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REPRESENTATION THEOREMS FOR TEMPERED ULTRADISTRIBUTIONS

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Abstract. We consider classes of spaces of Beurling and Roumieu type tempered ultradistributions containing some spaces of quasianalytic tempered and all spaces of non-quasianalytic tempered ultradistributions. We prove that every ultradistribution f in a space of the considered classes has the form

$$f = P(\Delta)u_1 + u_2,$$

where P is an ultradifferential operator, u_1 is a smooth function, u_2 is a real analytic function, and both of them satisfy some exponential growth conditions. Also, we give the boundary value representations for elements in the spaces of considered classes. Precisely, we prove that every solution of the heat equation, with appropriate exponential growth rate, defines an element in a space of the corresponding class, and conversely, that every element in a space of the considered classes is a boundary value of a solution of the heat equation with appropriate exponential growth rate.

1. Introduction

Tempered ultradistributions spaces have appeared in papers of many authors in the last three decades and even earlier. Among others we mention Gel'fand and Shilov [**7**], Björk [**2**], Wloka [**21**], Grudzinski [**8**], Avantaggiati [**1**], De Roever [**20**], Kashpirovskij [**10**], Pathak [**17**] and Pilipović [**19**]. In general, besides Beurling and Roumieau, for the theory of boundary value problems and of ultradistributions we should mention Köthe, Tilmann and their pupils, Sebastião e Silva and his school, H. Komatsu and the Japanese school, C.C. Chou, J. Ciorănescu, V.V.Žarinov, S. Pilipovic and many others who have contributed much to the theory. In this paper, we give new representation theorems for the spaces in classes of tempered ultradistributions containing some quasianalytic tempered and all non-quasianalytic tempered ultradistributions of Beurling type as well as of Roumieu type. Examples of spaces of tempered ultradistributions which are considered in the paper

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are: (in the Roumieu case) dual spaces of Gelfand–Shilov space $\mathcal{S}^{\alpha}_{\beta}$, $\alpha, \beta > 1/2$, dual spaces of the generalized Gelfand–Shilov spaces of S–type (introduced in [**7**]), and (in the Beurling case) the space Σ'_{α} $\alpha > 1/2$, (introduced in [**19**]).

In a series of papers [**14**], [**15**] and [**16**], Matsuzawa developed a calculus approach to the theory of hyperfunctions by treating an element $u(x)$ of a space of generalized functions as the initial value of a unique solution $U(x,t)$ of the heat equation

$$\begin{aligned} &\Big(\frac{d}{dt} - \Delta\Big)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \\ &U(x,0) = u(x), \end{aligned} \tag{1}$$

which satisfies the appropriate growth rate condition determined by the space. In this way the elements of the following spaces were characterized: the space of distributions $\mathcal{D}'$, spaces of Gevrey ultradistributions with compact support $\mathcal{E}'^{\{s\}}$, $\mathcal{E}'^{(s)}$, $s > 1$ [**15**]; spaces of tempered distributions $\mathcal{S}'$ [**16**]; spaces of hyperfunctions $\mathcal{B}$ [**14**]; spaces of Gevrey tempered ultradistributions (of Roumieu type) $\mathcal{S}'^{\beta}_{\alpha}$; spaces of Beurling type ultradistributions with compact support $\mathcal{E}'^{(M_p)}$; spaces of Fourier hyperfunctions $\mathcal{F}'$ [**11**] and extended Fourier hyperfunctions $\mathcal{G}'$ [**5**].

In this paper we use the heat kernel tehnique to characterize classes of Beurling as well as Roumieu type tempered ultradistributions. Our interest lies in the quasianalytic case, although the theorems do not exclude the non-quasianalytic case. We prove that every ultradistribution f in the considered classes has the form

$$f(x) = P(\Delta)u_1(x) + u_2(x),$$

where P is an ultradifferential operator, $u_1(x)$ is a smooth function, $u_2(x)$ is a real analytic function, and both functions satisfy some exponential growth conditions. Also, we give the boundary value representations for elements of the considered classes. Precisely, we prove that every solution of the heat equation, with appropriate exponential growth rate, defines an element of the corresponding class, and conversely, that every element of the considered classes is a boundary value of a solution of the heat equation with appropriate exponential growth rate.

Our results concerning boundary value representations for elements of Roumieu type tempered ultradistributions spaces generalize results of the paper [**6**]. This generalization is not a trivial one, since instead of Gevrey sequences ($\{p!^s,\ p \in \mathbb{N}\}$), satisfying strong conditions, we deal with more general class of defining sequences.

2. Preliminaries

We use multi-index notation $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$, $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$, where $d \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$, $\varphi(x) \in C^\infty(\mathbb{R}^d)$,

$$\varphi^{(\alpha)}(x) = (\partial/\partial x)^\alpha \varphi(x) = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \cdots (\partial/\partial x_d)^{\alpha_d}.$$

Let $\{M_p,\ p \in \mathbb{N}_0\}$ and $\{N_p,\ p \in \mathbb{N}_0\}$ be sequences of positive numbers, where $M_0 = N_0 = 1$. The following conditions will be used: (for their detailed analysis see, for example [**12**])

(M.1) (*logarithmic convexity*)

$$M_p^2 \leq M_{p-1} M_{p+1}, \quad p = 1, 2, \dots$$

(M.2) (*stability under ultradifferential operators*) There are constants A and H such that

$$M_p \leq A H^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \dots$$

(M.3) (*strong non-quasi-analyticity*) There is a constant A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq A \frac{p M_p}{M_{p+1}}, \quad p = 1, 2, \dots$$

Some results remain valid, however, when (M.2) and (M.3) are replaced by the following weaker conditions:

(M.2)' (*stability under differential operators*) There are constants A and H such that

$$M_{p+1} \leq A H^p M_p, \quad p = 0, 1, \dots;$$

(M.3)' (*non-quasi-analyticity*)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

The corresponding conditions for the sequence $\{N_p,\ p \in \mathbb{N}_0\}$ will be denoted by (N.1), (N.2), (N.3), (N.2)'and (N.3)'.

The so-called associated functions for the sequence $\{M_p,\ p \in \mathbb{N}_0\}$ are

$$M(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p}{M_p}, \quad \widetilde{M}(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p p!}{M_p}, \quad \overline{M}(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p p!}{M_p^2},$$

where $\rho > 0$.

The corresponding associated functions for sequence $\{N_p,\ p \in \mathbb{N}_0\}$ will be denoted by $N(\cdot)$, $\widetilde{N}(\cdot)$ and $\overline{N}(\cdot)$.

Remark 1. The Gevrey sequence

$$p^{sp} \quad \text{or} \quad (p!)^s \quad \text{or} \quad \Gamma(1+sp), \quad p \in \mathbb{N}_0, \quad s > 1,$$

satisfies all the above conditions and $M(\rho) \sim \rho^{1/s}$, $\widetilde{M}(\rho) \sim \rho^{1/(s-1)}$, $\overline{M}(\rho) \sim \rho^{1/(2s-1)}$ (see [**7**]).

