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CELL DECOMPOSITIONS OF SATAKE COMPACTIFICATIONS

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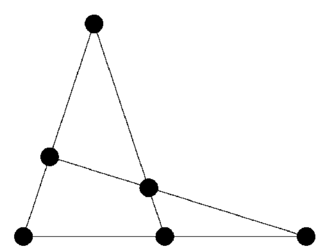
Introduction

(0.1). We begin in (0.1)–(0.4) by defining some partially ordered sets, using projective geometry. We build cell complexes out of these posets, and say a few words about their importance.

Let $\mathbf{P}^2(\mathbf{Q})$ be the rational projective plane (the set of lines through the origin in $\mathbf{Q}^3$). Let $\mathbf{A}$ be the following configuration of six points in this plane, arranged along four lines as shown:



Any arrangement of six points in this pattern is called a *complete quadrilateral*. Any two complete quadrilaterals differ by a projective transformation of the plane.

Let $\mathcal{F}^*$ be the poset of all non-empty subsets of the six points of $\mathbf{A}$, partially ordered by inclusion. The order complex (5.6) of this poset is a barycentric subdivision of the closed simplex of dimension five. We call this 5-simplex F^*. It has one j-dimensional face for each subset of $\mathbf{A}$ with $j+1$ elements.

The subsets of $\mathbf{A}$ which lie on a line determine a closed subcomplex of the 2-skeleton of F^*. Let F be the complement of this closed subcomplex in F^*.

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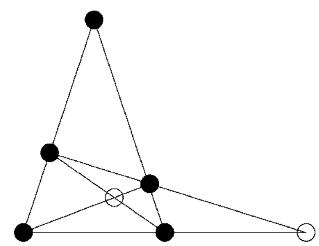
A *Carnot configuration* is any arrangement of seven points (five colored $\bullet$ and two colored $\circ$) in this pattern:



Two complete quadrilaterals, $\mathbf{A}$ and (say) $\mathbf{A}'$, can be formed from the Carnot configuration shown by filling in one of the $\circ$ points and removing the other. Let $\mathcal{F}'^*$ be the poset of non-empty subsets of $\mathbf{A}'$. Its order complex is, again, a barycentric subdivision of a closed 5-simplex F'^*. Since $\mathbf{A}$ and $\mathbf{A}'$ meet in five points, F^* and F'^* can be glued together, canonically, along a facet (a codimension-one face), which is a closed 4-simplex. Define $F' \subset F'^*$ as we defined $F \subset F^*$; the gluing of F^* and F'^* restricts to a gluing of F and F'.

Now take all the complete quadrilaterals in the plane, form the 5-simplex for each one, and glue the 5-simplices together in pairs along facets, using all the Carnot configurations. You obtain a topological space. The group of projective transformations of $\mathbf{P}^2$ acts on the space, since it acts on the configurations; in fact, it acts transitively on the 5-simplices. The space has infinitely many connected components, but they are all equivalent because of the transitivity. Pick one connected component and call it $\mathcal{X}^*$. Let $\mathcal{X}$ be the subspace of $\mathcal{X}^*$ defined by replacing every $F^* \subset \mathcal{X}^*$ by the corresponding F.

We now state some facts which will not be obvious, but which we will prove in this paper. The space $\mathcal{X}$ is homeomorphic to a Euclidean space of dimension five. In fact, it is homeomorphic to the space X of all positive definite symmetric bilinear forms on $\mathbf{R}^3$, modulo multiplication by positive scalars (the *homotheties*); equivalently, this is the space of 3×3 positive definite symmetric matrices over $\mathbf{R}$, modulo homotheties. The largest subgroup of the projective transformations that preserves the triangulation of $\mathcal{X}$ is, up to conjugation, $\mathrm{GL}_3(\mathbf{Z})$, the group of 3×3 invertible matrices over $\mathbf{Z}$. It acts on $\mathbf{P}^2$ by acting on the lines in $\mathbf{Q}^3$; this coincides with the usual action of $\mathrm{GL}_3(\mathbf{Z})$ on bilinear forms. The space $\mathcal{X}^*$ corresponds, in the vector space of all symmetric bilinear forms on $\mathbf{R}^3$ (mod homotheties), to X together with the positive semidefinite forms whose kernels are proper rational subspaces of $\mathbf{R}^3$. For subgroups $\Gamma \subset \mathrm{GL}_3(\mathbf{Z})$, there are only finitely many (open) simplices in $\mathcal{X}$ modulo Γ. If Γ has large enough finite index in $\mathrm{GL}_3(\mathbf{Z})$, we get a finite decomposition of $\mathcal{X}/\Gamma \cong X/\Gamma$ into (open) simplices. Furthermore, $\mathcal{X}^*/\Gamma$ is a compact Hausdorff space. It is one of the two *minimal Satake compactifications* of X/Γ.

(0.2). Let us say a bit about X/Γ. More pictures will be coming in (0.3).

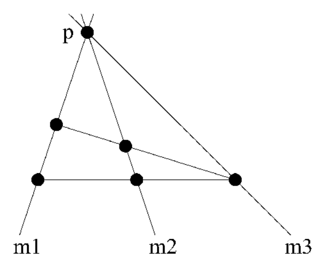
The cohomology of X/Γ (with constant coefficients $\mathbf{C}$, say) is closely tied to number theory, since the cohomology classes are often attached to automorphic forms for SL_3. A family of Hecke correspondences acts on X/Γ and its cohomology. When one has an eigenclass α in cohomology for the Hecke correspondences, the Hecke eigenvalues give extremely important arithmetic information. Examples have provided strong evidence that α can have attached Galois representations [A-P-T] [A-M2], or even motives [vG-T].

Given X and Γ, it is in general hard to compute the cohomology $H^*(X/\Gamma)$. Much of the importance of $\mathcal{X}/\Gamma$ is that you can use the finite cell structure to compute $H^*(X/\Gamma)$ and the Hecke operators on it—see (0.7).

For an overview of these ideas, see any of the cited references, or the survey article [M2].

(0.3). By playing a slightly different game with projective configurations, we will obtain a different compactification $\mathcal{X}^*_{\max}/\Gamma$ of $\mathcal{X}/\Gamma$ which gives more detailed information near the boundary.

A line *respects* a subset of $\mathbf{P}^2$ if it meets at least two points of the subset (5.9). A *rake* for $\mathbf{A}$ is a pair $(p, \{m_\ell\})$ consisting of one point p of $\mathbf{A}$ and anywhere from one to three distinct lines m_ℓ passing through p, such that all the lines respect $\mathbf{A}$.



As a set, let $\mathcal{F}^*_{\max}$ be the union of $\mathcal{F}^*$ and the set of all rakes for $\mathbf{A}$, with the part of $\mathcal{F}^*$ corresponding to one-element subsets of $\mathbf{A}$ removed. We define a partial order $S \sqsubseteq T$ on $\mathcal{F}^*_{\max}$. If S and T are both in $\mathcal{F}^*$, define the relation to be $S \subseteq T$ as usual. If $S = (p, \{m_\ell\})$ is a rake and T is not, then $S \sqsubseteq T$ if and only if $p \in T$ and each m_ℓ respects T. If T is a rake and S is not, then $S \not\sqsubseteq T$. If S and T are both rakes, then $S \sqsubseteq T$ if and only if they have the same point p and every line in S is a line in T.

The order complex $F^*_{\max}$ of $\mathcal{F}^*_{\max}$ is homeomorphic to a closed 5-simplex, except that each vertex has been removed and "blown up", being replaced by a 2-simplex. (We get a 2-simplex because, if T is a rake with three lines, then the order complex of $\{S \in \mathcal{F}^*_{\max} \mid S \sqsubseteq T\}$ is the barycentric subdivision of a 2-simplex.) The subsets of $\mathbf{A}$ which lie on a line, together with the rakes, gives a closed subcomplex of the 2-skeleton of $F^*_{\max}$ whose complement is, canonically, F.

As before, take all the complete quadrilaterals, form the $F^*_{\max}$ for each one, and glue them together along closed facets using all the Carnot configurations.

Pick one connected component of the result and call it $\mathcal{X}^*_{\max}$. Up to canonical isomorphism, the same $\mathcal{X}$ as before is the subspace of $\mathcal{X}^*_{\max}$ corresponding to the F's. The group $\mathrm{GL}_3(\mathbf{Z})$ acts on $\mathcal{X}^*_{\max}$ as before, and this action restricts to the previous one on $\mathcal{X}$. The quotient $\mathcal{X}^*_{\max}/\Gamma$ is the *maximal Satake compactification* of $\mathcal{X}/\Gamma \cong X/\Gamma$.

The maximal Satake compactification is closely related to the Borel-Serre compactification $\bar{X}/\Gamma$, which is the biggest compactification coming from the geometry of the group $\mathrm{SL}_3(\mathbf{R})$ acting on X (see (2.6)). The inclusion $\partial\bar{X}/\Gamma \hookrightarrow \bar{X}/\Gamma$ gives a pullback map in cohomology. For $\mathrm{SL}_3(\mathbf{R})$, [A-G-G, p. 415] shows the kernel of this pullback on H^3 is precisely the *cuspidal cohomology* $H^3_{\mathrm{cusp}}(X/\Gamma)$, which corresponds to the cuspidal automorphic forms. We will see later that the maximal Satake compactification is a quotient of the Borel-Serre by a map that restricts to the identity on X/Γ and where the fibers of the quotient map over the boundary are nilmanifolds. The pullback map on cohomology to the maximal Satake boundary still gives information about the cuspidal cohomology.

In Example 1 of (5.10), we indicate how Voronoï reduction theory for $\mathrm{GL}_3(\mathbf{Z})$, together with the results of this paper, will produce $\mathcal{X}^*_{\max}$. The $\mathcal{X}^*$ of (0.1) is actually derived as a quotient of $\mathcal{X}^*_{\max}$: see (6.6).

(0.4). Our projective configurations can actually produce many of the spaces X/Γ and their compactifications directly, without going through $\mathcal{X}$ first. Let p be a prime $\neq 2, 3$, and let $\mathbf{F}_p$ be the field of p elements. Consider $\mathbf{P}^2(\mathbf{F}_p)$, the projective plane over $\mathbf{F}_p$ (the set of lines through the origin in $\mathbf{F}_p^3$). We make the same kind of definitions as in (0.1)–(0.3), but with $\mathbf{F}_p$ replacing $\mathbf{Q}$ everywhere. Let $\mathbf{A}$ be a fixed complete quadrilateral in $\mathbf{P}^2(\mathbf{F}_p)$, and define $\mathcal{F}^*_{\max}$, $F^*_{\max}$, and F as before. These are isomorphic (resp., homeomorphic) to the same posets (resp., spaces) as in (0.1) and (0.3): for instance, $F^*_{\max}$ is the same blow-up of the 5-simplex.

Form a space by gluing all the $F^*_{\max}$'s along facets using Carnot configurations. This space has only finitely many cells, since the projective plane is finite. The space has one or three connected component(s), depending on whether $p \equiv 5$ or 1 mod 6; let $\mathcal{X}^*_{\max}(p)$ be the normalization (7.2) of one connected component. Let Γ be the extended principal congruence subgroup $\tilde{\Gamma}(p) = \{\gamma \in \mathrm{GL}_3(\mathbf{Z}) \mid \gamma \equiv \lambda I \pmod{p}, \lambda \in \mathbf{F}_p^\times\}$.

The result is that $\mathcal{X}^*_{\max}(p)$ is, up to homeomorphism, the maximal Satake compactification of $X/\tilde{\Gamma}(p)$. The subspace of $\mathcal{X}^*_{\max}(p)$ corresponding to the F's is identified with $X/\tilde{\Gamma}(p)$.

In Section 7 below, we indicate how to prove this result, building on [M1].

For a survey of all the ideas in (0.1)–(0.4), emphasizing the projective geometry of the configurations, see [M-M0].

(0.5). The present paper treats a wider class of groups and spaces than we have seen so far. Let $\mathcal{V}$ be a vector space over $\mathbf{Q}$, and let $C \subset \mathcal{V}$ be a self-adjoint homogeneous cone defined over $\mathbf{Q}$ which is indecomposable over $\mathbf{R}$ (see (1.2) for discussion and exceptions). Then C is the cone of positive definite symmetric matrices over $\mathbf{R}$, or the positive definite Hermitian matrices over $\mathbf{C}$ or over the quaternions $\mathbf{H}$. Let X be the symmetric space given by C modulo the positive real scalars (the homotheties). We have $X = K\backslash G$, where G is a semi-simple Lie group and K a maximal compact subgroup. Then X is a Riemannian symmetric space under a G-invariant metric. In fact, we will call X a *linear symmetric space*, because it has two geometries existing side by side—the linear geometry as an open convex set in $\mathcal{V}/(\text{homotheties})$, and the geometry of the G-invariant metric. A major point of the paper is to compare the two geometries.

Let $\Gamma \subset G$ be an arithmetic group. It typically lies in $\mathrm{SL}_n(\mathfrak{o})$, where $\mathfrak{o}$ is an order in a division algebra $\mathfrak{D}$ over $\mathbf{Q}$. We treat the cases where $\mathfrak{D}$ is $\mathbf{Q}$, an imaginary quadratic number field, or one of certain non-commutative division algebras. The different $\mathfrak{D}$ and $\mathfrak{o}$ put different rational structures on C and X.

Since X is contractible, X/Γ is a $K(\Gamma, 1)$ space when Γ is torsion-free. This ties us in to the group cohomology of Γ (e.g., [Sou1]).

One can find Γ-admissible polyhedral decompositions $\mathcal{R}$ of X (5.1). Roughly speaking, these are decompositions of X, without gaps or overlaps, into rational polyhedral sets F (1.12) such that, up to Γ-equivalence, $\mathcal{R}$ has only finitely many different elements. The $F \in \mathcal{R}$ descend mod Γ to give a decomposition of X/Γ. An appropriate finite union of elements of $\mathcal{R}$ is a fundamental domain D for Γ.

The F's are open convex sets, so they are homeomorphic to open cells. One would like to say that they make X and X/Γ into cell complexes. The difficulty is that the F's are ideal polyhedra: they typically have vertices and other low-dimensional faces off at infinity. Furthermore, the closure of F in a Satake compactification of X or X/Γ is generally not the same as the "naïve" closure in $\mathcal{V}/(\text{homotheties})$. The Satake compactification blows up the regions at infinity, and it is important to see how the decomposition $\mathcal{R}$ looks after the blow-up—in particular, whether the closures of the open cells are actually closed cells. The F's are defined using the linear geometry; we must see how they look in the symmetric space geometry out near infinity.

