**, Proposition 2.5] we characterized the space $\mathcal{O}'_{\mu,\#}$ constituted by the elements of H'_μ that generate convolution operators in H'_μ.

The Hankel convolution is studied on χ'_μ in [**8**]. If $T \in \chi'_\mu$ and $\phi \in \chi_\mu$ the $\#$-convolution $T\#\phi$ of T and ϕ is also defined by (1.1).

This paper, where we analyze new properties of the distributions in H'_μ and χ'_μ, is organised as follows. In Section 2 we represent the generalized functions in H'_μ and χ'_μ as initial values of solutions of the Kepinski type equations [**22**, p. 99]

$$S_{\mu,x}U = \frac{\partial}{\partial t}U,$$

where $S_{\mu,x} = x^{-\mu-\frac{1}{2}}Dx^{2\mu+1}Dx^{-\mu-\frac{1}{2}}$. The Hankel positive definite functions and generalized functions are studied in Section 3. Finally, in Section 4 we obtain characterizations of the distributions in H'_μ having bounded above or bounded below support in $(0,\infty)$.

Throughout this paper C will always represent a positive constant not necessarily the same in each occurrence.

2. Hankel transformable generalized functions as initial values of solutions of Kepinski type equations. In this Section we obtain representations of the elements of H'_μ and χ'_μ as the initial values of solutions of the Kepinski type equation [**22**, p. 99].

Firstly we need to prove a result that will be essential in the sequel.

We will denote by E the function defined by

$$E(x,t) = x^{\mu+\frac{1}{2}}(2t)^{-\mu-1}\exp\Big(-\frac{x^2}{4t}\Big), \quad x,t \in (0,\infty).$$

According to [**14**, (10), p. 29] the following useful formula

$$h_\mu(E(.,t))(y) = e^{-ty^2}, \;\; y,t \in (0,\infty), \tag{2.1}$$

holds.

Lemma 2.1 *(i) If $\phi \in H_\mu$ then*

$$E(.,t)\#\phi \to \phi, \;\; \textit{as } t \to 0^+, \;\; \textit{in } H_\mu. \tag{2.2}$$

(ii) If $\phi \in \chi_\mu$ then

$$E(.,t)\#\phi \to \phi, \;\; \textit{as } t \to 0^+, \;\; \textit{in } \chi_\mu. \tag{2.3}$$

Proof. (i) For every $t \in (0,\infty)$, $E(.,t) \in H_\mu$. Then, according to [**19**, Proposition 2.2, (i)], $E(.,t)\#\phi \in H_\mu$, for each $t \in (0,\infty)$.

Let $\phi \in H_\mu$. By invoking the interchange formula [**19**, (1.3)] and [**23**, Lemma 8], (2.2) is equivalent to

$$x^{-\mu-\frac{1}{2}}h_\mu(E(.,t))h_\mu(\phi) \to h_\mu(\phi), \text{ as } t \to 0^+, \text{ in } H_\mu. \tag{2.4}$$

Write $\psi = h_\mu(\phi)$. By (2.1) to see (2.4) we have to show that

$$e^{-ty^2}\psi(y) \to \psi(y), \text{ as } t \to 0^+, \text{ in } H_\mu.$$

Let $m,k \in N$ and $\varepsilon > 0$. Leibniz rule leads to

$$\begin{aligned} &(1+y^2)^m\Big(\frac{1}{y}D\Big)^k(y^{-\mu-\frac{1}{2}}\psi(y)(e^{-ty^2}-1)) \qquad (2.5)\\ &= (1+y^2)^m\Big(\frac{1}{y}D\Big)^k(y^{-\mu-\frac{1}{2}}\psi(y))(e^{-ty^2}-1)\\ &\quad+\sum_{j=0}^{k-1}\binom{k}{j}(1+y^2)^m\Big(\frac{1}{y}D\Big)^j(y^{-\mu-\frac{1}{2}}\psi(y))(-2t)^{k-j}e^{-ty^2}, \;\; t,y \in (0,\infty). \end{aligned}$$

It is clear that

$$\frac{1}{1+y^2}|e^{-ty^2}-1| \le \frac{2}{1+y^2}, \quad t,y \in (0,\infty).$$

Hence, there exists $y_0 \in (0,\infty)$ such that, for every $y > y_0$ and $t \in (0,\infty)$,

$$\frac{1}{1+y^2}|e^{-ty^2} - 1| \leq \varepsilon. \tag{2.6}$$

Moreover, we can find $\delta > 0$, for which

$$\frac{1}{1+y^2}|e^{-ty^2} - 1| \leq \varepsilon, \;\; y \leq y_0 \text{ and } 0 < t < \delta. \tag{2.7}$$

By combining (2.5), (2.6) and (2.7) it concludes that

$$\gamma^{\mu}_{m,k}(\psi(e^{-ty^2} - 1)) \to 0, \;\text{ as } t \to 0^+.$$

Thus the proof of (i) is finished.

(ii) Let $\phi \in \chi_\mu$. Since $E(.,t) \in \chi_\mu$, for each $t \in (0,\infty)$, according to [**8**, Proposition 3.2], $E(.,t)\#\phi \in \chi_\mu$, for every $t \in (0,\infty)$.

To see (2.3), by [**8**, Theorem 2.1], it is sufficient to prove that

$$e^{-ty^2}\psi(y) \to \psi(y), \text{ as } t \to 0^+, \text{ in } \mathcal{Q}_\mu, \tag{2.8}$$

where $\psi = h_\mu(\phi)$.