We shall also use the following condition which is equivalent to the fact that the sequence $N_p = M_p^2$, $p \in \mathbb{N}_0$ satisfies condition (N.3) and which follows from (M.3) see [**18**, p. 300].

(C) There exists a positive integer k such that

$$\liminf_{p\to\infty}\left(\frac{m_{kp}}{m_p}\right) > k,$$

where $m_p = M_p/M_{p-1}$, $p = 1, 2, \ldots$.

Remark 2. 1. The equence $M_p = p!^s$, $p \in \mathbb{N}_0$, $s > 1/2$, satisfies condition (C), but not (M.3), $(1/2 < c \leq 1)$.

2. If $m_p = p(\log p)^\alpha$, $\alpha > 0$, $p = 1, 2, \ldots$ then the sequence $\{M_p = m_2 \cdots m_p,\ p \in \mathbb{N}_0\}$ satisfies (C).

An operator of the form

$$P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha, \ \ a_\alpha \in \mathbb{C},$$

is called an ultradifferential operator of class (M_p) (respectively of class $\{M_p\}$) if there are positive constants L and C (respectively for every $L > 0$ there is a constant $C > 0$) such that

$$|a_\alpha| \leq \mathcal{C} L^\alpha / M_\alpha, \quad \alpha \in \mathbb{N}_0^d.$$

$\mathcal{R}$ denotes a family of increasing sequences $\{h_p,\ p \in \mathbb{N}\}$, with positive elements, tending to infinity.

3. Spaces $\mathcal{S}^{(M_p)}_{(N_p)}$, $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$ and their duals

Definition 1. The space $\mathcal{S}^{M,m}_{N,n}$, $m, n > 0$, is the space of smooth functions φ on $\mathbb{R}^d$, such that

$$|x^\beta \varphi^{(\alpha)}(x)| \leq C_\varphi \frac{M_{|\alpha|} N_{|\beta|}}{m^{|\alpha|} n^{|\beta|}}, \ \text{ for every } \alpha, \beta \in \mathbb{N}_0^d, \tag{3.1}$$

where the constant C_φ depends only on φ. It is a Banach space with the norm

$$s_{m,n}(\varphi) = \sup_{\alpha,\beta \in \mathbb{N}_0^d} \frac{m^{|\alpha|} n^{|\beta|}}{M_{|\alpha|} N_{|\beta|}} \|x^\beta \varphi^{(\alpha)}(x)\|_\infty. \tag{3.2}$$

Let

$$\mathcal{S}^{(M_p)}_{(N_p)} = \operatorname*{proj\,lim}_{\substack{m\to\infty\\ n\to\infty}} \mathcal{S}^{M,m}_{N,n}, \qquad \mathcal{S}^{\{M_p\}}_{\{N_p\}} = \operatorname*{ind\,lim}_{\substack{m\to 0\\ n\to 0}} \mathcal{S}^{M,m}_{N,n}.$$

The notation $\mathcal{S}^*_\dagger$ denotes $\mathcal{S}^{(M_p)}_{(N_p)}$ or $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$.

Remark 3. The inclusion $i : \mathcal{S}^*_\dagger \to \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions, is continuous.

Remark 4. The space $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$ is a generalized Gel'fand–Shilov space of S–type as defined in [**G**]. In particular, if $M_p = p^{rp}$ and $N_p = p^{sp}$, $p \in \mathbb{N}_0$, $s, r > 0$, the Gelfand–Shilov space $\mathcal{S}^s_r$ is equal to $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$.

Remark 5. If conditions (M.1), (M.3)', (N.1), (N.2)' and (N.3)' are satisfied, the spaces $\mathcal{S}^{(M_p)}_{(N_p)}$ and $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$ are test spaces for tempered ultradistibutional spaces of Beurling and Roumieu type, respectively. More precisely (see [**13**]),

1. The inclusion $i : \mathcal{D}^* \to \mathcal{S}^*_\dagger$ is continuous.
2. The set $\mathcal{S}^*_\dagger \backslash \mathcal{D}^*$ is nonempty.
3. The space $\mathcal{D}^*$ is dense in $\mathcal{S}^*_\dagger$.

Here $\mathcal{D}^*$ denotes the spaces $\mathcal{D}^{(M_p)}$ or $\mathcal{D}^{\{M_p\}}$. For the definition and properties of $\mathcal{D}^*$ see [**12**].

Remark 6. If conditions (M.1), (M.3)', (N.1) and (N.3)' are satisfied, the inclusions $i : \mathcal{F} \to \mathcal{S}^{\{M_p\}}_{\{N_p\}}$, and $i : \mathcal{G} \to \mathcal{S}^{(M_p)}_{(N_p)}$, are continuous. Here $\mathcal{G}$ is a test space for the space $\mathcal{G}'$ of extended Fourier hyperfunctions (defined as in [**5**]), and $\mathcal{F}$ is a test space for the space $\mathcal{F}'$ of Fourier hyperfunctions (defined as in [**11**]). (For the proof see [**13**].)

The following theorem characterizes the topology in the spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$.

THEOREM 1. *Let conditions (M.1), (C), (N.1), (N.2)' and (N.3)' be satisfied. A sequence φ_j in the space $\mathcal{S}^{(M_p)}_{(N_p)}$ (respectively $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$), converges to zero in the space $\mathcal{S}^{(M_p)}_{(N_p)}$ (respectively in $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$), as $j \to \infty$, if and only if, for every $m, n > 0$ (respectively for some $m, n > 0$)*

$$\sigma_{m,n}(\varphi_j) = \sup_{\substack{\alpha \in \mathbb{N}_0^d \\ x \in \mathbb{R}^d}} \frac{m^{|\alpha|}}{M_{|\alpha|}} \big|\varphi_j^{(\alpha)}(x) \exp[N(n|x|)]\big| \to 0, \quad as \;\; j \to \infty. \tag{3.3}$$

Proof. Let us prove the theorem for the space $\mathcal{S}^{(M_p)}_{(N_p)}$.