The boundary components of the Satake compactification are themselves linear symmetric spaces, smaller than X but of the same type. If F' is a component of the closure of F in a Satake boundary component, then our main technical result, Proposition 4.1, shows that F' is itself a rational polyhedral set in the boundary component, and it gives a formula for the vertices of F' in terms of the vertices of F. Thus F' is also an open cell. Our main result (5.6)–(5.7) is that the F's and F''s do give a regular[1] cell decomposition $\mathcal{R}^*$ of the maximal Satake compactification of X. For suitable arithmetic $\Gamma' \subseteq \Gamma$, this descends to make the maximal Satake compactification of X/Γ' into a regular cell complex with only finitely many cells.

We only obtain our results under a combinatorial hypothesis. Consider the order complex of the closure F^* of F, what we called $\mathcal{F}^*$ above. The hypothesis is that, for all $F \in \mathcal{R}^*$, the $\mathcal{F}^*$ must be shellable (see (a) and (b) in (5.6)). In

[1]A *regular* cell complex is a CW-complex in which the attaching map for each closed cell is an embedding.

other words, if F^* looks combinatorially like a closed ball, then it is one. It is hard to avoid hypotheses like these, for a simple reason: if a space looks topologically like a closed ball, it may not be one. The Poincaré conjecture and the Schoenflies problem, both still unsolved, are examples of the difficulty. One needs hard data, like a shelling, to prove that F^* is a closed ball.

Of course, one also needs local topological information saying that the candidate cells fit together nicely. We show in (4.7)–(4.8) that the local topological structure of F near F' is constant, and that the link of F in F' is a ball. We also show the whole $\mathcal{R}^*$ is a Whitney stratification. We can then use a double induction technique developed in [M-M1, (8.3)], using the shellability as input, to show that the F^* are actual closed cells.

We look especially at the case where $\mathcal{R}$ is the Voronoï decomposition of X. When $G = \mathrm{SL}_n(\mathbf{R})$, this was introduced by Voronoï in 1908 as a tool for studying densest lattice packings; Voronoï proved it was a Γ-admissible rational polyhedral decomposition. It has since been generalized to all self-adjoint homogeneous cones. It is straightforward (if slow) to compute the decomposition for a given X and classify its cells up to Γ-equivalence. In the cases $G = \mathrm{SL}_2(\mathbf{R})$ and $\mathrm{SL}_3(\mathbf{R})$, we check the shellability conditions and obtain our main theorems for $\mathcal{R}^*$ without an outstanding hypothesis.

(0.6). We now discuss the contents of the various sections. Section 1 sets up background from algebraic groups, and Section 2 summarizes the theory of Satake compactifications. In Section 3, we describe neighborhood bases for points on the Satake boundary; to the best of our knowledge, this result has not appeared in the literature before. The technical heart of the paper is Section 4, which shows how the closure of a rational polyhedral set meets a boundary component. These results, plus other technical material, are assembled in Section 5 to prove our main theorems.

Sections 2–5 focussed on the maximal Satake compactification. There are actually finitely many different Satake compactifications of X and X/Γ. For a reasonable range of $\mathfrak{D}$, these are classified by the non-empty subsets of ${}_{\mathbf{Q}}\Delta$, the rational restricted fundamental system of roots for G. The compactifications fit together in a poset structure, where a given compactification is a quotient of the compactifications above it. Section 6 extends our results to all these compactifications. Finally, Section 7 outlines how to prove the results in (0.4).

See (5.10) and (6.6) for examples of our techniques.

(0.7). As we have indicated, part of the motivation for this work has been to compute the cohomology of the compactifications of X/Γ and their boundaries, to compute the Hecke action on these classes, and to find applications in number theory and automorphic forms. A small sample of papers along such lines is [A-G-G] [A-M1] [Gu-M] and those cited in (0.2). However, several other methods could also be used. Harder, Schwermer, Franke, and others have investigated the questions extensively in a direct way, using Eisenstein series and other analytic techniques.

For computations with cell complexes, the methods of [A-M3] handle the Borel-Serre boundary (not only the Satake boundary), and for arbitrary $\mathfrak{D}$ (including all number fields).

We expect our methods could be extended to give cell decompositions of the Borel-Serre compactification $\bar{X}/\Gamma$. Each of our F' would be blown up into $F' \times$ (a cell in the unipotent part). Basically, one would have to show that the set of unipotent elements in (4.3), Lemma 2 is a cell. We have resisted the temptation to do this, because the Satake compactifications have their own elegance. Every Satake boundary component comes from a smaller symmetric space, so every F' can be expressed in the same terms as the F's they come from. For instance, it would be less natural to classify cells in $\bar{X}/\Gamma$ by projective configurations.

In our setting, the maximal Satake compactification coincides with the reductive Borel-Serre compactification.

(0.8). This issue of the *Publications de l'Institut Mathématique* (Belgrade) is connected with the "Geometric Combinatorics" conference held in Kotor in 1998. My talk in Kotor concerned toric varieties and the obstacles to lifting their natural intersection products to ring structures on their intersection cohomology. This material will be written up in a forthcoming book about applications of Macaulay 2. I believe the present paper is in the spirit of the Kotor conference, and I am grateful to the editors for allowing me to present it here.

I would like to thank A. Ash, R. Edwards, P. Gunnells, M. Goresky, J. R. Myers, and S. Zucker for helpful conversations. This paper is a version, with stronger results, of the second half of my Ph. D. thesis, the first half of which appeared as [M1]. I am especially grateful to my advisor, Bob MacPherson, for all his help over the years. I thank the Institute for Advanced Study for their hospitality while this paper was being put into final form. I thank the NSF for support.

Section 1—Background

In this section, we summarize facts about algebraic groups, linear symmetric spaces, parabolic subgroups, and rational polyhedral sets. We generally follow [Z], specializing the material to the cases at hand. Parts (1.1)–(1.5) follow [S2, §3], which in turn relies on [W1].

(1.1). Let $\mathfrak{A}$ be a finite-dimensional simple associative algebra over $\mathbf{Q}$. This has the form $\mathfrak{A} = M_n(\mathfrak{D})$, where $M_n(R)$ is the ring of $n \times n$ matrices with entries in R and $\mathfrak{D}$ is a finite-dimensional division algebra over $\mathbf{Q}$. Fix such an expression.

In general, the $\mathbf{R}$-algebra $\mathfrak{D}_\mathbf{R} = \mathfrak{D} \otimes_\mathbf{Q} \mathbf{R}$ breaks up as a direct sum of simple algebras. *We assume $\mathfrak{D}_\mathbf{R}$ is simple* for the rest of the paper. This assumption is quite restrictive: see Remark 2 in (1.2).

The rank of $\mathfrak{D}$ over its center equals m^2 for some m. Then $\mathfrak{D}_\mathbf{R} = M_{\bar{m}}(\bar{\mathfrak{D}})$, where $\bar{\mathfrak{D}}$ is either $\mathbf{R}$, $\mathbf{C}$, or the quaternions $\mathbf{H}$, and the number $\bar{m}$ is m if $\bar{\mathfrak{D}} = \mathbf{R}$ or $\mathbf{C}$, and $m/2$ if $\bar{\mathfrak{D}} = \mathbf{H}$.

Let $\mathfrak{A}_{\mathbf{R}}$ be the $\mathbf{R}$-algebra $\mathfrak{A} \otimes_{\mathbf{Q}} \mathbf{R}$. The identifications above imply

$$\mathfrak{A}_{\mathbf{R}} = M_n(\mathfrak{D}_{\mathbf{R}}) = M_{n\bar{m}}(\bar{\mathfrak{D}}).$$

We write $x \in \mathfrak{A}_{\mathbf{R}}$ as $x = (x_{pq}) \in M_{n\bar{m}}(\bar{\mathfrak{D}})$ unless otherwise noted.

An involution on an algebra is a linear map $x \to x'$ which is an anti-automorphism ($(xy)' = y'x'$) and satisfies $x'' = x$. The involution is *positive* if $\operatorname{trace} x'x > 0$ for all $x \neq 0$. We fix the positive involution $'$ of $\mathfrak{A}_{\mathbf{R}}$ given by the standard involution in $M_{n\bar{m}}(\bar{\mathfrak{D}})$; this is the transpose when $\bar{\mathfrak{D}} = \mathbf{R}$, and conjugate transpose when it is $\mathbf{C}$ or $\mathbf{H}$.

We make the additional hypothesis that the involution on $\mathfrak{A}_{\mathbf{R}}$ comes from an involution on $\mathfrak{A}$, denoted by the same symbol $'$. That is, $'$ is defined over $\mathbf{Q}$. An element $x \in \mathfrak{A}$ or $\mathfrak{A}_{\mathbf{R}}$ is *symmetric* if $x = x'$.

The group $\mathbf{R}_+ = (0, \infty)$ acts on $\mathfrak{A}_{\mathbf{R}}$ as multiplication by the positive scalars. This is the group of *homotheties*.

(1.2). Let G be the group of inner automorphisms of $\mathfrak{A}_{\mathbf{R}}$. This is the group of invertible elements in $\mathfrak{A}_{\mathbf{R}}$, modulo the center. It is a semi-simple Lie group. Let $G_{\mathbf{Q}}$ be the group of rational points in G; this is the group of invertible elements of $\mathfrak{A}$, modulo the center of this group.

Take the Cartan involution $x \mapsto (x')^{-1}$, and let K be its fixed point set in G. This is a maximal compact subgroup of G.

Definition. The *symmetric space* for G is $X = K\backslash G$.

Inside the vector space $\mathcal{V}$ of symmetric matrices in $\mathfrak{A}_{\mathbf{R}}$, there is a cone of positive definite matrices, whose quotient modulo the homotheties is naturally identified with X. Equivalently, X is (the homothety classes of) the set of $x \in \mathfrak{A}_{\mathbf{R}}$ which are symmetric and satisfy the following positivity condition [W1, p. 3]: $\operatorname{trace}(y'xy)$ is positive definite as a function of y. The rational points of $\mathcal{V}$ are the symmetric elements in $\mathfrak{A}$; the rational points of X are (the homothety classes of) $X \cap \mathfrak{A}$.

Convention. The phrase "the homothety classes of" will often be omitted from now on.

An element $g \in G$ acts on $x \in X$ by $x \mapsto g'xg$. The group K is the stabilizer of the (rational) basepoint given by the identity matrix.

Under the inner product $\operatorname{trace} x'y$, the positive elements of $\mathcal{V}$ form a *self-adjoint homogeneous cone* C defined over $\mathbf{Q}$ [A-M-R-T, Ch. 2] [A1]. Our X is C mod homotheties.

Convention. We sometimes abuse notation by replacing G up to isogeny (i.e., with subquotients differing from G by finite index). In all our cases, K changes in a corresponding way, so $K\backslash G$ does not change at all.

Remark 1. Over $\mathbf{R}$, any self-adjoint homogeneous cone decomposes as a direct sum of indecomposables of the following types [A-M-R-T, p. 63]. We give a description of the cone, followed by a $K\backslash G$ presentation.

(1) the positive definite $n \times n$ symmetric matrices over $\mathbf{R}$; $\mathrm{SO}_n(\mathbf{R}) \backslash \mathrm{GL}_n^\circ(\mathbf{R})$;

(2) the positive definite $n \times n$ Hermitian symmetric matrices over $\mathbf{C}$; $\mathrm{U}_n \backslash \mathrm{GL}_n(\mathbf{C})$;

(3) the positive definite $n \times n$ Hermitian symmetric matrices over $\mathbf{H}$; $\mathrm{HU}_n \backslash \mathrm{GL}_n(\mathbf{H})$;

(4) one of the two "inside pieces" of the light cone in an $\mathbf{R}^{n+1}$ with an inner product of signature $(n,1)$; $(\mathrm{SO}(n) \times \mathrm{SO}(1)) \backslash \mathrm{SO}^\circ(n,1)$ (after homotheties);

(5) the 27-dimensional cone of positive definite 3×3 symmetric matrices over the octonions; a real form of E_6 modulo a compact form of F_4 (after homotheties).

The present paper covers the self-adjoint homogeneous cones that are defined over $\mathbf{Q}$, indecomposable over $\mathbf{R}$, and of type (1), (2) or (3).

The fourth type of C can be treated quickly, since it is of $\mathbf{R}$-rank one for all n. Here X has only one Satake compactification up to isomorphism (compare (6.4)). A Γ-admissible polyhedral cone decomposition $\mathcal{R}$ of C provides a similar decomposition $\mathcal{R}^*$ just by taking closures in $\mathcal{V}$. The closure of each $F \in \mathcal{R}^*$ is a closed convex set, hence a closed ball. The analogues of the theorems of (5.6)–(5.7) hold, without shellability hypotheses.

Remark 2. The range of division algebras $\mathfrak{D}$ we work with is fairly narrow, for two reasons. First, $'$ must be defined over $\mathbf{Q}$, so that we have a good theory of boundary components for C as in (1.11). When $\mathfrak{D}$ is a number field, for instance, this forces $\mathfrak{D}$ to be either totally real or a CM field.[2]

Second, if we allowed $\mathfrak{D}_{\mathbf{R}}$ to split over $\mathbf{R}$, then C would be a sum of two or more self-adjoint homogeneous cones over $\mathbf{R}$. The positive real scalars would act independently on each of the summands; this group is the identity component of the $\mathbf{R}$-split radical [B-S, (0.3)] of the automorphism group of C. The subgroup where a common scalar acts on all the summands would be the identity component of the $\mathbf{Q}$-split radical; X would be C mod the $\mathbf{Q}$-split radical. The quotient of the two radicals would be the $\mathbf{Q}$-*anisotropic homotheties*, say H_a. The automorphism group G of X would be reductive, consisting of a semi-simple part times H_a. The difficulty is that Satake compactifications have not been developed when G is reductive, only when G is semi-simple. It certainly seems possible to construct such a theory, along the lines of [S1] [S2] [Z]. However, it would be more appropriate for a separate paper. In (2.8), we outline how the theory might go.