Let $m, k \in \mathbb{N}_0$ and $\varepsilon > 0$. We can write

$$|e^{-ty^2} - 1|(1+|y|^2)^m|y^{-\mu-\frac{1}{2}}\psi(y)| \leq C(1+|y|^2)^m|y^{-\mu-\frac{1}{2}}\psi(y)|(e^{-t(\mathrm{Re}\ y)^2}+1),$$
$$0 < t < 1, \;\; |\operatorname{Im} y| \leq k.$$

Hence, there exists $a > 0$ such that, for every $y \in \mathbb{C}$ being $|y| \geq a$ and $|\operatorname{Im} y| \leq k$,

$$|e^{-ty^2} - 1|(1+|y|^2)^m|y^{-\mu-\frac{1}{2}}\psi(y)| \leq \varepsilon, \;\; 0 < t < 1.$$

Moreover, we can find $t_0 \in (0,1)$ for which

$$|e^{-ty^2} - 1|(1+|y|^2)^m|y^{-\mu-\frac{1}{2}}\psi(y)| \leq \varepsilon, \;\; |\operatorname{Im} y| \leq k, \;\; |y| \leq a \text{ and } 0 < t < t_0.$$

Hence, if $0 < t < t_0$, then $w^{\mu}_{m,k}(\psi(e^{-ty^2} - 1)) \leq \varepsilon$.

Thus (2.8) is established. $\square$

Now we characterize the elements of H'_μ as the initial values of solutions of Kepinski-type equations.

THEOREM 2.2 *Let $u \in H'_\mu$. Define the function U by*

$$U(x,t) = (u\#E(.,t))(x), \;\; x, t \in (0,\infty).$$

Then (i) U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ and

$$S_{\mu,x}U(x,t)=\frac{\partial}{\partial t}U(x,t),\ \ x,t\in(0,\infty). \tag{2.9}$$

(ii) For every $T\in(0,\infty)$ there exists $C>0$ and $r\in\mathbb{N}_0$ such that

$$|U(x,t)|\leq Cx^{\mu+\frac{1}{2}}t^{-(\mu+1+2r)}(1+x^2)^r,\ \ x\in(0,\infty)\ \textit{and}\ 0<t<T.$$

(iii) $U(x,t)\to u$, as $t\to 0^+$, in the weak $*$ topology of H'_μ, that is

$$\langle u,\phi\rangle=\lim_{t\to 0^+}\int_0^\infty U(x,t)\phi(x)dx,\ \ \phi\in H_\mu.$$

Conversely, if U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ such that (i) and (ii) hold, then there exists a unique $u\in H'_\mu$ for which

$$U(x,t)=(u\#E(.,t))(x),\ \ x,t\in(0,\infty).$$

Proof. Let $u\in H'_\mu$. Since $E(.,t)\in H_\mu$, $t\in(0,\infty)$, by [**19**, Proposition 3.5] the function U defined by $U(x,t)=(u\#E(.,t))(x)$, $t,x\in(0,\infty)$, is infinitely differentiable and $x^{-\mu-\frac{1}{2}}U$ is a multiplier of H_μ [**3**, Theorem 2.3]. Moreover, [**19**, (3.18)],

$$\int_0^\infty U(x,t)\phi(x)dx=\langle u,E(.,t)\#\phi\rangle,\ \ \phi\in H_\mu \text{ and } t\in(0,\infty).$$

Hence, according to Lemma 2.1(i), (iii) holds.

To see that U satisfies (2.9) we take into account that, by [**19**, Proposition 4.7(ii)],

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-S_{\mu,x}\Big)U(x,t)&=\Big\langle u(y),\frac{\partial}{\partial t}(\tau_xE(.,t))(y)-\tau_x(S_{\mu,x}E(.,t))(y)\Big\rangle\\&=\Big(u\#\Big(\frac{\partial}{\partial t}E(.,t)-S_{\mu,x}E(.,t)\Big)\Big)(x),\ \ x,t\in(0,\infty).\end{aligned}$$

Since $(\frac{\partial}{\partial t}-S_{\mu,x})E(x,t)=0$, $x,t\in(0,\infty)$, we conclude (i).

We now will prove (ii). Since $u\in H'_\mu$ there exists $C>0$ and $r\in\mathbb{N}_0$ such that

$$|\langle u,\phi\rangle|\leq C\max_{0\leq m,k\leq r}\gamma^\mu_{m,k}(\phi),\ \ \phi\in H_\mu. \tag{2.10}$$

Let $k\in\mathbb{N}_0$ and $T\in(0,\infty)$. We can write

$$x^{-\mu-\frac{1}{2}}S^k_{\mu,k}=\sum_{i=0}^k a_{i,k}x^{2i}\Big(\frac{1}{x}D\Big)^{k+i}x^{-\mu-\frac{1}{2}},$$

where $a_{i,k}$, $i = 0, \dots, k$, are suitable real numbers.

Then

$$x^{-\mu-\frac{1}{2}} S_{\mu,x}^k E(x,t) = \sum_{i=0}^{k} a_{i,k} (-1)^{k+i} x^{2i} (2t)^{-(\mu+1+k+i)} \exp\Big(-\frac{x^2}{4t}\Big), \ \ x, t \in (0,\infty),$$

and, if $T \in (0,\infty)$,

$$x^{-\mu-\frac{1}{2}} |S_{\mu,x}^k E(x,t)| \le C t^{-(\mu+1+2k)} (1+x^2)^k \exp\Big(-\frac{x^2}{4t}\Big), \ \ x \in (0,\infty) \text{ and } 0 < t < T.$$

Hence, according to [**19**, Proposition 2.1(ii)] and [**18**, (2), p. 310], we have

$$\begin{aligned} |S_{\mu,y}^k \tau_x(E(.,t))(y)| &\le \int_{|x-y|}^{x+y} D_\mu(x,y,z) |S_{\mu,z}^k E(z,t)| dz \\ &\le C t^{-(\mu+1+2k)} \int_{|x-y|}^{x+y} D_\mu(x,y,z) z^{\mu+\frac{1}{2}} (1+z^2)^k \exp\Big(-\frac{z^2}{4t}\Big) dz \\ &\le C t^{-(\mu+1+2k)} (xy)^{\mu+\frac{1}{2}} \exp\Big(-\frac{(x-y)^2}{8t}\Big), \\ &\qquad\qquad x, y \in (0,\infty) \text{ and } 0 < t < T. \end{aligned} \tag{2.11}$$

From (2.10) and (2.11) it deduces that

$$\begin{aligned} |U(x,t)| &\le C t^{-(\mu+1+2k)} x^{\mu+\frac{1}{2}} \sup_{0<y<\infty} (1+y^2)^r \exp\Big(-\frac{(x-y)^2}{8t}\Big) \\ &\le C t^{-(\mu+1+2k)} x^{\mu+\frac{1}{2}} (1+x^2)^r, \ \ x \in (0,\infty) \text{ and } 0 < t < T. \end{aligned}$$

To prove the converse we proceed as in the proof of [**13**, Theorem 2.4].