(a) First, we prove that $\sigma_{m,n}(\varphi_j) \to 0$, as $j \to 0$, implies that $s_{m,n}(\varphi_j) \to 0$, as $j \to 0$. Let $m_1, n_1 > 1$, and let $j \in \mathbb{N}$ be fixed. Since $\varphi_j \in \mathcal{S}^{(M_p)}_{(N_p)}$, for every $m, n > 0$,

$$m_1^{|\alpha|} n_1^{|\beta|} \frac{m^{|\alpha|} n^{|\beta|}}{M_{|\alpha|} N_{|\beta|}} \big|x^\beta \varphi_j^{(\alpha)}(x)\big| < \infty, \quad \alpha, \beta \in \mathbb{N}_0^d,$$

it follows that

$$\frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)| \text{ converges to zero uniformly in } x \in \mathbb{R}^d, \text{ as } |\alpha+\beta| \to \infty. \tag{3.4}$$

By condition (N.2)' for every $m, n > 0$, there exists a $C > 0$, such that for every $\alpha, \beta \in \mathbb{N}_0^d$,

$$\frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)| \le C\frac{m^{|\alpha|}}{M_{|\alpha|}}\frac{(1+nH)^{|\beta|+1}}{N_{|\beta|+1}}|x^{|\beta|+1}\varphi_j^{(\alpha)}(x)|\frac{1}{|x|}.$$

From conditions (M.1), (C), (N.1) and (N.3)' it follows that the sequences $\{m^{|\alpha|}/M_{|\alpha|}, \alpha \in \mathbb{N}_0^d\}$ and $\{(1+nH)^{|\beta|+1}/N_{|\beta|+1},\ \beta \in \mathbb{N}_0^d\}$ are bounded. Therefore, there exists a constant $C > 0$ such that

$$\frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)| \le \frac{C}{k}, \qquad |x| \ge k > 1,$$

which implies that

$$\frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)| \text{ converges to zero uniformly in } (\alpha,\beta) \in \mathbb{N}^{2d}, \text{ as } |x| \to \infty. \tag{3.5}$$

Now, (3.4) and (3.5) imply that there exist $x_0 \in \mathbb{R}^d$, $\alpha_0, \beta_0 \in \mathbb{N}_0^d$, such that

$$\begin{aligned} s_{m,n}(\varphi_j) &= \sup_{\alpha,\beta\in\mathbb{N}_0^d}\sup_{x\in\mathbb{R}^d}\frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)| = \frac{m^{|\alpha_0|}n^{|\beta_0|}}{M_{|\alpha_0|}N_{|\beta_0|}}|x_0^{\beta_0}\varphi_j^{(\alpha_0)}(x_0)| \\ &\le \sup_{\alpha\in\mathbb{N}_0^d}\frac{m^{|\alpha|}}{M_{|\alpha|}}\|\varphi_j^{(\alpha)}(x)\exp[N(n|x|)]\|_\infty = \sigma_{m,n}(\varphi_j). \end{aligned} \tag{3.6}$$

(b) Let us prove that $s_{m,n}(\varphi_j) \to 0$ as $j \to 0$, implies that $\sigma_{m,n}(\varphi_j), \to 0$, as $j \to 0$. Let $j \in \mathbb{N}$ be fixed. Similarly as above one can prove that $\varphi_j \in \mathcal{S}_{(N_p)}^{(M_p)}$ implies

1. $\dfrac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)|$ converges to zero uniformly in $(x,\alpha) \in \mathbb{R}^d \times \mathbb{N}_0^d$, as $|\beta| \to \infty$;
2. $\dfrac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)|$ converges to zero uniformly in $(x,\beta) \in \mathbb{R}^d \times \mathbb{N}_0^d$, as $|\alpha| \to \infty$;
3. $\dfrac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}|x^{\beta}\varphi_j^{(\alpha)}(x)|$ converges to zero uniformly in $(\alpha,\beta) \in \mathbb{N}_0^d \times \mathbb{N}_0^d$, as $|x| \to \infty$.

From these facts there exist α_1, $\beta_1 \in \mathbb{N}_0^d$ and $x_1 \in \mathbb{R}^d$ such that

$$\begin{aligned} \sigma_{m,n}(\varphi_j) &= \sup_{\alpha\in\mathbb{N}_0^d}\frac{m^{|\alpha|}}{M_{|\alpha|}}\|\varphi_j^{(\alpha)}(x)\exp[N(n|x|)]\|_\infty = \frac{m^{|\alpha_1|}n^{|\beta_1|}}{M_{|\alpha_1|}N_{|\beta_1|}}|x_1^{\beta_1}\varphi_j^{(\alpha_1)}(x_1)| \\ &\le \sup_{\alpha,\beta\in\mathbb{N}_0^d}\frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}}\|x^{\beta}\varphi_j^{(\alpha)}(x)\|_\infty = s_{m,n}(\varphi_j). \end{aligned} \tag{3.7}$$

Analogously, the assertion can be proved for the space $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$. $\square$

From the proof of the second part of the previous theorem we have:

THEOREM 2. *Let conditions (M.1), (C), (N.1), (N.2)' and (N.3)' be satisfied. If* $\varphi \in \mathcal{S}^{(M_p)}_{(N_p)}$ *(respectively* $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$*) then there exists* $C > 0$ *such that for every* $m, n > 0$ *(respectively some* $m, n > 0$*)*

$$|\varphi^{(\alpha)}(x)| \le C \frac{M_{|\alpha|}}{m^{|\alpha|}} \exp[-N(n|x|)], \quad x \in \mathbb{R}^d. \tag{3.8}$$

Let us give the heat kernel characterization of the spaces $\mathcal{S}^*_\dagger$. Denote by $E(x,t)$ the heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-d/2} \exp[-|x|^2/4t], & t > 0, \\ 0, & t < 0. \end{cases} \tag{3.9}$$

The function $E(x,t)$ is an entire function of order 2 for $t > 0$ and has the following properties [**15**].

(E0) $E(x,t)$ satisfies the heat equation.

(E1) $\int_{\mathbb{R}^d} E(x,t)dx = 1, \; t > 0$.

(E2) There are positive constants C and a' such that

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} E(x,t)\right| \le C^{|\alpha|+1} t^{-(|\alpha|+d)/2} \alpha!^{1/2} \exp[-a'|x|^2/4t], \quad t > 0, \tag{3.10}$$

where $a' \in (0,1)$ can be taken as close as desired to 1.

(E3) If conditions (M.1), (M.3)', (N.1) and (N.3)' are satisfied, $E(\cdot, t)$ is an element of $\mathcal{S}^*_\dagger$, for every $t > 0$.

THEOREM 3. *Let conditions (M.1), (M.2)', (C), (N.1), (N.2)' and (N.3)' be satisfied and* $\varphi \in \mathcal{S}^*_\dagger$. *For every* $t > 0$, *the function*

$$U(x,t) = \int_{\mathbb{R}^d} E(x-y,t)\varphi(y)dy \tag{3.11}$$

is an element of $\mathcal{S}^*_\dagger$, *and* $U(x,t)$ *converges to* $\varphi(x)$ *in* $\mathcal{S}^*_\dagger$, *as* $t \to 0$.

Proof. We shall prove this theorem for $\varphi \in \mathcal{S}^{(M_p)}_{(N_p)}$. The proof for $\varphi \in \mathcal{S}^{\{M_p\}}_{\{N_p\}}$ is analogous.