(1.3). *Examples.* (a) If $\mathfrak{D} = \mathbf{Q}$, then $G = \mathrm{PGL}_n(\mathbf{R})$, the group of invertible $n \times n$ matrices over $\mathbf{R}$ modulo the scalars. Up to isogeny, $G = \mathrm{SL}_n(\mathbf{R})$ and $K = \mathrm{SO}_n(\mathbf{R})$. Referring to Remark 1 of (1.2), X is case (1) mod homotheties. This is the classical case for reduction theory of positive definite real quadratic forms.

(b) If $\mathfrak{D}$ is an imaginary quadratic number field, then $G = \mathrm{PGL}_n(\mathbf{C})$; up to isogeny, $G = \mathrm{SL}_n(\mathbf{C})$ and $K = \mathrm{SU}_n$. Here X is as in (2), modulo the positive real scalars.

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[2]Any finite-dimensional division algebra $\mathfrak{D}$ over $\mathbf{Q}$ can be used in the reduction theory of [A3] [A-M3]. There, we retract everything away from infinity, which is safer. In the present paper, we make F run out toward infinity; requiring that F hit only rational boundary components puts strong conditions on $\mathfrak{D}$.

(c) If $\mathfrak{D}$ is a quaternion algebra defined over $\mathbf{Q}$, then X is as in (3), modulo the positive real scalars.

In examples (a)–(c), the conjugate transpose $'$ is clearly defined over $\mathbf{Q}$.

(d) Consider a $\mathbf{Q}$-vector space of dimension $2l$ with basis $e_1, \ldots, e_{2l}$. The *Clifford algebra* $\mathfrak{C}(2l)$ is the quotient of the free non-commutative $\mathbf{Q}$-algebra on these symbols by the relations $e_i e_j = -e_j e_i$ $(i \neq j)$ and $e_i^2 = -1$. This is a central simple algebra over $\mathbf{Q}$ of dimension 2^{2l}. (If $l = 1$, we get the standard quaternions.) If $\mathfrak{D} = \mathfrak{C}(2l)$, then $\mathfrak{D}_{\mathbf{R}} = \mathbf{R}, \mathbf{H}, M_2(\mathbf{H})$, and $M_8(\mathbf{R})$ when $l = 0, 1, 2, 3$, respectively, and $\mathfrak{C}(2l+8) \otimes \mathbf{R} \cong \mathfrak{C}(2l) \otimes M_{16}(\mathbf{R})$ [P, pp. 132–3]. Thus $\bar{\mathfrak{D}} = \mathbf{R}$ and $\bar{m} = 2^l$ if $2l \equiv 0, 6 \pmod 8$, and $\bar{\mathfrak{D}} = \mathbf{H}$ and $\bar{m} = 2^{l-1}$ if $2l \equiv 2, 4 \pmod 8$. There is an involution on $\mathfrak{C}(2l)$ characterized by $e_i' = -e_i$. This is defined over $\mathbf{Q}$; it is positive because, if x is expressed as a $\mathbf{Q}$-linear combination of the 2^{2l} basis elements, then $\operatorname{trace} x'x$ is the sum of the squares of the coefficients.

(1.4). Fix a lattice $L \subset \mathcal{V}$—that is, a finitely-generated additive subgroup of the rational points of $\mathcal{V}$ that generates $\mathcal{V}$ as $\mathbf{Q}$-vector space. Let Γ_0 be the subgroup of G that preserves the lattice $(Lg = L)$. Let Γ be any subgroup of G commensurable with Γ_0. Then Γ is an *arithmetic group*, and every arithmetic subgroup of G arises in this way for some choice of L.

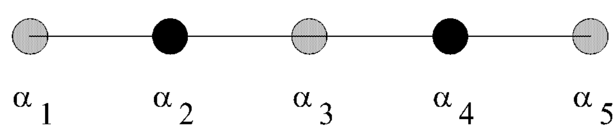
Example. Fix an order $\mathfrak{o}$ in $\mathfrak{D}$, and let $\Gamma_0 = \mathrm{GL}_n(\mathfrak{o})$ be the subgroup of $G_{\mathbf{Q}}$ consisting of the $x \in \mathfrak{A}$ such that both x and x^{-1} have entries in $\mathfrak{o}$. This arises for an appropriate L which contains, as a sublattice of finite index, the points in $\mathcal{V}$ with coordinates in $\mathfrak{o}$.

(1.5). The Lie algebra $\mathfrak{g}$ of G is identified with $\mathfrak{A}_{\mathbf{R}}$ (with the usual bracket $[x, y] = xy - yx$) modulo its center. Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification.

The set of all real diagonal elements in $\mathfrak{g}$—the classes (modulo center) of $x = (x_{pq})$ with $x_{pp} \in \mathbf{R}$ and $x_{pq} = 0$ if $p \neq q$—forms a maximal $\mathbf{R}$-split torus ${}_{\mathbf{R}}\mathfrak{a}$ in $\mathfrak{g}$. Let ${}_{\mathbf{R}}\Phi$ be the root system of $\mathfrak{g}_{\mathbf{C}}$ relative to ${}_{\mathbf{R}}\mathfrak{a}$. A restricted fundamental system ${}_{\mathbf{R}}\Delta$ in ${}_{\mathbf{R}}\Phi$ is the set of α_p defined by

$$\alpha_p = x_{p+1,p+1} - x_{p,p} \qquad \text{for } p = 1, \ldots, n\bar{m} - 1.$$

The $\alpha_{p\bar{m}}$ for $p = 1, \ldots, n-1$ are called *critical*; the others are *non-critical*. ${}_{\mathbf{R}}\Phi$ is a root system of type $A_{n\bar{m}-1}$. We draw the case $n = 3$, $\bar{m} = 2$, with the critical and non-critical roots pictured as solid and shaded circles, respectively.



We fix a maximal $\mathbf{Q}$-split torus ${}_{\mathbf{Q}}\mathfrak{a}$ in $\mathfrak{g}$, namely the subalgebra of ${}_{\mathbf{R}}\mathfrak{a}$ cut out by

$$\alpha_p(x) = 0 \qquad \text{for all non-critical } \alpha_p.$$

The $\mathbf{Q}$-rank of G and X is $n-1$.

Let ${}_{\mathbf{Q}}\Phi$ be the roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to ${}_{\mathbf{Q}}\mathfrak{a}$. The inclusion ${}_{\mathbf{Q}}\mathfrak{a} \hookrightarrow {}_{\mathbf{R}}\mathfrak{a}$ induces a pullback ${}_{\mathbf{Q}}\rho_{\mathbf{R}} : {}_{\mathbf{R}}\mathfrak{a}^* \twoheadrightarrow {}_{\mathbf{Q}}\mathfrak{a}^*$. Under this map, the non-critical roots of ${}_{\mathbf{R}}\Delta$ go to 0. The critical roots map to a restricted fundamental system ${}_{\mathbf{Q}}\Delta$ in ${}_{\mathbf{Q}}\Phi$, which is of type A_{n-1}. The map is bijective when restricted to the critical roots $\alpha_p \in {}_{\mathbf{R}}\Delta$, and it preserves the order of the Dynkin diagram.

Example. Many objects in this paper have pre-subscripts ${}_{\mathbf{Q}}$ or ${}_{\mathbf{R}}$. If $\mathfrak{D}$ is a number field, we may ignore these, because ${}_{\mathbf{Q}}\Delta = {}_{\mathbf{R}}\Delta$, etc.

Example. Let $\mathfrak{C}(4)$, the *sedenions*, be as in (d) of (1.3). Let $\mathfrak{A} = M_2(\mathfrak{C}(4))$. Then $\mathfrak{A}_{\mathbf{R}} = M_4(\mathbf{H})$, ${}_{\mathbf{R}}\Delta$ is an A_3, and ${}_{\mathbf{Q}}\Delta$ is an A_1. On Dynkin diagrams, ${}_{\mathbf{Q}}\rho_{\mathbf{R}}$ sends the middle dot of the A_3 to the dot in the A_1, and sends the other two dots to zero.

(1.6). Let $\mathfrak{h}$ be the usual real form of the Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ [Z, (1.1(2))], chosen so that ${}_{\mathbf{Q}}\mathfrak{a} \subseteq {}_{\mathbf{R}}\mathfrak{a} \subseteq \mathfrak{h}$. Let ${}_{\mathbf{C}}\Delta$ be a system of positive simple roots for $\mathfrak{h}$, compatible with ${}_{\mathbf{R}}\Delta$ under the pullback ${}_{\mathbf{R}}\rho_{\mathbf{C}}$ induced by ${}_{\mathbf{R}}\mathfrak{a} \hookrightarrow \mathfrak{h}$; these roots are all real-valued on $\mathfrak{h}$. The restriction of the Killing form to all three subalgebras is positive definite.

Let $\mathbf{F} = \mathbf{Q}$ or $\mathbf{R}$. Let ${}_{\mathbf{F}}A = \exp {}_{\mathbf{F}}\mathfrak{a}$, the identity component of a maximal $\mathbf{F}$-split torus of G. If $b = \exp a$, any $\alpha \in {}_{\mathbf{F}}\mathfrak{a}^*$ defines a character b^α on ${}_{\mathbf{F}}A$ that takes values in $(0, \infty)$.

For any $\Theta \subseteq {}_{\mathbf{F}}\Delta$, let ${}_{\mathbf{F}}\mathfrak{a}_\Theta$ be the subalgebra of ${}_{\mathbf{F}}\mathfrak{a}$ cut out by the equations $\alpha(x) = 0$ for all $\alpha \in \Theta$. Let ${}_{\mathbf{F}}A_\Theta$ be the corresponding subgroup of ${}_{\mathbf{F}}A$.

If we write A or $\mathfrak{a}$ without the ${}_{\mathbf{F}}$, then ${}_{\mathbf{R}}A$ and ${}_{\mathbf{R}}\mathfrak{a}$ are meant.

Definition. A set $\Theta \subseteq {}_{\mathbf{R}}\Delta$ is $\mathbf{Q}$-*rational* if there is an $\Upsilon \subseteq {}_{\mathbf{Q}}\Delta$ such that $\Theta = {}_{\mathbf{Q}}\rho_{\mathbf{R}}{}^{-1}(\Upsilon \cup \{0\})$. We write $\tilde{\Upsilon}$ for ${}_{\mathbf{Q}}\rho_{\mathbf{R}}{}^{-1}(\Upsilon \cup \{0\})$.

In our setting, the $\mathbf{Q}$-rational sets are precisely the subsets of ${}_{\mathbf{R}}\Delta$ that contain all the non-critical roots. If $\Theta = \tilde{\Upsilon}$, then ${}_{\mathbf{Q}}\mathfrak{a}_\Upsilon = {}_{\mathbf{R}}\mathfrak{a}_\Theta$.

(1.7). We set up some combinatorics surrounding the Dynkin diagrams. Let S be a subset of a vector space with an inner product. The *graph* of S has a vertex for each element of S; two vertices are joined by an edge if and only if the inner product of the corresponding elements is non-zero. We say S is *connected* if its graph is connected. We also speak of the connected components of S.

Let $\mathbf{F} = \mathbf{Q}$ or $\mathbf{R}$. Any $\lambda \in {}_{\mathbf{F}}\mathfrak{a}^*$ has a unique expression as a linear combination of the elements of ${}_{\mathbf{F}}\Delta$. The *support* $\operatorname{supp}(\lambda)$ of λ is the set of roots in ${}_{\mathbf{F}}\Delta$ for which the coefficients in this expression are non-zero.[3]

Let $\operatorname{supp}^*(\lambda)$ be the set of elements of ${}_{\mathbf{F}}\Delta$ which are not orthogonal to λ under the Killing form. If λ is translated by the Weyl group so that it lies in the Weyl

[3] supp depends on $\mathbf{F}$, but we drop the ${}_{\mathbf{F}}$ from the notation. The same holds for other symbols.

chamber, giving $\mathrm{supp}^*(\lambda)$ is equivalent to giving which walls of the Weyl chamber λ lies on.

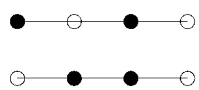
Let $\tau : G \to \mathrm{SL}(V)$ be an irreducible representation of G, with finite kernel, on a finite-dimensional real vector space V. Let $\lambda_0 \in \mathfrak{h}^*$ be the highest weight of τ with respect to ${}_{\mathbf{C}}\Delta$, and let $\mu_0 = {}_{\mathbf{F}}\rho_{\mathbf{C}}(\lambda_0)$ be its restriction to ${}_{\mathbf{F}}\mathfrak{a}^*$.

Definition. We say $\Theta \subset {}_{\mathbf{F}}\Delta$ is *τ-connected* if $\Theta \cup \{\mu_0\}$ is connected with respect to the Killing form. (This was called *τ-open* in [S1] [S2].) Every subset of ${}_{\mathbf{F}}\Delta$ has a largest τ-connected subset, its *τ-connected component.*

(1.8). For our G's, the parabolic subgroups have the familiar block upper-triangular form. We now make this precise. Fix $\Theta \subseteq {}_{\mathbf{R}}\Delta$.

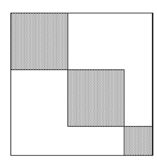
Definition. The *Θ-blocks* are a sequence of non-overlapping square blocks down the diagonal of the matrices in $\mathfrak{A}_{\mathbf{R}}$. They are characterized by the rule: the p-th and $(p+1)$-st diagonal entries lie in the same Θ-block if and only if $\alpha_p \in \Theta$.

Example. For SL_5 in a case with no non-critical roots, here are two examples of $\Theta \subset {}_{\mathbf{R}}\Delta$ (the solid dots) and the Θ-blocks.

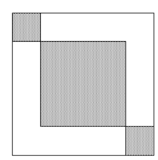


(1)

(2)



(1)



(2)

The *standard parabolic subalgebra* $\mathfrak{q}_\Theta$ is the subalgebra of $\mathfrak{g}$ with 0's to the lower left of the Θ-blocks, and arbitrary entries elsewhere. We have

$$\mathfrak{q}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{u}_\Theta. \tag{1.8.1}$$

The nilpotent radical $\mathfrak{u}_\Theta$ is the subalgebra with 0's in the Θ-blocks as well as below, and arbitrary entries to the upper right. The subalgebra having non-0's only inside the Θ-blocks splits as $\mathfrak{a}_\Theta$ and its orthocomplement under the Killing form, which is $\mathfrak{m}_\Theta$.