Let $m \in \mathbb{N}$ and $T \in (0,\infty)$. Define the function f_m by

$$f_m(t) = 0, \ \ t < 0, \ \text{ and } f_m(t) = \frac{t^{m-1}}{\Gamma(m)}, \ \ t \ge 0.$$

As it is well-known, we can write

$$\Big(\frac{d}{dt}\Big)^m v(t) = \delta(t) + w(t), \tag{2.12}$$

where v is an infinitely differentiable function on $\mathbb{R}$ such that $v(t) = f_m(t)$, $t \le \frac{T}{4}$; and $v(t) = 0$, $t \ge \frac{T}{2}$, and w is an infinitely differentiable function on $\mathbb{R}$ having its support contained in $[\frac{T}{4}, \frac{T}{2}]$. Here, as usual, δ denotes the Dirac functional.

We now define

$$\widetilde{U}(x,t)=\int_0^\infty U(x,t+s)v(s)ds,\ \ 0<t<\frac{T}{2}\text{ and }x\in(0,\infty).$$

Thus $\widetilde{U}$ is an infinitely differentiable function on $(0,\infty)\times(0,\frac{T}{2})$. Moreover, since U satisfies (ii) there exist $C>0$ and $r\in\mathbb{N}_0$ such that

$$|U(x,t)|\leq Cx^{\mu+\frac{1}{2}}t^{-(\mu+1+2r)}(1+x^2)^r,\ \ x\in(0,\infty)\text{ and }0<t<T.$$

Hence, if $m>\mu+2r-1$ it follows

$$\begin{aligned}|\widetilde{U}(x,t)|&\leq Cx^{\mu+\frac{1}{2}}(1+x^2)^r\int_0^{\frac{T}{2}}(t+s)^{-(\mu+1+2r)}|v(s)|ds\\&\leq Cx^{\mu+\frac{1}{2}}(1+x^2)^r\int_0^{\frac{T}{2}}s^{-(\mu-m+2r)}ds\\&\leq Cx^{\mu+\frac{1}{2}}(1+x^2)^r,\ \ 0<t<\frac{T}{2}\text{ and }x\in(0,\infty).\end{aligned}$$

Note that it is also deduced that $\widetilde{U}$ can be continuously extended to $(0,\infty)\times[0,\frac{T}{2})$.

Since $(\frac{\partial}{\partial t}-S_{\mu,x})\widetilde{U}(x,t)=0$, $0<t<\frac{T}{2}$ and $x\in(0,\infty)$, and by (2.12), one has

$$(-S_{\mu,x})^m\widetilde{U}(x,t)=\Big(-\frac{\partial}{\partial t}\Big)^m\widetilde{U}(x,t)=U(x,t)+\int_0^\infty U(x,t+s)w(s)ds,\quad(2.13)$$

for every $0<t<\frac{T}{2}$ and $x\in(0,\infty)$.

Now we introduce the function H defined by

$$H(x,t)=-\int_0^\infty U(x,t+s)w(s)ds,\ \ 0<t<\frac{T}{2}\text{ and }x\in(0,\infty).$$

By proceeding as above we can see that $(\frac{\partial}{\partial t}-S_{\mu,x})H(x,t)=0$, and $|H(x,t)|\leq Cx^{\mu+\frac{1}{2}}(1+x^2)^r$, $0<t<\frac{T}{2}$ and $x\in(0,\infty)$. Also H can be continuously extended to $(0,\infty)\times[0,\frac{T}{2})$.

If we define $g(x)=\widetilde{U}(x,0)$, $x\in(0,\infty)$, and $h(x)=U(x,0)$, $x\in(0,\infty)$, the uniqueness of solution implies that

$$\widetilde{U}(x,t)=(g\#E(.,t))(x),\ \ 0<t<\frac{T}{2}\text{ and }x\in(0,\infty),$$

and

$$H(x,t)=(h\#E(.,t))(x),\ \ 0<t<\frac{T}{2}\text{ and }x\in(0,\infty).$$

Define $u = (-S_\mu)^m g + h$. It is clear that $u \in H'_\mu$. Moreover, by taking into account [**19**, Proposition 4.7(iii)] and (2.13), it infers

$$\begin{aligned}(u\#E(.,t))(x) &= (-S_\mu)^m (g\#E(.,t))(x) + (h\#E(.,t))(x)\\ &= (-S_\mu)^m \widetilde{U}(x,t) + H(x,t) = U(x,t),\ \ 0 < t < \frac{T}{2} \text{ and } x \in (0,\infty).\end{aligned}$$

Furthermore, Lemma 2.1(i) implies that

$$U(.,t) = u\#E(.,t) \to u, \text{ as } t \to 0^+,$$

in the weak $*$ topology of H'_μ.

Hence u is the unique element of H'_μ fulfilling

$$U(x,t) = (u\#E(.,t))(x),\ \ x,t \in (0,\infty).$$

□

As a consequence of Theorem 2.2 we can obtain the following.

Corollary 2.3 *If $u \in H'_\mu$ then there exist $C > 0$, $r, m \in \mathbb{N}_0$ and two continuous functions g and h such that*

$$\begin{aligned}|h(x)| &\le C x^{\mu+\frac{1}{2}}(1+x^2)^r,\ \ x \in (0,\infty),\\ |g(x)| &\le C x^{\mu+\frac{1}{2}}(1+x^2)^r,\ \ x \in (0,\infty),\end{aligned}$$

and for which $u = S_\mu^m g + h$.