Let us prove that

$$\sup_{\alpha \in \mathbb{N}_0^d} \frac{m^{|\alpha|}}{M_{|\alpha|}} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \big(U(x,t) - \varphi(x)\big) \exp[N(n|x|)] \right\|_\infty \to 0, \quad \text{as } t \to 0, \tag{3.12}$$

for every $m, n > 0$. Let δ be a small positive number. We have

$$\begin{aligned}\Big|\frac{\partial^\alpha}{\partial x^\alpha}\big(U(x,t)-\varphi(x)\big)\Big| &= \Big|\int_{\mathbb{R}^d} E(x-y,t)(\varphi^{(\alpha)}(y)-\varphi^{(\alpha)}(x))dy\Big| \\ &= \Big|\int_{\mathbb{R}^d} E(y,t)(\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x))dy\Big| \\ &\leq \int_{|y|\leq\delta} E(y,t)|\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)|dy+ \\ &\quad + \int_{|y|\geq\delta} E(y,t)|\varphi^{(\alpha)}(x-y)|dy \\ &\quad + \int_{|y|\geq\delta} E(y,t)|\varphi^{(\alpha)}(x)|dy \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.13}$$

Since $\varphi \in \mathcal{S}_{(N_p)}^{(M_p)}$, by the mean value theorem and Theorem 2, we have

$$\begin{aligned} I_1 &= \int_{|y|\leq\delta} E(y,t)|\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)|dy \\ &\leq \int_{|y|\leq\delta} E(y,t)|\varphi^{(\alpha+1)}(x-\theta y)|\ |y|\ dy \\ &\leq C\frac{M_{|\alpha|+1}}{\tilde{m}^{|\alpha|+1}}\ \ \delta\int_{|y|\leq\delta} E(y,t)\exp[-N(\tilde{n}|x-\theta y|)]dy, \end{aligned} \tag{3.14}$$

for some $\theta \in (0,1)$ and every $\tilde{m}, \tilde{n} > 0$. For δ small enough, $|y| \leq \delta$, and $x \in \mathbb{R}^d$, we have

$$N(\tilde{n}|x-\theta y|) \geq N\Big(\frac{\tilde{n}}{2}|x|\Big). \tag{3.15}$$

Now, inequality (3.14), (3.15) and condition (M.2)' and property (E1) imply that there exists $C > 0$ such that

$$I_1 \leq C\frac{M_{|\alpha|}}{\overline{m}^{|\alpha|}}\ \delta\ \exp[-N(\overline{n}|x|)], \tag{3.16}$$

where $\overline{m} = \tilde{m}/H$ and $\overline{n} = \tilde{n}/2$.

From (3.8), and the definition of the heat kernel, it follows that there exists $C > 0$ such that

$$\begin{aligned} I_2 &= \int_{|y|\geq\delta} E(y,t)|\varphi^{(\alpha)}(x-y)|dy \\ &\leq C\frac{M_{|\alpha|}}{\tilde{m}^{|\alpha|}}(4\pi t)^{-d/2}\int_{|y|\geq\delta}\exp[-y^2/4t]\exp[-N(\tilde{n}|x-y|)]dy, \end{aligned} \tag{3.17}$$

for every $\tilde{m}, \tilde{n} > 0$.

Condition (N.1) implies that the associated function $N(\cdot)$ satisfies (see [**3**])

$$N(\rho+\delta) \le N(2\rho) + N(2\delta), \quad \rho, \delta > 0. \tag{3.18}$$

Taking $2\rho = \tilde{n}|x-y|$ and $2\delta = \tilde{n}|y|$, we get

$$N(\tilde{n}|x-y|) \ge N\Big(\frac{\tilde{n}}{2}|x-y| + \frac{\tilde{n}}{2}|y|\Big) - N(\tilde{n}|y|) \ge N\Big(\frac{\tilde{n}}{2}|x|\Big) - N(\tilde{n}|y|). \tag{3.19}$$

By (N.1) and (N.3)' (see [**12**, Lemma 4.1, (4.7)]) we have

$$N(n|y|) \le n|y| \le n\, y^2, \tag{3.20}$$

for $|y|$ large enough. Therefore from (3.17), (3.19), (3.20), it follows that

$$\begin{aligned} I_2 &\le C\ (4\pi t)^{-d/2} \exp\Big[-\frac{\delta^2}{8t}\Big] \exp\Big[-N(\frac{\tilde{n}}{2}|x|)\Big] \frac{M_{|\alpha|}}{\tilde{m}^{|\alpha|}} \int_{|y|\ge\delta} \exp\Big[-\frac{y^2}{8t} + N(\tilde{n}|y|)\Big] dy \\ &\le C\ \varepsilon_t\ \frac{M_{|\alpha|}}{\tilde{m}^{|\alpha|}} \exp\Big[-N\Big(\frac{\tilde{n}}{2}|x|\Big)\Big] \int_{|y|\ge\delta} \exp\Big[-\frac{y^2}{8t} + \tilde{n}y^2\Big] \\ &\le \tilde{C}\ \varepsilon_t\ \frac{M_{|\alpha|}}{\tilde{m}^{|\alpha|}} \exp\Big[-N(\frac{\tilde{n}}{2}|x|)\Big], \end{aligned} \tag{3.21}$$

where $\varepsilon_t = (4\pi t)^{-d/2} \exp\Big[-\dfrac{\delta^2}{8t}\Big]$ tends to zero as $t \to 0$.

Finally, since $\varphi \in \mathcal{S}^{(M_p)}_{(N_p)}$, by (3.8) and by the properties of the function $E(x,t)$ we have that there exists a $C > 0$ such that

$$\begin{aligned} I_3 &= \int_{|y|\ge\delta} E(y,t)|\varphi^{(\alpha)}(x)|dy \le |\varphi^{(\alpha)}(x)| \int_{|y|\ge\delta} E(y,t)dy \\ &\le \tilde{\delta}_t C \frac{M_{|\alpha|}}{m^{|\alpha|}} \exp[-N(n|x|)], \end{aligned} \tag{3.22}$$

for every $m, n > 0$, where $\tilde{\delta}_t = \int_{|y|\ge\delta} E(y,t)dy$ tends to zero as $t \to 0$.