The *standard parabolic subgroup* Q_Θ is the identity component of the subgroup of G corresponding to $\mathfrak{q}_\Theta$. It breaks up as $M_\Theta A_\Theta U_\Theta$ with respect to the Θ-blocks, just as in (1.8.1). For example, $Q_\varnothing$ is the standard Borel subgroup, and $Q_{({}_{\mathbf{R}}\Delta)} = G$. The group M_Θ is semi-simple.

Every parabolic subgroup of G is conjugate by an element of G to a Q_Θ for a unique Θ.

The subgroup Q_Θ is defined over $\mathbf{Q}$ if and only if Θ is $\mathbf{Q}$-rational. The *rational parabolic subgroups* P are exactly the conjugates by elements of $G_{\mathbf{Q}}$ of the Q_Θ for the $\mathbf{Q}$-rational Θ.

Let Θ be $\mathbf{Q}$-rational. X can be coarsely divided into an $n \times n$ block matrix form, where each block corresponds to a single entry from $\mathfrak{D}_{\mathbf{R}} \cong M_{\bar{m}}(\bar{\mathfrak{D}})$. Say that the top left Θ-block has size equal to $\nu_1 \times \nu_1$ of these coarser blocks, the next Θ-block holds $\nu_2 \times \nu_2$ of the coarser blocks, etc., until the last Θ-block holds $\nu_r \times \nu_r$ of the coarser blocks. (In particular, $\nu_1 + \cdots + \nu_r = n$.) Let

$$W_2 = (\underbrace{0, \ldots, 0}_{\nu_1}, v_{\nu_1+1}, \ldots, v_n),$$
$$\ldots,$$
$$W_r = (\underbrace{0, \ldots, 0}_{\nu_1+\cdots+\nu_{r-1}}, v_{\nu_{r-1}+1}, \ldots, v_n),$$

where each v_j stands for an element of $\mathfrak{D}_{\mathbf{R}}$. Then

$$W_\Theta = \{\mathfrak{D}_{\mathbf{R}}^n = W_1 \supset W_2 \supset \cdots \supset W_r \supset W_{r+1} = 0\},$$

a descending flag of $\mathfrak{D}_{\mathbf{R}}$-modules, is called the *standard rational flag* for Θ. It is the largest flag preserved by the (right) action of Q_Θ.

The *rational flags* are the $G_{\mathbf{Q}}$-conjugates of W_Θ. There is a one-to-one correspondence between rational flags $W_P = W_\Theta \cdot g$ (for $g \in G_{\mathbf{Q}}$) and rational parabolics $P = g^{-1} Q_\Theta g$, where the latter is the normalizer of the former.

(1.9). Let $\mathbf{F} = \mathbf{Q}$ or $\mathbf{R}$. The characters $a \mapsto a^\alpha$ for $\alpha \in {}_{\mathbf{F}}\Delta$ give a canonical isomorphism ${}_{\mathbf{F}}A \cong (0, \infty)^{{}_{\mathbf{F}}\Delta}$. For $a, \tilde{a} \in {}_{\mathbf{F}}A$, we say $a \geqslant \tilde{a}$ if and only if the coordinates of the points in $(0, \infty)$ satisfy the $\geqslant$ relation, term by term; this is a partial order on ${}_{\mathbf{F}}A$. For any $c \in {}_{\mathbf{F}}A$, define $A(c) = \{a \in {}_{\mathbf{F}}A \mid a \geqslant c\} \cong [c, \infty)^{{}_{\mathbf{F}}\Delta}$.

For $\Theta \subseteq {}_{\mathbf{F}}\Delta$, the isomorphism restricts to an isomorphism ${}_{\mathbf{F}}A_\Theta \cong (0, \infty)^{{}_{\mathbf{F}}\Delta - \Theta}$ defined by the roots not in Θ. Let ${}_{\mathbf{F}}A_\Theta(c) = {}_{\mathbf{F}}A_\Theta \cap {}_{\mathbf{F}}A(c)$.

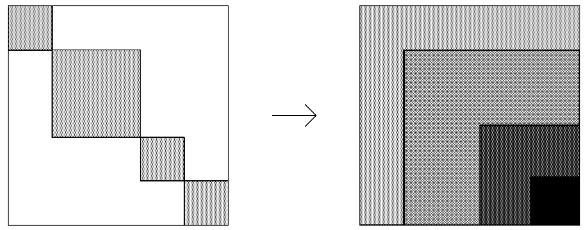
Let ${}_{\mathbf{F}}\bar{A}$ be the partial compactification of ${}_{\mathbf{F}}A$ corresponding to $(0, \infty]^{{}_{\mathbf{F}}\Delta}$ in the coordinates above. Let ${}_{\mathbf{F}}\bar{A}_\Theta$ be the closure of ${}_{\mathbf{F}}A_\Theta \subseteq {}_{\mathbf{F}}A$ in ${}_{\mathbf{F}}\bar{A}$; this has the obvious form $(0, \infty]^{{}_{\mathbf{F}}\Delta - \Theta}$. Let ${}_{\mathbf{F}}\bar{A}_\Theta(c) = \{a \in {}_{\mathbf{F}}\bar{A}_\Theta \mid a \geqslant c\} \cong [c, \infty]^{{}_{\mathbf{F}}\Delta - \Theta}$.

(1.10). We have the Iwasawa decomposition $G = KAN$, where $A = {}_{\mathbf{R}}A$ and $N = U_\varnothing$ as in (1.8). The map $(k, a, n) \mapsto kan$ is a bianalytic bijection. Identifying $X = K \backslash G \cong AN$, we get a left action of A on X making X into a principal A-bundle. This commutes with the ordinary (right) action of G. It is denoted $x \mapsto a \circ x$. It (or any G-translate of it) is called a *geodesic action*. There are also actions by the A_Θ, since these are all subgroups of A.

We need to know how the geodesic action for A_Θ looks with regard to the Θ-blocks in X. Let $a \in A_\Theta$, with coordinates $(a_1, a_2, a_3, \ldots)$ in $(0, \infty)^{{}_{\mathbf{R}}\Delta - \Theta}$.

Definition. Θ-blocks determine Θ-*chevrons* as in the figure on the next page.

The geodesic action $x \mapsto a \circ x$ is to multiply the first chevron by 1, the second by a_1, the next by $a_1 a_2$, the next by $a_1 a_2 a_3$, etc.



(1.11). Let $\Theta \subseteq {}_{\mathbf{R}}\Delta$ be $\mathbf{Q}$-rational. Let Y be the subspace of X with 0's everywhere except inside the lower right-hand (the r-th) Θ-block. Let ν_r be the size of the Θ-block relative to $\mathfrak{D}_{\mathbf{R}}$, as in (1.8). Such a Y will be called a *standard Peirce rational boundary component* of $\mathbf{Q}$-rank ν_r, and any $G_{\mathbf{Q}}$-translate will be called a *Peirce rational boundary component* of $\mathbf{Q}$-rank ν_r. These appear on the boundary of X as the sets of positive semidefinite forms having a given rational kernel of $\mathbf{Q}$-rank $n - \nu_r$.[4]

(1.12). We now define rational polyhedral sets. As in (1.2), the rational points of X are $X \cap \mathfrak{A}$ (mod homotheties). If a point on the boundary of X comes from $\mathfrak{A}$, it will lie in one of the Peirce rational boundary components, by [A1, p. 73].

Let $S = \varphi_1, \ldots, \varphi_s$ be non-zero symmetric points of $\mathfrak{A}$, with each φ_i lying on one of the Peirce rational boundary components.[5]

Definition. The *rational polyhedral set* $\mathrm{hull}^+(S)$ is the relative interior of the convex hull of S in $\mathfrak{A}_{\mathbf{R}}$, mod homotheties. We write

$$\mathrm{hull}^+(S) = \left\{ \sum_{i=1}^{s} \rho_i \varphi_i \;\middle|\; \rho_i > 0 \right\}. \tag{1.12.1}$$

Proposition. *Either every point in (1.12.1) corresponds to a positive definite form in X, or none of them do.*

Proof. Let $\rho_1, \ldots, \rho_s$ and $\hat\rho_1, \ldots, \hat\rho_s$ be two sets of positive numbers determining points $x, \hat x \in \mathrm{hull}^+(S)$. Assume x corresponds to a positive definite matrix in X. For any non-zero column vector $b \in (\bar{\mathfrak{D}})^{n\bar m}$, we have $b'xb = \sum \rho_i(b'\varphi_i b) > 0$. Each term $b'\varphi_i b \geqslant 0$, since φ_i is at least positive semi-definite. Multiplying the $\hat\rho_i$ by a common positive scalar (which is valid up to homothety), we may assume $\hat\rho_i \geqslant \rho_i$ for all i. Then $b'\hat x b = \sum \hat\rho_i(b'\varphi_i b) \geqslant \sum \rho_i(b'\varphi_i b) > 0$, as desired. $\square$

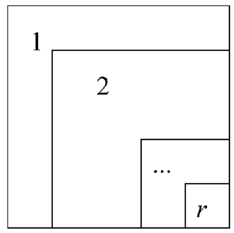
[4] In Section 2, we will define rational boundary components in any Satake compactification. In (6.4), we will see the Peirce definition is a special case of the later one. The Peirce decomposition [A-M-R-T, p. 80] is the main tool for constructing the boundary components for general self-adjoint homogeneous cones.

[5] "Boundary components" always includes X itself, with $\Theta = {}_{\mathbf{R}}\Delta$. This holds for Peirce, Satake, Borel-Serre, and crumpled corner components.

Corollary. *Either (1.12.1) is wholly contained in X, or it is disjoint from X.*

Every $\mathrm{hull}^+(S)$ is homeomorphic to an open cell, since it is a convex open set.

(1.13). Let Θ be $\mathbf{Q}$-rational. Number the Θ-chevrons as shown:



Say that φ_i *belongs to the j-th Θ-block* if it is contained in or to the lower right of the j-th chevron, but is not contained in or to the lower right of the $(j+1)$-st chevron. In this case, define $\tilde{\varphi}_i$ to have the same contents as φ_i inside the j-th Θ-block, and 0's elsewhere.

We sometimes re-index the φ_i as $\varphi_{j,k}$, where the j means φ_i belongs to the j-th block, and $\varphi_{j,k} = \varphi_{\nu_1+\cdots+\nu_{j-1}+k}$ $(k = 1, 2, \dots)$. In formulas like (1.12.1), we then rewrite the ρ_i as $\rho_{j,k}$.

(1.14). When we work with the Voronoï decomposition in (5.8)–(5.11), we will need a slight variation of the notation in (1.12). Let $\mathbf{x}$ denote non-zero (rational) points in $\mathfrak{D}^n$, or their images in $\mathfrak{D}^n_{\mathbf{R}}$, viewed as a row vector. Let $\varphi_{\mathbf{x}} = \mathbf{x}'\mathbf{x}$. This is an $n \times n$ symmetric matrix of elements of $\mathfrak{D}_{\mathbf{R}}$; since $'$ is defined over $\mathbf{Q}$, it is actually a rational element in $\mathcal{V}$. It lies on a Peirce rational boundary component of $\mathbf{Q}$-rank one. If $S = \{\mathbf{x}_1, \dots, \mathbf{x}_s\}$, we will abuse notation by writing

$$\mathrm{hull}^+(S) = \left\{ \sum_{i=1}^{s} \rho_i \varphi_{\mathbf{x}_i} \;\middle|\; \rho_i > 0 \right\}. \tag{1.14.1}$$

Let Θ be $\mathbf{Q}$-rational. Recall the rational flag W_Θ from (1.8). Say that $\mathbf{x}_i$ *belongs to the j-th Θ-block* if the member W_j of W_Θ contains $\mathbf{x}_i$, but W_{j+1} does not. That is, $\mathbf{x}$ belongs to the j-th Θ-block if it looks like

$$(\underbrace{0,\dots,0}_{\nu_1},\dots,\underbrace{0,\dots,0}_{\nu_{j-1}},\underbrace{\heartsuit,\dots,\heartsuit}_{\nu_j},\underbrace{*,\dots,*}_{\nu_{j+1}},\dots,\underbrace{*,\dots,*}_{\nu_r})$$

where the $\heartsuit$'s and $*$'s are arbitrary elements of $\mathfrak{D}$, and not all of the $\heartsuit$'s are zero. This is equivalent to saying $\varphi_{\mathbf{x}_i}$ belongs to the j-th Θ-block.

We often re-index $\mathbf{x}_i$ as $\mathbf{x}_{j,k}$ in the same manner as in (1.13). Define $\tilde{\mathbf{x}}_{j,k}$ by setting to zero all the entries of $\mathbf{x}_{j,k}$ outside the j-th group; that is, if

$$\mathbf{x}_{j,k} = (\underbrace{0,\dots,0}_{\nu_1},\dots,\underbrace{0,\dots,0}_{\nu_{j-1}},\underbrace{\heartsuit,\dots,\heartsuit}_{\nu_j},\underbrace{*,\dots,*}_{\nu_{j+1}},\dots,\underbrace{*,\dots,*}_{\nu_r})$$

then

$$\tilde{\mathbf{x}}_{j,k} = (\underbrace{0,\dots,0}_{\nu_1},\dots,\underbrace{0,\dots,0}_{\nu_{j-1}},\underbrace{\heartsuit,\dots,\heartsuit}_{\nu_j},\underbrace{0,\dots,0}_{\nu_{j+1}},\dots,\underbrace{0,\dots,0}_{\nu_r}).$$

We have $\varphi_{\tilde{\mathbf{x}}_{j,k}} = \tilde{\varphi}_{\mathbf{x}_{j,k}}$.

(1.15). For any set T inside a linear space (such as $\mathfrak{A}_{\mathbf{R}}$, $\mathcal{V}$, or $\mathcal{V}$ mod homotheties), the *relative interior* of T is the interior of T in the linear span of T. This is denoted $\operatorname{int} T$.

Let $B(t)$ and $B'(t)$ be matrices whose entries are real-valued functions for sufficiently large $t \in \mathbf{R}$. We say $B(t) \sim B'(t)$ as $t \to \infty$ if $\lim_{t\to\infty} B_{ij}(t)/B'_{ij}(t) = 1$ for each i, j.