Proof. Define the function U by

$$U(x,t) = (u\#E(.,t))(x),\ \ x,t \in (0,\infty).$$

Proposition 2.1 gives us the desired representation for u. □

By proceeding as in the proof of Theorem 2.2 and Corollary 2.3 and by using Lemma 2.1(ii) instead of Lemma 2.1(i) we can establish the corresponding result for the space χ'_μ.

Theorem 2.4 *Let $u \in \chi'_\mu$. Define the function*

$$U(x,t) = (u\#E(.,t))(x),\ \ x,t \in (0,\infty).$$

Then (i) U is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ and (2.9) holds.

(ii) For every $T > 0$ there exist $C > 0$ and $r \in \mathbb{N}_0$ such that

$$|U(x,t)| \le C x^{\mu+\frac{1}{2}} t^{-(\mu+1+2r)} e^{-rx},\ \ 0 < t < T \text{ and } x \in (0,\infty).$$

(iii) $U(.,t) \to u$, *as* $t \to 0^+$, *in the weak* $*$ *topology of* χ'_μ.

Conversely, if U *is an infinitely differentiable function on* $(0,\infty)\times(0,\infty)$ *such that (i) and (ii) hold, then there exists a unique* $u \in \chi'_\mu$ *for which*

$$U(x,t) = (u\#E(.,t))(x), \quad x,t \in (0,\infty).$$

Moreover, if $u \in \chi'_\mu$ *then there exist* $C > 0$, $r, m \in \mathbb{N}_0$ *and two continuous functions* g *and* h *such that*

$$|h(x)| \le Cx^{\mu+\frac{1}{2}}e^{rx}, \quad x \in (0,\infty),$$
$$|g(x)| \le Cx^{\mu+\frac{1}{2}}e^{rx}, \quad x \in (0,\infty),$$

and $u = S_\mu^m g + h$. $\square$

3. Positive definite Hankel transformable generalized functions. F.M. Cholewinski, D.T. Haimo and A.E. Nussbaum [**11**] and A.E. Nussbaum [**20**] have investigated the Bochner theorem for the Hankel transformation. Following [**11**] and [**20**] we say that a function $f \in x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$ is positive definite provided that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a}_j(\tau_{x_i}f)(x_j) \ge 0, \qquad ()$$

for every $n \in \mathbb{N}_0$, $a_i \in \mathbb{C}$, $x_i \in (0,\infty)$, $i = 1, 2, \ldots, n$.

After performing a suitable change of variables, from the results in [**11**] it follows that if f is a positive definite function then there exists a positive measure λ on $(0,\infty)$ such that $\int_0^\infty x^{\mu+\frac{1}{2}}d\lambda(x) < \infty$ and

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}}J_\mu(xy)d\lambda(y), \text{ a.e., } x \in (0,\infty).$$

Also if $u \in H'_\mu$ (respectively, χ'_μ) way say that u is a positive generalized function in H'_μ (respectively, χ'_μ) when

$$\langle u, \phi\#\overline{\phi}\rangle \ge 0, \quad \phi \in H_\mu \text{ (respectively, } \ \chi_\mu).$$

Note that if u is a positive definite generalized function in H'_μ then u is also a positive generalized function in χ'_μ.

The following result is a Hankel version of [**12**, Lemma 2.5] (see [**16**, pp. 153–155]).

THEOREM 3.1 *A positive definite continuous function is a positive definite generalized function in* H'_μ *(and hence in* χ'_μ*). Conversely, if a continuous function in* $x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$ *is a positive definite generalized function in* χ'_μ *then it is a* *positive definite function. Hence if a continuous function in $x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$ is a positive generalized function in H'_μ then it is a positive definite function.*

Proof. Let f be a positive definite continuous function. Since $f\in x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$, f is in H'_μ and according to [**19**, Proposition 3.5] we can write

$$\langle f,\phi\#\overline{\phi}\rangle=\langle f\#\phi,\overline{\phi}\rangle=\int_0^\infty\int_0^\infty(\tau_x f)(y)\phi(x)\overline{\phi}(y)dxdy,\ \ \phi\in H_\mu.$$

By writing each integral as limit of sums, from (3.1) we deduce that $\langle f,\phi\#\overline{\phi}\rangle\geq 0$, for each $\phi\in H_\mu$.

Suppose now that f is a continuous function that is a positive definite generalized function in χ'_μ. We will prove that if λ is a finite measure which is concentrated on a bounded set of $(0,\infty)$ then

$$\int_0^\infty\int_0^\infty(\tau_x f)(y)d\lambda(x)d\overline{\lambda}(y)\geq 0.$$

This property implies immediately that f is a positive definite function.

Let $\{\psi_n\}_{n\in\mathbb{N}_0}$ be a Hankel approximate identity in the sense of [**4**]. That is, there exists a sequence $\{a_n\}_{n\in\mathbb{N}_0}\subset(0,\infty)$ such that $a_n\downarrow 0$, as $n\to\infty$, and the following properties

(i) $\psi_n\in B_{\mu,a_n}$,

(ii) $\psi_n(x)\geq 0$, $x\in(0,\infty)$, and

(iii) $\int_0^\infty\phi_n(x)x^{\mu+\frac{1}{2}}dx=2^\mu\Gamma(\mu+1)$,

hold, for every $n\in\mathbb{N}_0$. Note that if $\{\psi_n\}_{n\in\mathbb{N}_0}$ and $\{\Psi_n\}_{n\in\mathbb{N}_0}$ are Hankel approximate identities then $\{\psi_n\#\Psi_n\}_{n\in\mathbb{N}_0}$ also is a Hankel approximate identity.