From (3.13), (3.16), (3.21) and (3.22), we obtain that $U(x,t)$ converges to $\varphi(x)$, in the space $\mathcal{S}^{(M_p)}_{(N_p)}$, as t tends to zero. $\square$

The dual spaces for $\mathcal{S}^{(M_p)}_{(N_p)}$ and $\mathcal{S}^{\{M_p\}}_{\{N_p\}}$, are denoted by $\mathcal{S}'^{(M_p)}_{(N_p)}$ and $\mathcal{S}'^{\{M_p\}}_{\{N_p\}}$, respectively, or for brevity, $\mathcal{S}'^{*}_{\dagger}$. Using Theorem 2 it is easy to prove the following theorem:

THEOREM 4. *Let conditions (M.1), (C), (N.1), (N.2)' and (N.3)' be satisfied. If $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$ (respectively $f \in \mathcal{S}'^{\{M_p\}}_{\{N_p\}}$) then for some $m, n > 0$ (respectively every $m, n > 0$) there exists $C > 0$ such that*

$$|\langle f, \varphi\rangle| \le C \sup_{\substack{\alpha\in\mathbb{N}_0^d \\ x\in\mathbb{R}^d}} \frac{m^{|\alpha|}}{M_{|\alpha|}} |\varphi^{(\alpha)} \exp N(n|x|)|. \tag{3.23}$$

4. Structure theorems for the spaces ${\mathcal{S}'}^{(M_p)}_{(N_p)}$ and ${\mathcal{S}'}^{\{M_p\}}_{\{N_p\}}$

We will use the following notations:

$$P_h(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} \Big(1 + \frac{\zeta}{hm_p}\Big), \quad h > 0,$$

$$P_{h_p}(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} \Big(1 + \frac{\zeta}{h_p m_p}\Big), \quad h_p \in \mathcal{R}.$$

If (M.1), (M.2) and (M.3) are satisfied, the operator $P_h(d/dx)$ is an ultradifferential operator of class (M_p) and the operator $P_{h_p}(d/dx)$ is an ultradifferential operator of class $\{M_p\}$, (see [**12**]).

The following lemma asserts that there exists a parametrix for the equation $P_{h_p}(d/dx)u = \delta$. Its proof is inspired by constructions in [**12**, Lemma 11.4] and [**4**, Lemma 2].

Lemma 5. *Let the sequence M_p satisfy conditions (M.1), (M.2), (M.3). For every $\varepsilon > 0$ and every constant $h > 0$ (respectively every sequence $h_p \in \mathcal{R}$) there exist smooth functions v and w such that:*

1. $\operatorname{supp} v \subset [0,\varepsilon]$,
2. $|v(x)| \le C \exp\Big[-\sup_p \log \dfrac{x^{-p}p!}{h^p M_p}\Big]$,
 $\Big($*resp.* $|v(x)| \le C \exp\Big[-\sup_p \log \dfrac{x^{-p}p!}{h_1\cdots h_p M_p}\Big]\Big)$, *for some* $C > 0$,
3. $|v^{(p)}(x)| \le C2^p h^p M_p$ (*resp.* $|v^{(p)}(x)| \le C2^p h_1\cdots h_p M_p$), *for some* $C > 0$,
4. $\operatorname{supp} w \subset [\varepsilon/2,\varepsilon]$,
5. $|w^{(p)}(x)| \le CL^p h^p M_p$ (*resp.* $|w^{(p)}(x)| \le CL^p h_1\cdots h_p M_p$), *for every* $L > 0$ *and some* $C > 0$,

and $\quad P_{h_p}(d/dx)v(x) = \delta + w(x)$.

Proof. We will prove the lemma in the Roumieu case. Let $\varepsilon > 0$, $h_p \in \mathcal{R}$ and $G_p = h_1\cdots h_p M_p$, $p \in \mathbb{N}_0$. By $G(\cdot)$ and $\widetilde{G}(\cdot)$ we denote the associated functions for the sequence $\{G_p,\ p\in\mathbb{N}_0\}$. Put

$$\Gamma(z) = \frac{1}{2\pi i}\int_0^{\infty} P_{h_p}(\zeta)^{-1} e^{z\zeta} d\zeta. \tag{4.1}$$

Note that in the proof of [**12**, Lemma 11.4] the following entire function was considered:

$$z \mapsto \frac{1}{2\pi}\int_0^{\infty} P_{h_p}(\zeta)^{-1} e^{iz\zeta} d\zeta.$$

Since the sequence $\{G_p, p\in\mathbb{N}_0\}$ satisfies conditions (M.1), (M.2) and (M.3)', by analogous arguments as in [**12**, Lemma 11.4] one can prove that:

(i) The integral (4.1) converges absolutely on $\{z|\arg z \in (\pi/2, 3\pi/2)\}$. It can be continued analytically on $\{z|\arg z \in (0, 3\pi/2)\}$ by

$$\Gamma_+(z) = \begin{cases} \frac{1}{2\pi i}\int_0^{\infty e^{i\alpha}} P_{h_p}(\zeta)^{-1} e^{z\zeta} d\zeta, & \arg z \in (0,\pi), \\ \Gamma(z), & \arg z \in (\pi/2, 3\pi/2), \end{cases}$$

where $\alpha \in [-\pi/2, \pi/2]$, and $\operatorname{Re}(ze^{i\alpha}) < 0$, and on $\{z|\arg z \in (\pi/2, 2\pi)\}$ by

$$\Gamma_-(z) = \begin{cases} \Gamma(z), & \arg z \in (\pi/2, 3\pi/2), \\ \frac{1}{2\pi i}\displaystyle\int_0^{\infty e^{i\beta}} P_{h_p}(\zeta)^{-1} e^{z\zeta} d\zeta, & \arg z \in (\pi, 2\pi), \end{cases}$$

where $\beta \in [-\pi/2, \pi/2]$, and $\operatorname{Re}(ze^{i\beta}) < 0$;

(ii)
$$P_{h_p}\Big(\frac{d}{dx}\Big)\Gamma(z) = -\frac{1}{2\pi i}\frac{1}{z}, \quad \text{for} \quad \operatorname{Re}(z) < 0; \tag{4.2}$$

(iii) The functions Γ_+ and Γ_- are bounded, satisfy (4.2), and

$$\gamma(x) = \Gamma_-(x - i0) - \Gamma_+(x + i0) == \begin{cases} \frac{1}{2\pi i}\displaystyle\int_{-i\infty}^{+i\infty} P_{h_p}(\zeta)^{-1} e^{x\zeta} d\zeta, & x > 0, \\ 0, & x < 0. \end{cases}$$

(iv) The function $g(x) = \operatorname{Re}(\gamma(x))$, $x \in \mathbb{R}$, is a real analytic function on $\mathbb{R}\backslash\{0\}$; $g(x) = 0$ for $x < 0$ and

$$|g(x)| \le A\sqrt{x}\exp[-\widetilde{G}(1/x)], \quad x \in \mathbb{R}_+. \tag{4.3}$$

Let us now prove the estimates for the derivatives of the function g. Let $q \in \mathbb{N}$. Since for every $\xi > 0$,

$$|g^{(q)}(x)| \le |\gamma^{(q)}(x)| = \frac{1}{2\pi}\int_{\xi - i\infty}^{\xi + i\infty} \zeta^q P_{h_p}(\zeta)^{-1} e^{x\zeta} d\zeta,$$

using the estimation (see [**12**, p. 88])

$$\exp[G(|\zeta|)] \le \prod_{p=1}^{\infty}\Big(1 + \frac{\zeta}{h_p m_p}\Big), \quad \operatorname{Re}(\zeta) > 0,$$

we have

$$\begin{aligned} |g^{(q)}(x)| &\le \frac{1}{2\pi}\inf_{\xi>0}\Big(\int_{\xi-i\infty}^{\xi+i\infty}\frac{|\zeta^q||d\zeta|}{|P_{h_p}(\zeta)|}e^{x\zeta}\Big) \\ &\le \frac{1}{2\pi}\inf_{\xi>0}\Big(\int_{\xi-i\infty}^{\xi+i\infty}\frac{|\zeta|^q\exp[-G(|\zeta|)]}{|1+\zeta|^2}|d\zeta|e^{x\xi}\Big) \\ &\le \frac{1}{2\pi}\inf_{\xi>0}\Big(\int_{\xi-i\infty}^{\xi+i\infty}\frac{|\zeta|^q}{|1+\zeta|^2}\inf_{p\in\mathbb{N}_0}\frac{G_p}{|\zeta|^q}|d\zeta|e^{x\xi}\Big). \end{aligned}$$