Section 2—Satake Compactifications

We describe Satake's compactifications of X/Γ, again following [Z] and specializing to our cases. We focus on the maximal Satake compactifications, postponing the treatment of the other compactifications until Section 6. Satake introduced X^*, a compactification of the whole of X, in [S1]. However, X^*/Γ is not a Hausdorff space, since X^*'s irrational boundary components are poorly adapted to the action of Γ. One defines a subspace ${}_{\mathbf{Q}}X^*$ containing only X and the rational boundary components. One also refines the topology of ${}_{\mathbf{Q}}X^*$ near the rational boundary components, using the technology of Siegel sets, until Γ acts properly discontinuously on ${}_{\mathbf{Q}}X^*$. The *(maximal) Satake compactification of* X/Γ is ${}_{\mathbf{Q}}X^*/\Gamma$ [S2].

(2.1). Let τ be a representation on V as in (1.7). There is an *admissible inner product* on V compatible with the involution $'$; that is, $\tau({g'}^{-1})^* \cdot \tau(g) = I$ for all $g \in G$, where * is the adjoint with respect to the admissible inner product. It follows that the map $\tau_0(g) = \tau(g)^* \cdot \tau(g)$ descends to X, and takes values in the space $S(V)$ of self-adjoint endomorphisms of V. Modding out by the action of the scalars on $S(V)$, we get an embedding, also denoted $\tau_0 : X \to \mathbf{P}S(V)$.

Definition. [S1, §2.1] The *Satake compactification* $X^* = X^*_\tau$ of X is the closure in $\mathbf{P}S(V)$ of the image of X under τ_0.

The group G acts on X^* by homeomorphisms via $x \mapsto \tau(g)^* x \tau(g)$; with respect to this action, and the obvious action on X, the map $\tau_0 : X \to X^*$ is G-equivariant.

Example. Take example (a) of (1.3) with $n = 2$. The space $\mathcal{V}$ of all real symmetric 2×2 matrices, modulo homotheties, is the sphere S^2, and X is embedded as an open disc in this S^2. If we let τ be the standard representation (the representation by the matrices of (1.1)), then the embedding of the disc is by τ_0. This is the Klein model of the hyperbolic plane, where geodesics are straight line segments in the disc. The compactification X^* is the closure of the disc.

(2.2). Let $\mu_0 = {}_{\mathbf{R}}\rho_{\mathbf{C}}(\lambda_0)$ be the real restricted highest weight of τ, as in (1.7). The Satake compactifications are classified by how μ_0 lies on the walls of the Weyl chamber.

Theorem. *(a)* [S1, §4.4] *Up to homeomorphism as a topological G-space, X^*_τ depends only on* $\operatorname{supp}^*(\mu_0)$.

(b) [S1, pp. 101–2] *For every non-empty $\Theta \subseteq {}_{\mathbf{R}}\Delta$, there is a τ (and thus an X^*_τ) whose μ_0 satisfies* $\operatorname{supp}^*(\mu_0) = \Theta$.

In studying the compactifications, much complexity arises because certain constructions are only valid for the τ-connected subsets of ${}_{\mathbf{Q}}\Delta$ and ${}_{\mathbf{R}}\Delta$. For most of the paper, we will simplify matters by fixing a representation $\tau = \tau_{\max}$ for which every subset of ${}_{\mathbf{Q}}\Delta$ and ${}_{\mathbf{R}}\Delta$ is τ-connected. By the theorem, any two such representations give the same Satake compactification, the *maximal Satake compactification*. We will see later (6.2) that all Satake compactifications are quotients of this one by a map that restricts to the identity on X.

Proposition. *There exists a representation $\tau_{\max}$ of G which is defined over $\mathbf{Q}$ and such that every subset of ${}_{\mathbf{Q}}\Delta$ and ${}_{\mathbf{R}}\Delta$ is $\tau_{\max}$-connected.*

Proof. By general facts about semisimple Lie groups, there exists a representation τ_1 of G, defined over $\mathbf{R}$, whose highest weight lies in the interior of the Weyl chamber for ${}_{\mathbf{R}}\Delta$. The Galois group of $\mathbf{C}/\mathbf{Q}$ acts on ${}_{\mathbf{R}}\mathfrak{a}^*$, preserving both the Killing form and ${}_{\mathbf{R}}\Phi$. Hence every Galois translate of τ_1 has highest weight in the interior of the Weyl chamber for ${}_{\mathbf{R}}\Delta$. Take a set of representatives of the image of the Galois group in the permutation group of ${}_{\mathbf{R}}\Phi$, and let $\tau_{\max}$ be the tensor product of the translates of τ_1 by these representatives. Then the highest weight of $\tau_{\max}$ is Galois-invariant, implying $\tau_{\max}$ is defined over $\mathbf{Q}$. The highest weight will still lie in the interior of the Weyl chamber, which proves every subset of ${}_{\mathbf{R}}\Delta$ is $\tau_{\max}$-connected. That every subset of ${}_{\mathbf{Q}}\Delta$ is $\tau_{\max}$-connected follows from [Z, Cor. 2.4]. $\square$

(2.3). Let $K_\Theta = K \cap M_\Theta$. Set

$$X_\Theta = K_\Theta \backslash M_\Theta.$$

This is a Riemannian symmetric space of $\mathbf{R}$-rank $\#\Theta$.

In fact, X_Θ is a linear symmetric space. From (1.8), it is evident that $K_\Theta\backslash M_\Theta$ is the space Y_Θ of matrices that are positive definite, symmetric, and lie entirely within the Θ-blocks; here we mod out by the positive real scalars in each Θ-block separately.

We identify X_Θ and Y_Θ throughout the paper. The map is characterized by sending the basepoint $K_\Theta 1$ of X_Θ to the point of Y_Θ given by the identity matrix in each Θ-block; the M_Θ-action determines the rest of the identification.

Let $\bigoplus V_\mu$ be the weight space decomposition of V with respect to ${}_{\mathbf{R}}\mathfrak{a}$. One sees [S1, (2.4)] [Z, (2.9)] that τ induces a representation of M_Θ with finite kernel on

$V_\Theta = \oplus_{\mathrm{supp}(\mu_0-\mu)\subseteq\Theta} V_\mu$, inducing an embedding $X_\Theta \hookrightarrow \mathbf{P}S(V_\Theta)$ via τ_0. We identify $S(V_\Theta)$ as the linear subspace of $S(V)$ given by transformations which are zero on the weight spaces complementary to V_Θ, and thereby regard X_Θ as embedded in X^*.

Definition. The X_Θ for $\Theta \subseteq {}_{\mathbf{R}}\Delta$ are the *standard boundary components of* X^*. The G-translates of the X_Θ are the *boundary components* of X^* *of type* Θ.

The space X^* is the union of all the boundary components. The normalizer of $X_\Theta \cdot g$ is $g^{-1}Q_\Theta g$; this is a one-to-one correspondence between boundary components and parabolic subgroups. (In particular, "type Θ" is well defined.)

(2.4). We now summarize Satake's method of compactifying the quotients X/Γ.

A boundary component of X^* is *rational* if it meets certain technical conditions (cf. condition (Q) of [S2], or [Z, Def. 3.2]). In our case, the statement is simple.

Proposition. *The rational boundary components of* X^* *are those of the form* $X_\Theta \cdot g$, *where* $g \in G_{\mathbf{Q}}$ *and* $\Theta \subseteq {}_{\mathbf{R}}\Delta$ *is* $\mathbf{Q}$*-rational.*

Proof. Because our $\tau_{\max}$ is defined over $\mathbf{Q}$, Assumptions 1 and 2 of [Z, pp. 330–1] hold. Every Θ is $\tau_{\max}$-connected. Then [Z, Cor. 3.3] gives the result. □

Definition. Let ${}_{\mathbf{Q}}X^*$ be the union of all the rational boundary components (including X).

Example. In the Klein model example of (2.1), the rational boundary components are the rational points on the circle that bounds the disc.

(2.5). We now summarize how to topologize ${}_{\mathbf{Q}}X^*$. We will not give all the details, since in (2.7) we will give Zucker's alternative construction of ${}_{\mathbf{Q}}X^*$.

Definition. [W1, §§4–6] [S2, §2] [B-S, (6.1)] [Z, (3.9)] Let Θ be $\mathbf{Q}$-rational. A *standard Siegel set in* X *with respect to* Q_Θ *and* $x \in X$ is a set

$$\mathcal{S} = \mathcal{S}_{c,\omega} = A_\Theta(c)\,\mathbf{o}\,x \cdot \omega,$$

where $c > 0$ is a constant and ω is some compact subset in $M_\Theta U_\Theta$ with non-empty interior. A *Siegel set* is any $G_{\mathbf{Q}}$-translate of a standard Siegel set.

The closure of a standard Siegel set in X^* meets only the standard boundary components. Hence the closure in X^* of any Siegel set meets only finitely many boundary components, all of which are rational.

Definition. A *fundamental set* in X for Γ is a subset $\Omega \subseteq X$ such that

(1) $X = \Omega \cdot \Gamma$;
(2) there are only finitely many $\gamma \in \Gamma$ such that $\Omega \cdot \gamma$ meets Ω;
(3) let Ω^* denote the closure of Ω in X^*; then Ω^* meets only finitely many boundary components of X^*, all of which are rational.

One can choose $\mathcal{S}$ large enough so that $X = \mathcal{S} \cdot G_{\mathbf{Q}}$. One can take a fundamental set Ω which is a union of translates of $\mathcal{S}$ by finitely many elements of Γ. Let Ω^* be the closure of Ω in X^* (the "usual" topology). One can arrange the choices so that, as sets, ${}_{\mathbf{Q}}X^* = \Omega^* \cdot \Gamma$. The main theorem of [S2] (Thm. 1; cf. [Z, (3.9)]) is that there is a unique topology on ${}_{\mathbf{Q}}X^*$ which coincides with the "usual" topology when restricted to Ω^*, on which Γ acts as a group of homeomorphisms, with ${}_{\mathbf{Q}}X^*$ Hausdorff and ${}_{\mathbf{Q}}X^*/\Gamma$ compact Hausdorff, and such that the Γ-action satisfies certain finiteness conditions near the boundary components. The topology does not depend on the choice of Γ, $\mathcal{S}$, or Ω. We always give ${}_{\mathbf{Q}}X^*$ this topology.

Remark. As on [S2, p. 562], we have a description of neighborhood bases of points $x \in {}_{\mathbf{Q}}X^*$. Let $\{\mathcal{U}_\alpha\}$ be a neighborhood basis for x in Ω^*. Let Γ_x be the stabilizer of x in Γ. Then $\{\mathcal{U}_\alpha \cdot \Gamma_x\}$ forms a neighborhood basis of x in ${}_{\mathbf{Q}}X^*$.

(2.6). We recall facts about the Borel-Serre compactification [B-S, §§3,5]. If Θ is $\mathbf{Q}$-rational, define the *standard corner* $X(Q_\Theta) = \bar{A}_\Theta \times^{A_\Theta} X$, where A_Θ acts on the X factor by the geodesic action. The usual (right) action of Q_Θ on X extends to an action on the corner. If P is an arbitrary rational parabolic subgroup, with $P = g^{-1}Q_\Theta g$ for $g \in G_{\mathbf{Q}}$, then the *corner* $X(P)$ is by definition the translate $X(Q_\Theta) \cdot g$; this is still a fiber product in the obvious way.

If $P' \subseteq P$ are rational parabolics, there is a natural embedding $X(P) \hookrightarrow X(P')$. The *Borel-Serre bordification* is $\bar{X} = \bigcup_P X(P)$, where the union is over all rational parabolics P, with identifications coming from these embeddings, and with the induced topology.

Let $e(Q_\Theta)$ be the quotient of X by the geodesic action of A_Θ, with canonical projection $b_\Theta : X \twoheadrightarrow e(Q_\Theta)$. We identify $e(Q_\Theta)$ with the subset of $\bar{A}_\Theta \times^{A_\Theta} X$ corresponding throughout the fiber product to the corners $(\infty, \dots, \infty) \in A_\Theta$. For an arbitrary rational parabolic P, define $e(P)$ and b_P analogously, and identify $e(P)$ with the analogous part of X_P. The $e(P)$ are the *Borel-Serre boundary components*. Our identifications give a canonical set bijection

$$X(P) = \coprod_{P' \supseteq P} e(P') \tag{2.6.1}$$

for rational parabolic P'.

(2.7). We now define an auxiliary space ${}_{\mathbf{Q}}\tilde{X}^*$, specializing [Z, (3.6)] to our case.

Let Θ be $\mathbf{Q}$-rational. The usual action of U_Θ commutes with the geodesic action, so $e(Q_\Theta)$ is a principal U_Θ-bundle. Let $\hat{e}(Q_\Theta)$ be the quotient of $e(Q_\Theta)$ by this U_Θ-action. Conjugating by $g \in G_{\mathbf{Q}}$ gives analogous definitions and quotient maps

$$e(P) \twoheadrightarrow \hat{e}(P) \tag{2.7.1}$$

for any rational parabolic P.

The *crumpled corner* $X^*(P)$ is the quotient of the Borel-Serre corner $X(P)$ in which each term $e(P')$ of (2.6.1) is collapsed down to $\hat{e}(P')$ as in (2.7.1). Again, there is a natural embedding $X^*(P) \hookrightarrow X^*(P')$ whenever $P' \subseteq P$. The space ${}_{\mathbf{Q}}\tilde{X}^* = \bigcup_P X^*(P)$ is the union over all rational parabolics P, with identifications coming from these embeddings and with the induced topology. The quotient maps $X(P) \twoheadrightarrow X^*(P)$ paste together into a quotient map $\bar{p} : \bar{X} \to {}_{\mathbf{Q}}\tilde{X}^*$.

One checks that $\hat{e}(Q_\Theta) = X_\Theta$ canonically. In turn, X_Θ lies in ${}_{\mathbf{Q}}X^*$ by definition. Conjugating by elements of $G_{\mathbf{Q}}$ and applying the Proposition of (2.4), we obtain a bijection of sets $\tilde{p} : {}_{\mathbf{Q}}\tilde{X}^* \to {}_{\mathbf{Q}}X^*$. A key result of [Z] (3.10) is that $\tilde{p}$ is continuous, and that it descends to a homeomorphism ${}_{\mathbf{Q}}\tilde{X}^*/\Gamma \to {}_{\mathbf{Q}}X^*/\Gamma$. This realizes the Satake compactification ${}_{\mathbf{Q}}X^*/\Gamma$ as a quotient of the Borel-Serre compactification $\bar{X}/\Gamma$.