Moreover, if $\{\psi_n\}_{n\in\mathbb{N}_0}$ is a Hankel approximate identity and f is a continuous function on $(0,\infty)$ such that $f\in x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$ then, for every $x\in(0,\infty)$,

$$\int_0^\infty(\tau_x\psi_n)(y)f(y)dy\to f(x),\ \text{ as } n\to\infty. \tag{3.2}$$

Indeed, let $x\in(0,\infty)$. According to [**18**, (2), p. 310] we can write

$$\begin{aligned}\int_0^\infty(\tau_x\psi_n)(y)y^{\mu+\frac{1}{2}}dy&=\int_0^\infty y^{\mu+\frac{1}{2}}\int_0^\infty D_\mu(x,y,z)\psi_n(z)dzdy\\&=\int_0^\infty\psi_n(z)\int_0^\infty y^{\mu+\frac{1}{2}}D_\mu(x,y,z)dydz\\&=\frac{1}{2^\mu\Gamma(\mu+1)}x^{\mu+\frac{1}{2}}\int_0^\infty\psi_n(z)z^{\mu+\frac{1}{2}}dz=x^{\mu+\frac{1}{2}},\end{aligned}$$

for every $n\in\mathbb{N}_0$.

Hence, for every $n \in \mathbb{N}_0$, one has

$$\int_0^\infty (\tau_x\psi_n)(y)f(y)dy - f(x) = \int_0^\infty (\tau_x\psi_n)(y)y^{\mu+\frac{1}{2}}(y^{-\mu-\frac{1}{2}}f(y) - x^{-\mu-\frac{1}{2}}f(x))dy.$$

Let $\varepsilon > 0$. There exists $0 < \delta < x$ such that $|y^{-\mu-\frac{1}{2}}f(y) - x^{-\mu-\frac{1}{2}}f(x)| < \varepsilon$ provided that $|x-y| < \delta$. Hence, since $f \in x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$, we can find $n_0 \in \mathbb{N}$ such that

$$\Big|\int_0^\infty (\tau_x\psi_n)(y)f(y)dy - f(x)\Big| \leq C\Big(\int_0^{x-\delta} + \int_{x+\delta}^\infty\Big)(\tau_x\psi_n)(y)y^{\mu+\frac{1}{2}}dy + \varepsilon = \varepsilon,\ \ n \geq n_0.$$

In the last equality we have taken into account that

$$(\tau_x\psi_n)(y) = \int_{|x-y|}^{x+y} D_\mu(x,y,z)\psi_n(z)dz \leq \int_\delta^\infty D_\mu(x,y,z)\psi_n(z)dz,\ \ |x-y| > \delta,$$

and that $\psi_n \in B_{\mu,a_n}$, $n \in \mathbb{N}_0$, for some $\{a_n\}_{n\in\mathbb{N}_0} \subset (0,\infty)$ being $a_n \downarrow 0$, as $n \to \infty$.

Assume that λ is a finite measure and that it is concentrated on $(0,a)$. For every $n \in \mathbb{N}_0$, we define

$$\Psi_n(x) = \int_0^\infty (\tau_x\psi_n)(y)d\lambda(y),\ \ x \in (0,\infty).$$

Note that $\Psi_n(x) = 0$, $x > a + a_n$ and $n \in \mathbb{N}_0$. Indeed, let $n \in \mathbb{N}_0$. According to [**18**, (2), p. 308] we have that

$$\Psi_n(x) = \int_0^a \int_{|x-y|}^{x+y} D_\mu(x,y,z)\psi_n(z)dzd\lambda(y) = 0,\ \ x > a + a_n.$$

Moreover, by [**5**, (1.2)] and [**25**, (7)] one has, for every $x \in (0,\infty)$ and $n,k \in \mathbb{N}_0$,

$$\Big(\frac{1}{x}D\Big)^k(x^{-\mu-\frac{1}{2}}\Psi_n(x)) = \int_0^\infty h_\mu((xt)^{-\mu-k}J_{\mu+k}(xt)h_\mu(S_\mu^k\psi_n)(t))(y)d\lambda(y).$$

Hence, for every $n,k \in \mathbb{N}_0$, since that the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, and by taking into account [**25**, Lemma 5.4-1 and Theorem 5.4-1], we get

$$\sup_{x\in(0,\infty)}\Big|\Big(\frac{1}{x}D\Big)^k(x^{-\mu-\frac{1}{2}}\Psi_n(x))\Big| < \infty.$$

Thus we conclude that, for every $n \in \mathbb{N}_0$, $\Psi_n \in B_\mu$, and then $\Psi_n \in \chi_\mu$ and

$$\langle f, \Psi_n \# \overline{\Psi}_n\rangle \geq 0. \tag{3.3}$$

On the other hand, $\lambda \in \mathcal{O}'_{\mu,\#}$. In fact, for every $\phi \in H_\mu$, we have

$$\langle h'_\mu(\lambda), \phi\rangle = \langle \lambda, h_\mu(\phi)\rangle = \int_0^\infty h_\mu(\phi)(x)d\lambda(x) = \int_0^\infty \phi(y)\int_0^\infty (xy)^{\frac{1}{2}}J_\mu(xy)d\lambda(x)dy.$$

Hence

$$h'_\mu(\lambda)(y) = \int_0^\infty (xy)^{\frac{1}{2}}J_\mu(xy)d\lambda(x), \ \ y \in (0,\infty).$$

Let $k \in \mathbb{N}_0$. We have

$$\Big(\frac{1}{y}D\Big)^k(y^{-\mu-\frac{1}{2}}h'_\mu(\lambda)(y)) = \int_0^\infty x^{2k+\mu+\frac{1}{2}}(xy)^{-\mu-k}J_{\mu+k}(xy)d\lambda(x), \ \ y \in (0,\infty).$$

Since $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$ and λ is supported on a bounded set on $(0,\infty)$, it follows

$$\sup_{y\in(0,\infty)}\Big|\Big(\frac{1}{y}D\Big)^k(y^{-\mu-\frac{1}{2}}h'_\mu(\lambda)(y))\Big| < \infty.$$

Thus we prove that $y^{-\mu-\frac{1}{2}}h'_\mu(\lambda)$ is a multiplier of H_μ ([**3**, Theorem 2.3]). Hence, from [**19**, Proposition 4.2] we deduce that $\lambda \in \mathcal{O}'_{\mu,\#}$.