By putting $p = q$, we get

$$\begin{aligned}|g^{(q)}(x)| &\leq \frac{1}{2\pi}\inf_{\xi>0}\Big(\int_{\xi-i\infty}^{\xi+i\infty}\frac{G_q}{|1+\zeta|^2}|d\zeta|e^{x\xi}\Big)\\ &\leq \frac{G_q}{2\pi}\inf_{\xi>0}\Big(e^{x\xi}\int_{\xi-i\infty}^{\xi+i\infty}\frac{|d\zeta|}{|1+\zeta|^2})\\ &\leq \frac{G_q}{2}\inf_{\xi>0}\Big(\frac{e^{x\xi}}{1+\xi}\Big) \leq \frac{G_q}{2}x = \frac{h_pM_q}{2}x,\end{aligned}$$

for $q \in \mathbb{N}$ and $\xi = 1/x$.

Let u be an element of $\mathcal{E}^{(G_p)}(\mathbb{R})$, equal to 1 on $(-\infty, \varepsilon/2]$, and equal to 0 on $[\varepsilon, \infty)$. Put

$$v(x) = g(x)u(x), \qquad x \in \mathbb{R}. \tag{4.4}$$

By (4.3) and the fact that $\operatorname{supp} v \subset [0, \varepsilon]$, there exists a $C > 0$ such that

$$|v(x)| \leq C\exp[-\widetilde{G}(1/x)].$$

For $x \in [0, \varepsilon]$

$$|v^{(q)}(x)| \leq \sum_{k=0}^{q}\binom{q}{k}|g^{(k)}(x)u^{(q-k)}(x)| \leq C\sum_{k=0}^{q}\binom{q}{k}\frac{G_k}{2}\varepsilon G_{q-k} \leq C2^qG_q. \tag{4.5}$$

Let

$$\tilde{u}(x) = \begin{cases} 0, & x \in (-\infty, \varepsilon/2)\\ u(x) - 1, & x \in (\varepsilon/2, \infty)\end{cases}$$

and $\omega = g\tilde{u}$.

As real analytic functions are ultradifferentiable (see [**12**]), and $\mathcal{E}^{\{G_p\}}$ is closed under pointwise multiplication, it follows that $\omega \in \mathcal{E}^{\{G_p\}}$. Since the support of ω is compact, it follows that $\omega \in \mathcal{D}^{\{G_p\}}$.

The operator $P_{h_p}(d/dx)$ is an ultradifferential operator of class $\{G_p\}$ and therefore $P_{h_p}(d/dx)\omega \in \mathcal{D}^{(G_p)}$. Furthermore, since

$$P_{h_p}\Big(\frac{d}{dx}\Big)g(x) = \operatorname{Re}\Big(P_{h_p}\Big(\frac{d}{dx}\Big)\gamma(x)\Big) = \delta(x),$$

we have

$$\begin{aligned}P_{h_p}\Big(\frac{d}{dx}\Big)v(x) &= P_{h_p}\Big(\frac{d}{dx}\Big)g(x)(1 + u(x) - 1)\\ &= P_{h_p}\Big(\frac{d}{dx}\Big)g(x) + P_{h_p}\Big(\frac{d}{dx}\Big)\omega(x) = \delta + w(x),\end{aligned}$$

which completes the proof. $\square$

Theorem 6. 1. *Let conditions (M.1), (M.2)', (C), (N.1), (N.2)' and (N.3)' be satisfied and* $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$ *(respectively* $f \in \mathcal{S}'^{\{M_p\}}_{\{N_p\}}$*). The function*

$$U(x,t) = \langle f(y), E(x-y,t)\rangle$$

is well defined on $\mathbb{R}^{d+1}_+ = \{(x,t)|x \in \mathbb{R}^d,\ t > 0\}$*, belongs to* $C^\infty(\mathbb{R}^{d+1}_+)$ *and satisfies the heat equation*

$$\Big(\frac{d}{dt} - \Delta\Big)U(x,t) = 0. \tag{4.6}$$

Furthermore, for some $m, n > 0$*, (respectively for every* $m, n > 0$*) and arbitrary* $T > 0$*, there exists a positive constant* C *such that*

$$|U(x,t)| \le C \exp\Big[N(n|x|) + \frac{1}{2}\overline{M}\Big(\frac{m}{t}\Big)\Big],\ \ x \in \mathbb{R}^d_+,\ \ t \in (0,T). \tag{4.7}$$

Also, for any $\psi \in \mathcal{S}^{(M_p)}_{(N_p)}$ *(respectively any* $\psi \in \mathcal{S}^{\{M_p\}}_{\{N_p\}}$*), we have*

$$\int_{\mathbb{R}^d} U(x,t)\psi(x)dx \to \langle f, \psi\rangle,\ \ t \to 0. \tag{4.8}$$

2. *If conditions (M.1), (M.2), (C), (N.1), (N.2)' and (N.3)' are satisfied, the converse is also true: for every smooth function* $U(x,t)$ *defined on* $\mathbb{R}^{d+1}_+$*, satisfying conditions* (4.6) *and* (4.7)*, for some* $m, n > 0$*, (respectively for every* $m, n > 0$*) there exists unique* $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$*, (respectively* $f \in \mathcal{S}'^{\{M_p\}}_{\{N_p\}}$*) such that*

$$U(x,t) = \langle f(y), E(x-y,t)\rangle. \tag{4.9}$$

Proof. We shall prove the assertion of the theorem only in the Beurling case. The assertion in the Roumieu case can be proved analogously.