(2.8). Here is a sketch of the "reductive Satake" theory of (1.2), Remark 2. The X would still be a Riemannian symmetric space, but it would break up as $X = \breve{X} \times H_a$, where $\breve{X}$ is the symmetric space of non-compact type for the semi-simple part of G, and H_a is a Euclidean symmetric space. The arithmetic group Γ would contain a subgroup Γ_a consisting of units of $\mathfrak{o}$ times the identity matrix; this subgroup would generate a lattice in H_a, and H_a mod the lattice would be a compact torus. A group $\breve{\Gamma}$ commensurable with Γ/Γ_a would act on $\breve{X}$. Thus X/Γ would be a fiber bundle over $\breve{X}/\breve{\Gamma}$ with fiber a compact torus. The fiber directions would not need to be compactified, so we could simply define the Satake compactifications of X/Γ to be the Satake compactifications of $\breve{X}/\breve{\Gamma}$ with the fibers suitably extended across the boundary. Decompositions $\mathcal{R}^*$ of the compactifications of X/Γ could be constructed just as in Section 5. The layout of the cells would reflect both $\breve{\Gamma}$ and the lattice in H_a.

Section 3—Neighborhood Bases in the Maximal Satake

Our next goal is to construct specific neighborhood bases of points on the boundary of ${}_{\mathbf{Q}}X^*$. Assume $y \in \Omega^*$, where Ω^* is a fundamental set (2.5). In Proposition 3.2, we construct a neighborhood basis $\{\mathcal{U}_{c,\bar{\omega}}\}$ for y in Ω^*. This determines a neighborhood basis for y in ${}_{\mathbf{Q}}X^*$—see (3.3).

One knows neighborhood bases for points on the boundary of the Borel-Serre. We will use these to construct the neighborhood bases in Ω^* in ${}_{\mathbf{Q}}\tilde{X}^*$, and will show these give neighborhood bases in Ω^* in ${}_{\mathbf{Q}}X^*$.

Throughout the section, let $\Theta \subseteq {}_{\mathbf{R}}\Delta$ be $\mathbf{Q}$-rational.

(3.1). We describe a neighborhood basis of a point in the Borel-Serre [B-S, Prop. 6.2ii]. Choose $x \in X$ and a standard parabolic Q_Θ. Let $y = b_\Theta(x)$. Let c run through $(0, \infty)$, and let ω run through the compact neighborhoods of the identity in $M_\Theta U_\Theta$. Then the closures $\bar{\mathcal{S}}_{c,\omega}$ in $\bar{X}$ of the Siegel sets $\mathcal{S}_{c,\omega}$ coincide with their closures in $X(Q_\Theta)$, and the family of all these $\bar{\mathcal{S}}_{c,\omega}$ forms a neighborhood basis of y in $\bar{X}$. The analogue holds for any rational parabolic P.

(3.2). We now construct neighborhood bases in Ω^*. We will need to compare the topologies on the three closures, $\bar{\Omega}$ in $\bar{X}$, $\tilde{\Omega}^*$ in ${}_{\mathbf{Q}}\tilde{X}^*$, and Ω^* in ${}_{\mathbf{Q}}X^*$.

Proposition. *Let $\Theta \subseteq {}_{\mathbf{R}}\Delta$ be* $\mathbf{Q}$*-rational. Choose $x \in X$. Let $\bar{y} = b_\Theta(x)$, $\tilde{y} = \bar{p}(\bar{y})$, and $y = \tilde{p}(\tilde{y})$. Let Ω be a fundamental set such that Ω^* contains y. Let c run through $(0, \infty)$, and let $\bar{\omega}$ run through the compact neighborhoods of the identity in M_Θ. Then the closures $\mathcal{U}_{c,\bar{\omega}}$ in Ω^* of the sets*

$$\Omega \cap (A_\Theta(c) \mathbf{o}\, x \cdot (\bar{\omega} U_\Theta)) \tag{3.2.1}$$

run through a neighborhood basis of y in Ω^*.

Again, the analogue is obtained for any rational parabolic P by conjugating by elements of $G_{\mathbf{Q}}$.

Proof. Since Ω is covered by a finite union of Siegel sets, it follows that when we work near a point y on a fixed boundary component, we may replace Ω with a Siegel set $\mathcal{S}$ which approaches y. Define the closures $\bar{\mathcal{S}}$, $\tilde{\mathcal{S}}^*$, and $\mathcal{S}^*$ in $\bar{X}$, ${}_{\mathbf{Q}}\tilde{X}^*$, and ${}_{\mathbf{Q}}X^*$. All three of these are compact Hausdorff [Z, (3.9)–(3.10)]. Also, $\tilde{p}$ induces a homeomorphism $\tilde{\mathcal{S}}^* \to \mathcal{S}^*$, so we may replace y with $\tilde{y}$ and Ω^* with $\tilde{\Omega}^*$. Enlarging $\mathcal{S}$ if necessary, we may assume $\mathcal{S} = A_\Theta(c_0) \mathbf{o}\, x \cdot (M_{\mathcal{S}} U_{\mathcal{S}})$, where $c_0 > 0$ and $M_{\mathcal{S}}$ and $U_{\mathcal{S}}$ are large enough compact neighborhoods of the identity in M_Θ and U_Θ, respectively.

Let $Y = \bar{\mathcal{S}} \cap \bar{p}^{-1}(\tilde{y})$. Clearly this is compact. By general facts about quotient maps, our result will be proved if we can show that, if $\mathcal{N}$ is a neighborhood of Y in $\bar{\mathcal{S}}$ that is saturated for $\bar{p}$ (i.e., is a union of fibers of $\bar{p}$), then $\mathcal{N}$ contains $\bar{p}^{-1}(\mathcal{U}_{c,\bar{\omega}})$, where $\mathcal{U}_{c,\bar{\omega}}$ is as in (3.2.1). That is, we must show $\mathcal{N}$ contains

$$A_\Theta(c) \mathbf{o}\, x \cdot (\bar{\omega} U_{\mathcal{S}}) \tag{3.2.2}$$

for some large enough c and small enough $\bar{\omega}$. Here we use the fact that, for rational parabolics $P' \supsetneqq P$, the fibers of $e(P') \twoheadrightarrow \hat{e}(P')$ are subgroups of the fibers for $e(P) \twoheadrightarrow \hat{e}(P)$.

Let $\bar{y}_1 \in Y$. Since $\bar{\mathcal{S}}$ is homeomorphic to the product of three compact spaces, namely $\bar{A}_\Theta(c)$, $M_{\mathcal{S}}$ and $U_{\mathcal{S}}$, there is a large $c_1 > 0$, and small neighborhoods $\bar{\omega}_1$ of 1 in M_Θ and υ_1 of 1 in U_Θ, so that $\mathcal{N}$ contains $\bar{A}_\Theta(c_1) \mathbf{o}\, x \cdot (\bar{\omega}_1 \upsilon_1)$. The result now follows from a standard compactness trick: as $\bar{y}_1$ ranges through Y (with only finitely many $\bar{y}_1$ actually needed), let c be the maximum of all the c_1, and let $\bar{\omega}$ be the intersection of all the $\bar{\omega}_1$. $\square$

(3.3). *Remark.* We can describe neighborhood bases for y in ${}_{\mathbf{Q}}X^*$, not merely in Ω^*, as in the Remark of (2.5) (with y replacing x). The group $\Gamma_x \cap M_\Theta$ is finite, and we can ignore it if we replace $\bar{\omega}$ by the (finite) intersection of all its translates by $\Gamma_x \cap M_\Theta$. The group $\Gamma_x \cap A_\Theta(c)$ is trivial for large enough c. So $\mathcal{U}_{c,\bar{\omega}} \cdot \Gamma_x$ is a union of translates of $\mathcal{U}_{c,\bar{\omega}}$ by a subgroup of U_Θ. Since $\mathcal{U}_{c,\bar{\omega}}$ is stable under the latter group, and assuming Ω^* is large enough, one can show $\mathcal{U}_{c,\bar\omega}\cdot\Gamma_x = \mathcal{U}_{c,\bar\omega}\cdot U_\Theta$. Thus a neighborhood basis for y in ${}_{\mathbf{Q}}X^*$ is

$$\{\bar A_\Theta(c)\,\mathbf{o}\,x\cdot(\bar\omega U_\Theta)\}$$

with $\bar\omega$ and c varying as in Proposition 3.2.

This remark is not needed in Section 4. Every F is contained in a suitable Ω by (3.5), so the sets $\Omega\cap\ldots$ of (3.2.1) are suitable.

(3.4). Proposition 3.2 will be most useful if, for a given $y\in X_\Theta$, we may choose the basepoint $x\in X$ so its coordinates are related to the coordinates of y in Y_Θ.

Proposition. *Let $y\in Y_\Theta$. Fix a representative of each Θ-block in its homothety class, and let x be the point in X determined by these blocks. Then $\tilde p(\bar p(b_\Theta(x)))$ coincides with y under the isomorphism $X_\Theta\cong Y_\Theta$.*

Proof. Since M_Θ acts equivariantly on all the spaces involved, it suffices to let x be given by a positive scalar multiple of the identity matrix in each Θ-block, and to show it maps under $\tilde p\bar p b_\Theta$ to the y given by the identity matrix. By the last sentence of (2.4), $\tilde p(\bar p(b_\Theta(x)))$ should be the identity coset in $K_\Theta\backslash M_\Theta$. But x corresponds to the coset Ka in $K\backslash G$ for some $a\in A_\Theta$. We find

$$\begin{aligned}\tilde p(\bar p(b_\Theta(Ka))) &= \tilde p(\bar p(K(A_\Theta)1 \text{ in } K(A_\Theta)\backslash G))\\ &= \tilde p(K(A_\Theta)1(U_\Theta) \text{ in } K(A_\Theta)\backslash G/(U_\Theta))\\ &= \tilde p(K_\Theta 1 \text{ in } K_\Theta\backslash M_\Theta)\\ &= K_\Theta 1 \text{ in } K_\Theta\backslash M_\Theta. \quad\square\end{aligned}$$

(3.5). To end the section, we show that

Proposition. *Any rational polyhedral set in X (1.12) is contained in a finite union of Siegel sets.*

Proof. Let the set be $\mathrm{hull}^+(S)$, with $S=\{\varphi_1,\ldots,\varphi_s\}$ for φ_i as in (1.12). This is a union of $s!$ pieces; one piece is

$$\left\{\sum_{i=1}^{s}\rho_i(\varphi_1+\cdots+\varphi_i)\;\middle|\;\rho_i>0,\varphi_i\in S\right\}, \tag{3.5.1}$$

and the others arise by permuting $\{\varphi_1,\ldots,\varphi_s\}$. The $\varphi_1+\cdots+\varphi_i$ are positive definite or semi-definite forms on $\mathfrak{D}^n$; hence they lie in Peirce boundary components. As $i=1,\ldots,s$, the kernels of the forms $\varphi_1+\cdots+\varphi_i$ define a rational flag. With this set-up, the fact that (3.5.1) is contained in a Siegel set is the main theorem of Avner Ash's thesis, which appeared in [A-M-R-T, p. 113]. $\square$

Section 4—Rational Polyhedral Cones and Boundary Components

The main theorems of the paper say essentially that the closures of the sets in a Γ-admissible polyhedral decomposition of X give a cell decomposition of the Satake compactification ${}_{\mathbf{Q}}X^*/\Gamma'$ for suitable arithmetic $\Gamma' \subseteq \Gamma$. We prove this in Section 5, using a double induction and shelling argument developed in [M-M1, (8.3)]. We need three ingredients for the argument, which Section 4 prepares. First, if F is any rational polyhedral set, we must identify the set F' in which its closure meets a given rational boundary component, say X_Θ. Our main technical result, Proposition 4.1, shows that if F' lies on the boundary, then its vertices are easily expressed in X_Θ in terms of the vertices of F. This is proved in (4.2)–(4.4).

Second, we must show that our candidate cell complex $\mathcal{R}^*$ is a Whitney stratification (4.8). Though we don't prove this until Section 5, since $\mathcal{R}^*$ is not set up until then, we do the main part of the work here. Third, we show that the link of each stratum meets each larger nearby stratum in an open ball. This is treated in (4.6)–(4.8); see (4.5) for an introduction to these subsections.

(4.1). Let F be an rational polyhedral set in X. Let $S = \{\varphi_1, \dots, \varphi_s\}$ be such that $F = \mathrm{hull}^+(S)$. Fix a $\mathbf{Q}$-rational $\Theta \subseteq {}_{\mathbf{R}}\Delta$. Re-index the $\varphi_i \in S$ as in (1.13) so that

$$F = \left\{ \sum_{j=1}^{r} \sum_{k=1}^{j_s} \rho_{j,k}\varphi_{j,k} \,\middle|\, \rho_{j,k} > 0 \right\}. \tag{4.1.1}$$

Define the $\tilde{\varphi}_{j,k}$ as in (1.13). Consider the following subset of Y_Θ:

$$E = \left\{ \sum_{j=1}^{r} \sum_{k=1}^{j_s} \rho_{j,k}\tilde{\varphi}_{j,k} \,\middle|\, \rho_{j,k} > 0 \right\} \tag{4.1.2}$$

Let F^* be the closure of F in ${}_{\mathbf{Q}}X^*$.

Proposition. *With notation as above, the isomorphism $Y_\Theta \cong X_\Theta$ identifies E with the relative interior of $F^* \cap X_\Theta$.*

Remarks. For $y \in Y_\Theta$, each Θ-block of y must be a positive definite matrix. So if there is a j such that $\sum_{k=1}^{s_j} \tilde{\rho}_{j,k}\tilde{\varphi}_{j,k}$ is positive definite for no $\tilde{\rho}_{j,1}, \dots, \tilde{\rho}_{j,s_j} > 0$, then E is empty, and F^* does not meet X_Θ.

As usual, there is an analogous result for arbitrary boundary components, obtained by acting by $G_{\mathbf{Q}}$.

We will often tacitly fix a point in its homothety class.