According to [**19**, Proposition 4.7] it follows

$$\begin{aligned}\langle f, \Psi_n\#\overline{\Psi}_n\rangle &= \langle f, (\lambda\#\psi_n)\#(\overline{\lambda}\#\psi_n)\rangle = \langle f, (\lambda\#\overline{\lambda})\#(\psi_n\#\psi_n)\rangle \\ &= \langle f\#\lambda, \overline{\lambda}\#(\psi_n\#\psi_n)\rangle \\ &= \int_0^\infty (f\#\lambda)(x)\int_0^\infty \tau_x(\psi_n\#\psi_n)(y)d\overline{\lambda}(y)dx \\ &= \int_0^\infty\int_0^\infty \tau_y(\psi_n\#\psi_n)(x)(f\#\lambda)(x)dxd\overline{\lambda}(y), \ \ n \in \mathbb{N}_0.\end{aligned}$$

On the other hand, $f\#\lambda$ is a continuous function on $(0,\infty)$ and we can write

$$|x^{-\mu-\frac{1}{2}}(f\#\lambda)(x)| \le \int_0^\infty x^{-\mu-\frac{1}{2}}|(\tau_x f)(y)|d|\lambda|(y) \le C\int_0^\infty y^{\mu+\frac{1}{2}}d|\lambda|(y), \ \ x \in (0,\infty).$$

Hence $f\#\lambda \in x^{\mu+\frac{1}{2}}L_\infty(0,\infty)$. By (3.2) and by using dominated convergence theorem we conclude, since $\{\psi_n\#\psi_n\}_{n\in\mathbb{N}_0}$ is a Hankel approximate identity, that

$$\int_0^\infty\int_0^\infty \tau_y(\psi_n\#\psi_n)(x)(f\#\lambda)(x)dxd\overline{\lambda}(y) \to \int_0^\infty\int_0^\infty (\tau_x f)(y)d\lambda(x)d\overline{\lambda}(y), \text{ as } n\to\infty.$$

Then, from (3.3) it infers that

$$\int_0^\infty\int_0^\infty (\tau_x f)(y)d\lambda(x)d\overline{\lambda}(y) \ge 0.$$

Thus the proof is finished. $\Box$

Next we give a characterization of the positive definite generalized functions that involve the heat kernel function E.

Proposition 3.2. *Let $u \in H'_\mu$. Then the following two properties are equivalent.*

(i) u is a positive definite generalized function in H'_μ.

(ii) The function $U(x,t) = (u\#E(.,t))(x)$, $x \in (0,\infty)$, is a positive definite function, for every $t \in (0,\infty)$.

Proof. $(i) \Rightarrow (ii)$. By virtue of Theorem 3.1 it is sufficient to prove that

$$\langle U(.,t), \phi\#\overline{\phi}\rangle \geq 0, \;\; \phi \in H_\mu \text{ and } t \in (0,\infty).$$

By taking into account the interchange formula [**19**, (1.3)] and [**14**, (10), p. 29] it is not hard to see that $E(.,t) = E(.,\frac{t}{2})\#E(.,\frac{t}{2})$, $t \in (0,\infty)$.

Then, since $E(.,t) \in H_\mu$, $t \in (0,\infty)$, [**19**, Proposition 3.5] leads to

$$\langle U(.,t), \phi\#\overline{\phi}\rangle = \langle u\#E(.,t), \phi\#\overline{\phi}\rangle = \langle u, (\phi\#E(.,t/2))\#(\overline{\phi\#E(.,t/2)})\rangle \geq 0,$$
$$t \in (0,\infty) \text{ and } \phi \in H_\mu,$$

beacuse u is a positive definite generalized function in H'_μ.

$(ii) \Rightarrow (i)$. Let $\phi \in H_\mu$. According to Theorem 3.1 one has

$$\langle U(.,t), \phi\#\overline{\phi}\rangle \geq 0, \;\; t \in (0,\infty).$$

Hence, [**19**, Proposition 2.2,(i)] and Theorem 2.2,(iii) imply that

$$\langle u, \phi\#\overline{\phi}\rangle \geq 0.$$

Thus we prove that (i) holds. $\Box$

In a similar way we can establish the corresponding property for χ'_μ.

Proposition 3.3. *Let $u \in \chi'_\mu$. The following two properties are equivalent.*

(i) u is a positive definite generalized function in χ'_μ.

(ii) The function $U(x,t) = (u\#E(.,t))(x)$, $x \in (0,\infty)$, is a positive definite function, for every $t \in (0,\infty)$. $\Box$

An immediate consequence of Propositions 3.2 and 3.3 is the following.

Corollary 3.4. *Let $u \in H'_\mu$. Then u is a positive definite generalized function in H'_μ if, and only if, u is a positive definite generalized function in χ'_μ.* $\Box$

4. Distributions in H'_μ having Hankel transforms zero outside the interval $(0,a]$ or outside the interval (a,∞). In [**4**] it was established a Hankel version of the Paley–Wiener theorem. We introduced in [**4**] the space E_μ that consists of all those complex valued and smooth functions ϕ on $(0,\infty)$ such that, for every $k\in\mathbb{N}$, there exists

$$\lim_{x\to 0^+}\Big(\frac{1}{x}D\Big)^k(x^{-\mu-\frac{1}{2}}\phi(x)).$$

E_μ is endowed with the topology generated by the family $\{p^\mu_{m,k}\}_{m\in\mathbb{N}_0\setminus\{0\},k\in\mathbb{N}_0}$ of seminorms, where

$$p^\mu_{m,k}(\phi)=\sup_{x\in(0,m)}\Big|\Big(\frac{1}{x}D\Big)^k(x^{-\mu-\frac{1}{2}}\phi(x))\Big|,\ \ \phi\in E_\mu,$$

for every $m\in\mathbb{N}_0\setminus\{0\}$ and $k\in\mathbb{N}_0$. A functional $T\in H'_\mu$ is in E'_μ, the dual space of E_μ, if and only if, there exists $a=a(T)>0$ such that $\langle T,\phi\rangle=0$, for every $\phi\in E_\mu$ being $\phi(t)=0$, $t<b$, for some $b>a$ [**4**, Proposition 4.4]. Moreover the elements of E'_μ were characterized as follows. A functional $T\in H'_\mu$ is in E'_μ if, and only if, the Hankel transform $F=h'_\mu T$ of T satisfies the following two properties