1. Let $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$. The function $U(x,t) = \langle f(y), E(x-y,t)\rangle$ obviously belongs to $C^\infty(\mathbb{R}^{d+1}_+)$. Using Theorem 4, the estimate (3.10), condition (M.2)', the fact that $E(\cdot,t) \in \mathcal{S}^{(M_p)}_{(N_p)}$, for every fixed $t > 0$, and

$$N(h|y|) - N(2h|x-y|) \le N(h|x|), \quad h > 0,$$

(which follows from (3.18) by a similar argument as in (3.19)), we obtain

$$
\begin{aligned}
|U(x,t)| &= |\langle f(y), E(x-y,t)\rangle| \\
&\le C_1 \sup_{\alpha\in\mathbb{N}_0^d} \frac{k^{|\alpha|}}{M_{|\alpha|}} \Big\| E^{(\alpha)}(x-y,t)\exp[N(h|y|)]\Big\|_{L^\infty(\mathbb{R}_y^d)} \\
&\le C_1 \sup_{\alpha\in\mathbb{N}_0^d} \frac{k^{|\alpha|}}{M_{|\alpha|}} \tilde{C}^{|\alpha|+1} t^{-(|\alpha|+d)/2}\alpha!^{1/2} \Big\| \exp\Big[\frac{-4|x-y|^2}{(16t/a')}\Big] + N(h|y|)]\Big\|_{L^\infty(\mathbb{R}_y^d)} \\
&\le C_1 \sup_{\alpha\in\mathbb{N}_0^d} (1+k)^{|\alpha|+1}\tilde{C}^{|\alpha|+1} t^{-(|\alpha|+d)/2}(\alpha+1)!^{1/2}\frac{A(1+H)^{|\alpha|+1}}{M_{|\alpha|+1}} \times \\
&\qquad \times (4\cdot 16\pi T/a')^{d/2}\big\|E(2|x-y|,16T/a')\exp N[h|y|]\big\|_{L^\infty(\mathbb{R}_y^d)} \le \cdots \\
&\le C_2\Big(C_3^{|\alpha|+d}\frac{t^{-(|\alpha|+d)}(\alpha+d)!}{M^2_{|\alpha|+d}}\Big)^{1/2}\big\|E(2|x-y|,16T/a')\exp[N(h|y|)]\big\|_{L^\infty(\mathbb{R}_y^d)} \\
&= C_2 \exp\Big[\frac{1}{2}\overline{M}\Big(\frac{C_3}{t}\Big)\Big]\Big\|E\,(2|x-y|,16T/a')\ \exp[N(h|y|)]\Big\|_{L^\infty(\mathbb{R}_y^d)} \\
&\le C_4 \exp\Big[\frac{1}{2}\overline{M}\Big(\frac{C_3}{t}\Big)\Big]\Big\|\exp[-N(2h|x-y|)]\exp[N(h|y|)]\Big\|_{L^\infty(\mathbb{R}_y^d)} \\
&\le C\exp\Big[N(n|x|)+\frac{1}{2}\overline{M}\Big(\frac{m}{t}\Big)\Big],
\end{aligned}
$$

for some $k,h>0$, $n=h$, $m=C_3=(1+k)(1+H)\tilde{C}$, where $\tilde{C}$ is a constant in (E2).

2. Assume that $U(x,t)$ satisfies (4.6) and (4.7). Note that the sequence $\{M_p^2,\ p\in\mathbb{N}_0\}$ satisfies conditions (M.1), (M.2), (M.3), since the sequence $\{M_p,\ p\in\mathbb{N}_0\}$ satisfies (M.1), (M.2) and (C).

Let $\varepsilon>0$. Applying Lemma 5 to the sequence $\{M_p^2,\ p\in\mathbb{N}_0\}$, we get that for every $h>0$ there exist smooth functions $v,w\in C_0^\infty(\mathbb{R})$, with properties

$$
\operatorname{supp} v\subset[0,\varepsilon],\quad \operatorname{supp} w\subset[\varepsilon/2,\varepsilon], \tag{4.10}
$$

$$
|v(t)|\le C\sup_{p\in\mathbb{N}_0}\log\Big(\frac{1}{ht}\Big)^p\frac{p!}{M_p^2}=C\exp\Big[-\overline{M}\Big(\frac{1}{ht}\Big)\Big],\quad t>0, \tag{4.11}
$$

and such that

$$
P_h\Big(\frac{d}{dt}\Big)v=\delta+w, \tag{4.12}
$$

where

$$
P_h\Big(\frac{d}{dt}\Big)=\Big(1+\frac{d}{dt}\Big)^2\prod_{p=1}^{\infty}\Big(1+\frac{1}{hm_p^2}\frac{d}{dt}\Big).
$$

Let

$$
\begin{aligned}
g(x,t)&=\int_0^\infty U(x,t+s)v(s)ds,\\
h(x,t)&=\int_0^\infty U(x,t+s)w(s)ds.
\end{aligned} \tag{4.13}
$$

Put $h = 1/m$. Since $\operatorname{supp} v \subset [0,\varepsilon]$, we have that

$$|g(x,t)| \leq C\exp[N(n|x|)]\int_0^\varepsilon \exp\Big[\frac{1}{2}\overline{M}\Big(\frac{m}{t+s}\Big) - \overline{M}\Big(\frac{1}{hs}\Big)\Big]ds \tag{4.14}$$
$$\leq C\exp[N(n|x|)]\int_0^\varepsilon \exp\Big[-\frac{1}{2}\overline{M}\Big(\frac{m}{s}\Big)\Big]ds \leq \tilde{C}\exp[N(n|x|)],$$

for $(x,t) \in \mathbb{R}^{d+1}_+$.

It is easy to see that

$$|h(x,t)| \leq \tilde{C}\exp[N(n|x|)], \quad (x,t) \in \mathbb{R}^{d+1}_+. \tag{4.15}$$

The functions $g(x,t), h(x,t)$ are smooth on $\mathbb{R}_+$ and satisfy the heat equation. It can be easily proved, by using (4.10) and (4.11), that $g(x,t)$ and $h(x,t)$ can be continuously extended on $\overline{\mathbb{R}^{d+1}_+} = \{(x,t) | x \in \mathbb{R}^d,\ t \geq 0\}$. Put

$$g_0(x) = \lim_{t\to 0} g(x,0), \qquad h_0(x) = \lim_{t\to 0} h(x,0), \qquad x \in \mathbb{R}^d.$$

The functions $g_0(x)$ and $h_0(x)$ are $C^\infty(\mathbb{R}^{d+1})$.

Since $\operatorname{supp} w \subset [\varepsilon/2, \varepsilon]$, the function $h(x,t)$ can be continued analytically to $\{(x,t) \in \mathbb{R}^{d+1},\ x \in \mathbb{R}^d,\ t > -\varepsilon/2\}$. Thus $h_0(x)$ is a real analytic function.

From (4.14) and (4.15) it follows

$$|g_0(x)| \leq C\exp[N(n|x|)] \quad \text{and} \quad |h_0(x)| \leq C\exp[N(n|x|)]. \tag{4.16}$$

for some $n > 0$. Since $g(x,t)$ satisfies the heat equation, it follows from (4.12), that

$$U(x,t) + h(x,t) = P_h\Big(\frac{d}{dt}\Big)g(x,t) = P_h(-\Delta)g(x,t). \tag{4.17}$$

Define

$$f(x) = P_h(-\Delta)g_0(x) - h_0(x). \tag{4.18}$$

From (4.16) it follows that $g_0, h_0 \in \mathcal{S}'^{(M_p)}_{(N_p)}$. Since P_h is an ultradifferential operator of class (M_p^2), we have that $P_h(-\Delta)g_0 \in \mathcal{S}'^{(M_p)}_{(N_p)}$. Therefore, $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$.