(4.2). *Proof of Proposition 4.1.* In this subsection we prove that the subset of X_Θ corresponding to E is contained in $F^* \cap X_\Theta$. Choose $y \in X_\Theta$ corresponding in Y_Θ to a point of E. Fixing a representative of the homothety class in each Θ-block determines an $x \in X$ as in (3.4). Let $\tilde\rho_{j,k}$ be fixed positive constants which represent x as in (4.1.2). (These constants are not necessarily unique, even up to scalar multiples.) Let $\mathcal{U}$ be any member of the family $\mathcal{U}_{c,\bar\omega}$ of neighborhoods of y constructed in (3.2). Define a path $\zeta(t)$ as follows:

$$\zeta(t) = \sum_{j=1}^{r} \sum_{k=1}^{s_j} t^j \tilde\rho_{j,k} \varphi_{j,k}. \tag{4.2.1}$$

This path lies in F for all $t \in \mathbf{R}$, since rational polyhedral sets are convex in the linear coordinates. We want to show that as $t \to \infty$, the path $\zeta(t)$ eventually lies in $\mathcal{U}$.

Divide the matrix of $\zeta(t)$ into Θ-chevrons as in (1.10). Let the *j-th strip* be the side part of the j-th chevron, the part that is not in the j-th Θ-block. Since y corresponds to a point of E, we have $E \neq \varnothing$; hence for *all* t, each Θ-block in $\zeta(t)$ contains a positive definite matrix, by a simple argument. Positive definite matrices are invertible. Hence there is a unique $u(t) \in U_\Theta$ such that $\zeta(t) \cdot u(t)$ is in Θ-block diagonal form.

The contents of the j-th Θ-block of $\zeta(t)$ are

$$\left(\sum_{k=1}^{s_j} \tilde\rho_{j,k} \tilde\varphi_{j,k} \right) \cdot t^j + (\text{terms of degree} < j \text{ in } t).$$

We claim that the contents of the j-th Θ-block of $\zeta(t) \cdot u(t)$ are

$$\begin{aligned} &\left(\sum_{k=1}^{s_j} \tilde\rho_{j,k} \tilde\varphi_{j,k} \right) \cdot t^j \\ &\qquad + \sum_{\text{finite}} (\text{homogeneous rational functions of } t \text{ of degree} < j). \end{aligned} \tag{4.2.2}$$

To prove the claim, recall the semi-direct product decomposition $U_\Theta = U_\Theta^{[1]} \cdots U_\Theta^{[r]}$, where $U_\Theta^{[j]}$ is the subgroup whose only non-zero entries outside the Θ-blocks lie in the j-th strip. Write $u(t) = u_1(t) \cdots u_r(t)$ with $u_j(t) \in U_\Theta^{[j]}$. Then $\zeta(t) \cdot u_1(t) \cdots u_j(t)$ has zeroes throughout strips 1 through j. One checks that the action of u_j changes the j'-th strip, for all $j' > j$, by $O(t^j)$ as $t \to \infty$.

Now let $x(t)$ be the path given by acting on x by the diagonal geodesic flow for A_Θ. In the $(0,\infty)^{\mathbf{R}\Delta-\Theta}$ notation for A_Θ, this is $(t, \ldots, t) \mathbin{\mathbf{o}} x$, and it equals

$$\sum_{j=1}^{r} \sum_{k=1}^{s_j} t^j \tilde\rho_{j,k} \tilde\varphi_{j,k}.$$

The main point is that, in the j-th Θ-block, $x(t)$ is exactly the highest order term of (4.2.2). That is, $x(t) \sim \zeta(t) \cdot u(t)$ as $t \to \infty$. Thus we can choose $m(t) \in M_\Theta$ such that $x(t) = \zeta(t) \cdot m(t) u(t)$ and such that $m(t)$ approaches the identity as $t \to \infty$. Taking inverses, we find that $\zeta(t) \in \mathcal{U}$ for large enough t, as desired.

(4.3). Now take a $y \in X_\Theta$ which corresponds in Y_Θ to a point outside the closure of E. We will show $y \notin F^*$. Let x be a lift of y in the Θ-blocks of X, as in (3.4).

We will be focusing on just one of the irreducible factors of Y_Θ; this means concentrating on one Θ-block. Since y is not in the closure of E, there is a $j_0 \in \{1, \ldots, r\}$ such that the j_0-th block of x is not in the closure of the set of $\nu_{j_0\bar{m}} \times \nu_{j_0\bar{m}}$ matrices

$$\left\{ \sum_{k=1}^{s_{j_0}} \rho_{j_0,k} \tilde{\varphi}_{j_0,k} \;\middle|\; \rho_{j_0,k} > 0 \right\}. \tag{4.3.1}$$

Now (4.3.1) is the interior of the convex hull of a finite set of points. There will exist η, a linear function of the coordinates of the j_0-th Θ-block, such that any element z of (4.3.1) satisfies $\eta(z) > 0$, whereas $\eta(x) < 0$. We want to exhibit a set $\mathcal{U}$ in the family $\mathcal{U}_{c,\bar{\omega}}$ of (3.2.1) such that η is negative on $\mathcal{U} \cap F$. This will prove $y \notin F^*$.

As $a \in A_\Theta(c)$ varies, move $a\,\mathbf{o}\,x$ in its homothety class in such a way that the part of it in the j_0-th Θ-block remains constant. With this normalization, η has the constant negative value $\eta(x)$ on $A_\Theta(c)\,\mathbf{o}\,x$.

To study the action of U_Θ, we need two lemmas.

Lemma 1. *Let*
$$f = \frac{a_1x_1 + \cdots + a_qx_q}{b_1x_1 + \cdots + b_qx_q}$$
where the x_i are real variables, the a_i and b_i are real constants, and all $b_i > 0$. Then $|f|$ is bounded on the set $\{x_1 > 0, \ldots, x_q > 0\}$.

Proof. f is continuous on the compact set $S^{q-1} \cap [0, \infty)^q$. ☐

Lemma 2. *The set $\{u \in U_\Theta \mid (A_\Theta(c)\,\mathbf{o}\,x \cdot u) \cap F \neq \varnothing\}$ is contained in a compact subset $\mathcal{K}$ of U_Θ. (The $\mathcal{K}$ is independent of c.)*

Proof. Recall the semi-direct product decomposition $U_\Theta = U_\Theta^{[1]} \cdots U_\Theta^{[r]}$. By induction on j, it suffices to prove that $\{u \in U_\Theta \mid (A_\Theta(c)\,\mathbf{o}\,x \cdot u) \cap F \neq \varnothing\}$ is contained in a compact subset of $U_\Theta^{[j]}$.

Consider formula (4.1.1). Let i_1, i_2 satisfy $1 \leqslant i_1 < i_2 \leqslant n\bar{m}$. Every $\varphi_{j,k}$ has non-negative diagonal entries, since it is positive semi-definite; for the same reason, its i_1i_1- and i_2i_2-entries are positive whenever its i_1i_2-entry is non-zero. By Lemma 1, we find that for all $\rho_{j,k} > 0$,

$$\frac{i_1i_2\text{-entry of (4.1.1)}}{i_1i_1\text{-entry of (4.1.1)}} \tag{4.3.2}$$

is bounded in absolute value by a constant independent of the $\rho_{j,k}$'s.

Let $u \in U_\Theta^{[j]}$. Since x is fixed, we get an affine-linear map

$$\{\text{entries in } U_\Theta^{[j]} \text{ in the } j\text{-th strip}\} \to \{\text{entries in } x \cdot U_\Theta^{[j]} \text{ in the } j\text{-th strip}\}.$$

This is an isomorphism, because the j-th Θ-block of x is positive definite and hence invertible. Applying (4.3.2) for all i_1 in the j-th Θ-block, we find that $\{u \in U_\Theta^{[j]} \mid x \cdot u \in F\}$ is contained in a compact set.

Furthermore, take $a \in A_\Theta(c)$ given as a diagonal matrix by

$$a = \operatorname{diag}(\underbrace{a_1, \ldots, a_1}_{\text{1st block}}, \underbrace{a_2, \ldots, a_2}_{\text{2nd block}}, \ldots),$$

and assume the $i_1 i_1$-entry of F lies in the j-th block. Then

$$\begin{aligned}\frac{i_1 i_2\text{-entry of } a \,\mathbf{o}\, x \cdot u}{i_1 i_1\text{-entry of } a \,\mathbf{o}\, x \cdot u} &= \frac{a_j \cdot (i_1 i_2\text{-entry of } x \cdot u)}{a_j \cdot (i_1 i_1\text{-entry of } x \cdot u)} \\ &= (4.3.2).\end{aligned}$$

The lemma follows. □

Recall that we have normalized $a \,\mathbf{o}\, x$ up to homotheties so that the j_0-th block remains fixed. Then one checks that the j_0-th block of any element of $a \,\mathbf{o}\, x \cdot u$ differs from the j_0-th block of $a \,\mathbf{o}\, x$, in each entry, by a sum of homogeneous rational functions in c with terms of degrees -1 through $-(j_0 - 1)$. The coefficients of the sum depend on u and x. By the compactness statement in Lemma 2, the j_0-th Θ-block of any element of $a \,\mathbf{o}\, x \cdot \mathcal{K}$ still differs from the j_0-th block of $a \,\mathbf{o}\, x$, in each entry, by a sum of homogeneous rational functions in c, with terms of the same negative degrees and with coefficients depending only on x and $\mathcal{K}$. Thus for large enough c, the values $\eta(z)$ will be negative and bounded away from zero for all $z \in A_\Theta(c) \,\mathbf{o}\, x \cdot \mathcal{K}$.

Finally, it is clear that the action of a small enough compact neighborhood $\bar{\omega}$ of the identity in M_Θ will change η by an arbitrarily small amount. Hence we can choose c large enough and $\bar{\omega}$ small enough so that η will be negative and bounded away from zero on $\mathcal{U} \cap F$.

(4.4). To finish the proof, we use the following elementary fact. Let $f : A \to B$ be a homeomorphism. Let $U \subseteq A$ and $V \subseteq B$ be open sets. If $f(U) \subseteq V$ and $f(A - \bar{U}) \subseteq B - \bar{V}$, then $f(U) = V$.

This proves Proposition 4.1. □

(4.5). Throughout (4.5)–(4.8), let $F = \operatorname{hull}^+(S)$ be a rational polyhedral set in X. Let Θ be $\mathbf{Q}$-rational. Let $F' = \operatorname{int}(F^* \cap X_\Theta)$, and assume this is non-empty.

Our next goal is to study the normal structure of F' in $F \cup F'$: we want to show there is a neighborhood of F' in $F \cup F'$ homeomorphic to $F' \times N$, where N is an open ball together with one point n_0 on its boundary, and where F' corresponds to $F' \times \{n_0\}$.

The main tool for studying F', given in (4.2), was to flow toward X_Θ along a path $\zeta(t)$ that was asymptotic to the geodesic flow $x(t)$. A first idea for studying the normal structure would be to see if F is foliated by such paths, so that F is a fiber bundle over F'. Unfortunately, this can be false. If $y \in X_\Theta$ and x is a lift of y in X as in (3.4), and F is not a simplex, there can be more than one choice of $\tilde{\rho}_i$ giving x, and distinct paths of the form (4.2.1) starting at x and ending at y. In (4.6)–(4.8) we cover F by a collection of open sets called *simplicial tents*; in these, the paths from x to y are unique. A refinement of the argument of (4.2) will give an embedding from each tent into F', with good enough controls that we can establish the normal-structure result in (4.8).

(4.6). As a subset of $\mathcal{V}$ mod homotheties, F is the interior of a compact convex polytope $\hat{F}$ with vertices among the $\varphi_i \in S$. Let $\hat{\Sigma}$ be the abstract closed simplex with vertex set S. Let $\hat{\sigma} : \hat{\Sigma} \twoheadrightarrow \hat{F}$ be the linear map sending a vertex to the corresponding φ_i. Restricting to the interior Σ of $\hat{\Sigma}$ gives a map $\sigma : \Sigma \twoheadrightarrow F$.

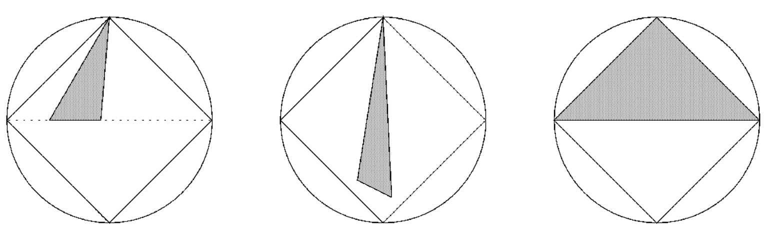
As in (1.13), partition the $\varphi_i \in S$ into $\varphi_{j,k}$ ($j = 1, \ldots, r$), and the ρ_i into $\rho_{j,k}$. View the $\rho_{j,k}$ as barycentric coordinates on Σ; there they are well-defined up to a common positive scalar multiple. For each j, define the *j-th Θ-face of $\hat{\Sigma}$* to be the span of the vertices for $\rho_{j,1}, \rho_{j,2}, \ldots$.

Take $j \in \{1, \ldots, r\}$. In $\hat{F}$, the convex hull of $\{\varphi_{j',k} \mid j' \geqslant j\}$ is a face $\hat{F}_j$ of $\hat{F}$ cut out by the Peirce boundary component for the j'-th Θ-chevrons, $j' \geqslant j$. Let $d_j = \dim \hat{F}_j$.

Definition. A *simplicial tent in $\hat{\Sigma}$* is any subset $\hat{T} \subseteq \hat{\Sigma}$ satisfying:

(1) $\hat{T}$ is the convex hull of $\hat{T}_1 \cup \cdots \cup \hat{T}_r$, where $\hat{T}_j$ is a simplex contained in the j-th Θ-face of $\hat{\Sigma}$;
(2) for each j, the dimension of the convex hull of $\hat{T}_j \cup \hat{T}_{j+1} \cup \cdots \cup \hat{T}_r$ is d_j;
(3) the restriction of $\hat{\sigma}$ to $\hat{T}$ is injective.