(i) $z^{-\mu-\frac{1}{2}}F(z)$ is an even and entire function, and

(ii) there exists $C,A>0$ and $r\in\mathbb{N}_0$ such that [**4**, Propositions 4.5 and 4.9].

$$|z^{-\mu-\frac{1}{2}}F(z)|\leq C(1+|z|)^r e^{A|\operatorname{Im} z|},\ \ z\in\mathbb{C}.$$

In this section, inspired by the paper of J.P. Gabardo [**15**], we obtain a new characterization of the elements $T\in H'_\mu$ that are also in E'_μ. Also we characterize the functionals in H'_μ that are, for some $a>0$, zero inside the interval $[a,\infty)$, that is, those $T\in H'_\mu$ such that $\langle T,\phi\rangle=0$, $\phi\in B_{\mu,a}$.

THEOREM 4.1. *Let $T\in H'_\mu$ and $a>0$. Then the following two properties are equivalent*

(i) $\langle h'_\mu T,\phi\rangle=0$, for every $\phi\in H_\mu$, such that supp $\phi\subset(a,\infty)$,

(ii) $\lim_{k\to\infty}R^{-2k}S^k_\mu T=0$, in the weak $*$ topology of H'_μ, for every $R>a$.

Proof. $(i)\Rightarrow(ii)$. Firstly note that by virtue of [**25**, Lemma 5.4-1,(6) and Theorem 5.4-1], for every $R>0$, the sequence $\{R^{-2k}S^k_\mu T\}_{k\in\mathbb{N}_0}$ converges to zero in the weak $*$ topology of H'_μ if, and only if, the sequence $\{R^{-2k}x^{2k}h'_\mu T\}_{k\in\mathbb{N}_0}$ converges to zero in the weak $*$ topology in H'_μ.

Let $a<\varepsilon<\eta<R$. Define $\psi\in C^\infty(0,\infty)$ such that $\psi(y)=1$, $y\in(0,\varepsilon)$, and $\psi(y)=0$, $y\in(\eta,\infty)$. Then $h'_\mu T=\psi h'_\mu T$.

Let now $m\in\mathbb{N}_0$ and $\phi\in H_\mu$. Leibniz rule leads to

$$\Big(\frac{1}{y}D\Big)^m(y^{-\mu-\frac{1}{2}}y^{2k}R^{-2k}\phi(y)\psi(y))$$
$$=R^{-2k}\sum_{j=0}^m\binom{m}{j}2k(2k-2)\cdots(2k-2j+2)y^{2(k-j)}\Big(\frac{1}{y}D\Big)^{m-j}(y^{-\mu-\frac{1}{2}}\phi(y)\psi(y)),$$
$$k\in\mathbb{N}_0 \text{ and } y\in(0,\infty).$$

Hence, since $\psi(y)=0$, $y>\eta$, where $\eta<R$, we conclude that

$$\gamma^{\mu}_{n,m}(y^{2k}R^{-2k}\phi\psi)\to 0,\ \text{as } k\to\infty,$$

for every $\phi\in H_\mu$, and then

$$\langle R^{-2k}y^{2k}h'_\mu(T),\phi\rangle=\langle h'_\mu(T),R^{-2k}y^{2k}\phi\psi\rangle\to 0,\ \text{as } k\to\infty,$$

for every $\phi\in H_\mu$. Thus (ii) is proved.

$(ii)\Rightarrow(i)$. Let $R>a$ and $\phi\in H_\mu$ such that $\phi(x)=0$, $x\le a+\varepsilon$, for some $\varepsilon>0$. Since $R^{-2k}S^k_\mu T\to 0$, as $k\to\infty$, in the weak $*$ topology of H'_μ, [**25**, Theorem 5.4-1] implies that the sequence $\{R^{-2k}y^{2k}h'_\mu T\}_{k\in\mathbb{N}_0}$ is weakly $*$ (or, equivalently, strongly) bounded in H'_μ.

Moreover, for every $k,l,m\in\mathbb{N}_0$ we can write that

$$\begin{aligned}&(1+x^2)^l\Big(\frac{1}{x}D\Big)^m(x^{-\mu-\frac12}x^{-2k}R^{2k}\phi(x))\\ &\qquad=R^{2k}\sum_{j=0}^{m}c_j(k)(1+x^2)^l\Big(\frac{1}{x}D\Big)^j(x^{-\mu-\frac12}\phi(x))x^{-2(k+m-j)},\ \ x\in(0,\infty),\end{aligned}$$

where $c_j(k)$ is a polynomial in k, for every $j=0,\dots,m$.

Hence, it follows

$$\begin{aligned}&\Big|(1+x^2)^l\Big(\frac{1}{x}D\Big)^m(x^{-\mu-\frac12}x^{-2k}R^{2k}\phi(x))\Big|\\ &\qquad\le C\sum_{j=0}^{m}\Big(\frac{R}{x}\Big)^{2k}|c_j(k)|\Big|(1+x^2)^l\Big(\frac{1}{x}D\Big)^j(x^{-\mu-\frac12}\phi(x))\Big|\\ &\qquad\le C\sum_{j=0}^{m}\Big(\frac{R}{a+\varepsilon}\Big)^{2k}|c_j(k)|\gamma^{\mu}_{l,j}(\phi),\ \ x\in(0,\infty)\text{ and } k,l,m\in\mathbb{N}_0.\end{aligned}$$

Then, we conclude that $R^{2k}x^{-2k}\phi\to 0$, as $k\to\infty$, in H_μ, provided that $R<a+\varepsilon$.