Let us prove that $U(x,t) = \langle f(y), E(x-y)\rangle$. Put

$$A(x,t) = \int_{\mathbb{R}^d} E(x-y,t)g_0(y)dy, \quad t > 0;$$
$$B(x,t) = \int_{\mathbb{R}^d} E(x-y,t)h_0(y)dy, \quad t > 0.$$

The functions $A(x,t)$ and $B(x,t)$ satisfy the heat equation and $A(x,t)$ and $B(x,t)$ converge locally uniformly to $g_0(x)$ and $h_0(x)$, respectively, as t converges to zero. Therefore, they can be continuously extended to $\overline{\mathbb{R}^{d+1}_+}$ and

$$\lim_{t\to 0} A(x,t) = g_0(x) = \lim_{t\to 0} g(x,t), \quad \text{and} \quad \lim_{t\to 0} B(x,t) = h_0(x) = \lim_{t\to 0} h(x,t).$$

Furthermore, the functions $A(x,t)$ and $B(x,t)$ satisfy the following growth conditions:

$$\begin{aligned} |A(x,t)| &\le C\exp[N(a|x|)] \le \widetilde{C}\exp[ax^2], \quad t\in(0,T),\\ |B(x,t)| &\le C\exp[N(a|x|)] \le \tilde{C}\exp[ax^2], \quad t\in(0,T). \end{aligned}$$

Let us prove the first inequality. For arbitrary $\delta > 0$,

$$\begin{aligned} |A(x,t)| &\le |\int_{\mathbb{R}^d} E(y,t)g_0(x-y)dy| \\ &\le C\int_{|y|\le\delta} E(y,t)\exp[N(n|x-y|)]dy + C\int_{|y|\ge\delta} E(y,t)\exp[N(n|x-y|)]dy \\ &= I_1 + I_2. \end{aligned}$$

Using (3.18) we get that

$$\begin{aligned} I_1 &\le C\exp[N(2n|x|)]\int_{|y|\le\delta} E(y,t)\exp[N(2n|y|)]dy \\ &\le C\exp[N(2n|x|)]\exp[N(2n\delta)]\int_{\mathbb{R}^d} E(y,t)dy \\ &\le C_1\exp[N(2n|x|)] \le \tilde{C}\exp[ax^2], \end{aligned}$$

for some $n > 0$. For $\delta > 0$ large enough, by (3.18), we have

$$\begin{aligned} I_2 &\le C(4\pi t)^{-d/2}\exp[N(2n|x|)]\int_{|y|\ge\delta}\exp[-\frac{y^2}{4t} + N(2n|y|)]dy \\ &\le C(4\pi t)^{-d/2}\exp[N(2n|x|)]\exp[-\frac{\delta^2}{4t}(1-\frac{1}{b})]\int_{|y|>\delta}\exp[-\frac{y^2}{4Tb} + 2ny^2]dy \\ &\le C_1\exp[N(2n|x|)] \le \tilde{C}\exp[2nx^2], \end{aligned}$$

for some $n > 0$ and $t \in (0,T)$, where $0 < b < \min(1, 1/8Tn)$.

By the uniqueness theorem for the initial-value heat equation (see [**9**, p. 216], it follows that the solution of the problem

$$u_t - \Delta u = 0, \qquad t\in(0,\infty), \qquad\qquad u(x,0) = f(x),$$

is unique, provided we restrict ourselves to a solution satisfying

$$|u(x,t)| \le C\exp[ax^2], \qquad t \in (0,T).$$

Therefore,

$$g(x,t) = A(x,t) = \int_{\mathbb{R}^d} E(x-y,t)g_0(y)dy,$$
$$h(x,t) = B(x,t) = \int_{\mathbb{R}^d} E(x-y,t)h_0(y)dy.$$

From above (4.18) and (4.17) it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} E(x-y,t)f(y)dy &= \int_{\mathbb{R}^d} E(x-y,t)[P(-\Delta)g_0(y) - h_0(y)]dy \\ &= P(-\Delta)\int_{\mathbb{R}^d} E(x-y,t)g_0(y)dy - \int_{\mathbb{R}^d} E(x-y,t)h_0(y)dy \\ &= P(-\Delta)g(x,t) - h(x,t) = U(x,t),\end{aligned}$$

i.e. $U(x,t) = \langle f(y), E(x-y,t)\rangle$. The uniqueness can be easily proved. □

THEOREM 7. *Let conditions (M.1), (M.2), (C), (N.1), (N.2)' and (N.3)' be satisfied and* $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$ *(respectively* $f \in \mathcal{S}'^{\{M_p\}}_{\{N_p\}}$*). There exist an ultradifferential operator* $P(d/dx)$ *of class* (M_p^2) *(respectively* $\{M_p^2\}$*), a smooth function* $u_1(x)$ *and a real analytic function* $u_2(x)$ *such that*

$$|u_1(x)| \le C\exp[N(n|x|)], \quad \textit{and} \quad |u_2(x)| \le C\exp[N(n|x|)],$$

for some $n > 0$*, (respectively every* $n > 0$*), and*

$$f(x) = P(\Delta)u_1(x) + u_2(x).$$

Proof. Let $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$ (respectively $\mathcal{S}'^{\{M_p\}}_{\{N_p\}}$) and let

$$U(x,t) = \langle f(y), E(x-y,t)\rangle.$$

By the first part of the Theorem 6, we have that $U(x,t) \in C^\infty(\mathbb{R}^{d+1}_+)$, (4.6), and (4.7). By using the same construction as in the proof of the second part of the Theorem 6 one can prove that

$$U(x,t) = P(-\Delta)g(x,t) - h(x,t), \tag{4.19}$$

where g and h are defined as in (4.13). Note that since $U(x,t)$ converges to $f(x)$ in the space $\mathcal{S}'^{(M_p)}_{(N_p)}$, as $t \to 0^+$ (see the proof of the first part of Theorem 6), the function $g(x,t)$ converges to a smooth function $g_0(x)$, and $h(x,t)$ converges to a real analytic function $h_0(x)$ as $t \to 0^+$ with

$$|g_0(x)| \leq C \exp[N(n|x|)], \quad \text{and} \quad |h_0(x)| \leq C \exp[N(n|x|)],$$

for some $n > 0$, (respectively every $n > 0$). This and (4.19) imply

$$f(x) = P(-\Delta)g_0(x) - h_0(x)$$

in the space $\mathcal{S}'^{(M_p)}_{(N_p)}$ (respectively $\mathcal{S}'^{\{M_p\}}_{\{N_p\}}$). Put $u_1(x) = g_0(x)$ and $u_2(x) = -h_0(x)$. This completes the proof. $\square$

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