Hence, for each $a<R<a+\varepsilon$,

$$\lim_{k\to\infty}\langle R^{-2k}x^{2k}h'_\mu T,R^{2k}x^{-2k}\phi\rangle=0.$$

Thus we prove that $\langle h'_\mu T,\phi\rangle=0$. $\square$

The following result can be seen as a dual version of Theorem 4.1.

THEOREM 4.2. *Let $T\in H'_\mu$ and $a>0$. The following two properties are equivalent.*

(i) $\langle h'_\mu T,\phi\rangle=0$, for every $\phi\in B_{\mu,b}$, with $b<a$.

(ii) There exists a unique sequence $\{L_k\}_{k\in\mathbb{N}_0}$ *in* H'_μ *such that* $L_0 = T$, $S_\mu L_{k+1} = L_k$, $k \in \mathbb{N}_0$, *and* $\lim_{k\to\infty} R^{-2k}h'_\mu L_k = 0$, *in the weak* $*$ *topology of* H'_μ, *for every* $R > \frac{1}{a}$.

Proof. $(i) \Rightarrow (ii)$. We define, for every $k \in \mathbb{N}_0$,

$$L_k = h'_\mu((-x^2)^{-k}h'_\mu T). \tag{4.1}$$

Note that, since $h'_\mu T$ vanishes in $(0, a)$, $L_k \in H'_\mu$, for each $k \in \mathbb{N}_0$. Let $R > \frac{1}{a}$ and $\frac{1}{R} < \delta < a$. Choose a function $\psi \in C^\infty(0,\infty)$ such that $\psi(x) = 0$, $0 < x < \delta$, and $\phi(x) = 1$, $x > \frac{a+\delta}{2}$. From (i) it deduces that

$$h'_\mu T = \psi h'_\mu T.$$

Moreover, for every $\phi \in H_\mu$,

$$\langle R^{-2k}h'_\mu L_k, \phi\rangle = \langle(-1)^k(Rx)^{-2k}h'_\mu T, \phi(x)\rangle = \langle h'_\mu T, (-1)^k(Rx)^{-2k}\phi(x)\psi(x)\rangle.$$

Since ψ is a multiplier of H_μ [**3**, Theorem 2.3], $\phi\psi \in H_\mu$. By taking into account that $\psi(x) = 0$, $0 < x < \delta$, being $R\delta > 1$, by proceeding as in the proof of part $(i) \Rightarrow (ii)$ in Theorem 4.1, we can conclude that

$$\langle R^{-2k}h'_\mu L_k, \phi\rangle \to 0, \text{ as } k \to \infty.$$

We now prove that $\{L_k\}_{k\in\mathbb{N}_0}$ where L_k is defined by (4.1), for every $k \in \mathbb{N}_0$, is the unique sequence satisfying the conditions in (ii).

Assume that, for every $k \in \mathbb{N}_0$, $\mathcal{L}_k \in H'_\mu$, being

$$\mathcal{L}_0 = 0,\ \ S_\mu\mathcal{L}_{k+1} = \mathcal{L}_k,\ \ k \in \mathbb{N}_0 \text{ and } \lim_{k\to\infty} R^{-2k}\mathcal{L}_k = 0,$$

in the weak $*$ topology of H'_μ, for every $R > \frac{1}{a}$.

We define the H'_μ-valued function F by

$$F(\lambda) = \sum_{k=0}^{\infty} \lambda^{2k}\mathcal{L}_{k+1} \tag{4.2}$$

that is holomorphic in $|\lambda| < a$. It is easy to see that

$$S_{\mu,x}F(\lambda) = \sum_{k=0}^{\infty} \lambda^{2k}\mathcal{L}_k = \lambda^2 F(\lambda),\ \ |\lambda| < a.$$

Then, according to [**25**, Theorem 5.5-2,(8)] it follows

$$-x^2 h'_\mu F(\lambda) = \lambda^2 h'_\mu F(\lambda),\ \ |\lambda| < a. \tag{4.3}$$

Since, for every $\lambda \neq 0$, the function $f(x) = x^2 + \lambda^2$, $x \in (0, \infty)$, is a multiplier in H_μ ([**25**, Lemma 5.3-1]) from (4.3) we infer that

$$\langle h'_\mu F(\lambda), \phi \rangle = \left\langle (x^2 + \lambda^2) h'_\mu F(\lambda), \frac{\phi}{x^2 + \lambda^2} \right\rangle = 0, \ \ \phi \in H_\mu \text{ and } 0 < \lambda < a.$$

Hence $F(\lambda) = 0$, $0 < |\lambda| < a$. From the representation (4.2) we conclude that $\mathcal{L}_k = 0$, $k \in \mathbb{N}_0$. Thus the uniqueness of the sequence $\{L_k\}_{k \in \mathbb{N}_0}$ satisfying the properties in (ii) is established.

$(ii) \Rightarrow (i)$. Let $\{L_k\}_{k \in \mathbb{N}_0}$ be a sequence in H'_μ satisfying the properties in (ii). Let $\phi \in B_{\mu,b}$, where $0 < b < a$, and let $R \in (\frac{1}{a}, \frac{1}{b})$. We can write

$$\langle h'_\mu T, \phi \rangle = \langle h'_\mu (S^k_\mu L_k), \phi \rangle = \langle (-x^2)^k h'_\mu L_k, \phi \rangle = \langle R^{-2k} h'_\mu L_k, (-1)^k (xR)^{2k} \phi \rangle, \ \ k \in \mathbb{N}_0.$$

Since $\phi(x) = 0$, $x \geq b$, being $Rb < 1$, $(xR)^{-2k}\phi \to 0$, as $k \to \infty$, in H_μ. Hence, by taking into account that $R^{-2k} h'_\mu L_k \to 0$, as $k \to \infty$, in the weak $*$ topology of H'_μ, it concludes that $\langle h'_\mu T, \phi \rangle = 0$, and the proof is completed. $\square$

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