Example. Let $\mathfrak{D} = \mathbf{Q}$ and $G = \mathrm{SL}_2(\mathbf{R})$ up to isogeny. Here are the images in $\hat{F}$ of some typical simplicial tents, where $\hat{F}$ is the square in $X =$ (Poincaré disc) given by $F = \mathrm{hull}^+(S)$ for $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in the notation of (1.14), and with $\Theta = \varnothing$. The first three elements of S are $\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}$; the last is $\mathbf{x}_{2,1}$. The vertex for $\mathbf{x}_{2,1}$ is at the top of the picture.



In part (1), the vertices of $\hat{T}_j$ can be anywhere inside the j-th Θ-face, not necessarily at vertices of $\hat{\Sigma}$.

If $\hat{T}$ is a simplicial tent in $\hat{\Sigma}$, we call $T = \hat{T} \cap \Sigma$ a *simplicial tent in* Σ. Note that $\hat{T}$ is in fact a closed simplex, and T is its interior.

By a dimension count, the restriction of $\hat{\sigma}$ to each $\hat{T}_j$ is injective. One can then check that every point of Σ lies in the interior of some simplicial tent T in Σ. Hence every point in F is in the image of some simplicial tent.

(4.7). Keep the notation of the preceding two subsections. Let T be an arbitrary simplicial tent in Σ. We now construct a map $\sigma(T) \to F'$.

The action of $\mathbf{R}_+ = (0, \infty)$ on $\hat{\Sigma}$ given by $(\rho_{j,k}) \mapsto (t^j \rho_{j,k})$ for $t \in \mathbf{R}_+$ is a one-parameter group of homeomorphisms of $\hat{\Sigma}$. Each $\hat{T}_j$ is defined by a set of linear equations in the barycentric coordinates $\rho_{j,1}, \rho_{j,2}, \ldots$ of the j-th Θ-face of $\hat{\Sigma}$. Since $t \in \mathbf{R}_+$ acts by a common factor t^j on $\rho_{j,1}, \rho_{j,2}, \ldots$, the $\mathbf{R}_+$-action preserves each $\hat{T}_j$. Hence the action gives a one-parameter group of diffeomorphisms of T.

By the same proof as in (4.2)–(4.3), the restriction to T of the formula

$$\{(\rho_{j,k}) \mid j = 1, \ldots, r,\, k = 1, \ldots, s_j\} \mapsto \lim_{t \to \infty} \sum_{j=1}^{r} \sum_{k=1}^{s_j} t^j \rho_{j,k} \varphi_{j,k}$$

is a continuous surjection $\zeta_T : T \twoheadrightarrow F'$. Thus $\zeta_T \circ \sigma^{-1} : \sigma(T) \twoheadrightarrow F'$ is well-defined (recall that σ is injective on T) and continuous. (Note: ζ_T does not extend to $\hat{T}$ in general.)

Let E be as in (4.1), defined with respect to F and Θ. There is a continuous map $\tilde{\sigma} : \Sigma \to E$ given by $\sum_{j=1}^{r} \sum_{k=1}^{j_s} \rho_{j,k} \tilde{\varphi}_{j,k}$. We have a commutative diagram

$$\begin{tikzcd}
 & T \arrow[ddl, dashed, "\tilde{\sigma}|_T"'] \arrow[d, dashed, "\sigma|_T"] \arrow[ddr, dashed, "\zeta_T"] & \\
 & \sigma(T) \arrow[dl] \arrow[dr] & \\
E \arrow[rr, "\text{homeom.}"] & & F'
\end{tikzcd}$$

where the arrow $\sigma(T) \twoheadrightarrow E$ is $\tilde{\sigma} \circ \sigma^{-1}$ restricted to $\sigma(T)$.

Obviously $\tilde{\sigma}$ is an open map on Σ. If $z \in E$ and $(\rho_{j,k})$ is a point of $\tilde{\sigma}^{-1}(z)$, we could choose a simplicial tent T in Σ containing this point. Then $\tilde{\sigma}(T)$ would be an open neighborhood of z. By the commutative triangle of solid lines above, we see that for every $y \in F'$ there is a simplicial tent T such that $\zeta_T(T)$ is an open neighborhood of y in F'.

The map $\tilde{\sigma}|_T : T \to E$ is linear in the ρ's, so the pre-image of any point z of the image is a slice of T by some flat (i.e., affine-linear space). Such a slice is homeomorphic to an open ball of dimension $\ell = \dim F - \dim F'$. The pre-image of a small perturbation of z will be a slice of T by a nearby flat; since T is open and convex, the new slice will be homeomorphic to the original slice. In other words, $\tilde{\sigma}$ is a fiber bundle over its image, with fiber homeomorphic to an open ℓ-ball. The base is contractible, so the bundle is trivial. By the commutative diagram, $\zeta_T \circ \sigma^{-1} : \sigma(T) \to F'$ is a fiber bundle over its image F'_T, with the same fiber. We can actually say more: since points in the image of ζ_T are the limits of flows along these fibers, ζ_T induces a (trivial) bundle structure $\sigma(T) \cup F'_T \to F'_T$, where the fiber is homeomorphic to the subset $N = (0, \infty)^\ell \cup \{\mathbf{0}\}$ of $\mathbf{R}^\ell$.

(4.8). We now finish assembling the ingredients for the proof of the main theorem (5.6). Recall [G-M, pp. 36–37] that a *Whitney stratification* of a closed subset Z of a smooth manifold M is a decomposition of Z as a locally finite disjoint union of connected and locally closed smooth submanifolds $S_i \subseteq M$ (called the *strata*), satisfying two conditions:

(i) If S_i, S_j are strata of Z and $S_i \cap \bar{S}_j \neq \varnothing$, then $S_i \subseteq \bar{S}_j$.

(ii) Whitney's Condition B. Assume $S_i \subset \bar{S}_j$ are strata of Z. Let $y \in S_i$, let $\{x_k\}$ be a sequence in S_j converging to y, and let $\{y_k\}$ be a sequence in S_i converging to y. Assume that (with respect to some local coordinate system on M) the secant lines $\ell_k = \overline{x_k y_k}$ converge to some limiting line ℓ, and that the tangent planes $T_{x_k} S_j$ to S_j at x_k converge to some limiting plane τ. Then $\ell \subseteq \tau$.

Let F and F' be as in (4.5). We would like to see if the pair F', F satisfies Whitney's Condition B. The question makes no sense if F', F are regarded as subsets of ${}_{\mathbf{Q}}X^*$, where there is no obvious manifold M containing them. But we can choose a fundamental set Ω such that Ω^* contains a neighborhood of F^*. The image of Ω^* under τ_0 will lie in the manifold $M = \mathbf{P}S(V)$. In this way we can define a differentiable structure on a neighborhood of F' in $F \cup F'$; the structure will be independent of the choices of Ω and τ involved.

Proposition 1. *The pair F', F satisfies Whitney's Condition B.*

Proof. Let T_{x_l} be the tangent planes in $\mathbf{P}S(V)$ to F at a sequence of points $x_l \in F$ converging to $y \in F'$. Each T_{x_l} is spanned by a small perturbation of the coordinates $\rho_{j,k}$ in F near x_l. It is clear from (4.2)–(4.3) that $T = \lim_{l\to\infty} T_{x_l}$ exists, and that its intersection with F' is the tangent plane to F' at y. Whitney's Condition B is immediate. $\square$

In the same way, it makes sense to say that F^* is a subanalytic subset in this differentiable structure. By [G-M, p. 43], then, F^* has *some* Whitney stratification $\mathcal{W}$. Recall the definition of *link* from [G-M, p. 41].

Proposition 2. *Let $y \in F'$ be contained in the stratum S of $\mathcal{W}$.*

(a) If $\dim S = \dim F'$, then the link of y in F^* with respect to $\mathcal{W}$ meets F in an open ball of dimension $\dim F - \dim F' - 1$.

(b) If $\dim S < \dim F'$, then the link of y in F^* with respect to $\mathcal{W}$ meets F in an open ball of dimension $\dim F - \dim S - 1$.

Proof. The link is defined using a *normal slice* at y [G-M, p. 41], which we may choose as small as we like, and in any position we like as long as the slice is transverse to F'. Choose a simplicial tent T which gives a neighborhood of y in $F \cup F'$, and take the normal slice to be along the fibers of ζ_T. Part (a) follows. Part (b) is deduced from (a) by a straightforward argument, because F' is a manifold. $\square$

(4.9). *Remark.* Let $\hat{e}$ and $\hat{e}'$ be boundary components in ${}_{\mathbf{Q}}X^*$, of types Θ and Θ', respectively, with $\hat{e}'$ in the closure of $\hat{e}$. Let F be a rational polyhedral set in $\hat{e}$, and F' a rational polyhedral set in $\hat{e}'$, with F' in the closure of F. We know $\hat{e}$ is a product of linear symmetric spaces coming from the Θ-blocks. The component $\hat{e}'$ is a product of smaller linear symmetric spaces, where each factor of $\hat{e}$ has a boundary component formed by a product of one or more of the factors of $\hat{e}'$. The techniques of (4.2) and (4.7) can be applied on each factor of $\hat{e}$ separately. In particular, the propositions of (4.1) and (4.8) still hold for F and F', even though they are both on the boundary of X. We will use this fact without comment from now on.

SECTION 5—THE MAIN THEOREMS

So far we have studied rational polyhedral sets one at a time. However, we really want to tile X with such sets. In (5.1), we define the notion of Γ-admissible polyhedral decomposition, which makes this precise. In (5.2)–(5.4), we show that a Γ-admissible polyhedral decomposition $\mathcal{R}$ of X extends in a locally finite way to a decomposition $\mathcal{R}^*$ of ${}_{\mathbf{Q}}X^*$.

For a given $F \in \mathcal{R}^*$, we let $\mathcal{F}^*$ be the set of all elements of $\mathcal{R}^*$ which are in the closure F^* of F. In (5.6)–(5.7) we prove our main theorems, which say roughly that if every $\mathcal{F}^*$ is shellable, then $\mathcal{R}^*$ is a regular cell complex structure on ${}_{\mathbf{Q}}X^*$, and, for suitable arithmetic $\Gamma' \subseteq \Gamma$, it descends to make the Satake compactification ${}_{\mathbf{Q}}X^*/\Gamma'$ a finite regular cell complex.

From (5.8) to the end of the section, $\mathcal{R}$ is the Voronoï decomposition, and $G = \mathrm{SL}_n(\mathbf{R})$. We present several examples in (5.10).

(5.1). Definition. [A-M-R-T, p. 117] A Γ-*admissible polyhedral decomposition* of X is a collection $\mathcal{R} = \{F\}$ of rational polyhedral sets in X such that

(1) if $F \in \mathcal{R}$, then every face F' of F (such that $F' \subset X$) also belongs to $\mathcal{R}$;
(2) the intersection of two members of $\mathcal{R}$, if non-empty, is a face of each of them;
(3) $\mathcal{R}$ is closed under the action of Γ;
(4) modulo Γ, there are only a finite number of elements of $\mathcal{R}$;
(5) $X = \cup_{F\in\mathcal{R}} F$.

Ash constructs several different Γ-admissible polyhedral decompositions of C into rational polyhedral cones [A-M-R-T, II.5]. They can be effectively computed. Modulo homotheties, each decomposition descends to a Γ-admissible polyhedral decomposition $\mathcal{R}$ of X into rational polyhedral sets.

Fix an $\mathcal{R}$ from now on. In practice, we take it to be Γ_0-admissible, since it will then be Γ-admissible for any arithmetic $\Gamma \subseteq \Gamma_0$.

(5.2). Let D be a maximal set of Γ-inequivalent top-dimensional elements of $\mathcal{R}$, together with all their faces in X. By the definition, the number of cells in D is finite.

Lemma. *D is a fundamental set for Γ in X, and is contained in a finite union of Siegel sets.*

Proof. It is immediate that D satisfies properties (1) and (2) of a fundamental set (2.5). By (3.5), D is contained in a finite union of Siegel sets. As in (2.5), the closure of a Siegel set in X^* meets only a finite number of rational boundary components; property (3) of a fundamental set follows. $\square$

Corollary. *For any $F \in \mathcal{R}$, the closures F^* of F in X^*, ${}_{\mathbf{Q}}X^*$, and D^* coincide.* $\square$

(5.3). Proposition. *Let $\hat{e}(P)$ be a rational boundary component in ${}_{\mathbf{Q}}X^*$. Let $y \in \hat{e}(P)$. Then the set $\{F^* \cap \hat{e}(P) \mid F \in \mathcal{R}, y \in F^*\}$ has only finitely many distinct members.*

Note. Infinitely many F's can give rise to the same $F^* \cap \hat{e}(P)$.

Proof. Assume the contrary. Then there is an infinite subset $\Psi \subseteq \mathcal{R}$ such that, for $F \in \Psi$, the sets $F^* \cap \hat{e}(P)$ are all distinct. Fix a Siegel set $\mathcal{S}$ whose closure contains y in the interior of its intersection with $\hat{e}(P)$. Then each $F \in \Psi$ must meet $\mathcal{S}$. Since only finitely many Γ-translates of $\mathcal{S}$ can meet $\mathcal{S}$, some infinite subset of Ψ has the property that all its members are pairwise Γ-inequivalent. But this contradicts the definition of $\mathcal{R}$. $\square$

The proposition implies that the set $\{F^* \cap \hat{e}(P) \mid F \in \mathcal{R}\}$ has a common refinement that is locally finite in $\hat{e}(P) \cong X_\Theta$. The refinement is a union of rational polyhedral sets; we always use the relative interior of the sets. If F' is in the refinement, we include in the refinement all the (relatively open) faces of the closure of F' in $\hat{e}(P)$.

(5.4). Definition. The set $\mathcal{R}^*$ is the union of $\mathcal{R}$ and, for each rational parabolic P, the common refinements in $\hat{e}(P)$ just described.

Proposition. *$\mathcal{R}^*$ has only finitely many elements modulo Γ.*

Proof. Since there are only finitely many rational boundary components mod Γ, it suffices to prove that the part of $\mathcal{R}^*$ lying in a given $\hat{e}(P)$ is finite mod Γ. By Proposition 5.3, it suffices in turn to prove that $\{F^* \cap \hat{e}(P) \mid F \in \mathcal{R}\}$ has only finitely many distinct elements mod Γ, for then the refinement will be Γ-finite also.

Assume the contrary. Then there is an infinite subset Ψ of $\mathcal{R}$ which is pairwise inequivalent under $\Gamma \cap P$. By the Γ-finiteness, there is an infinite subset $\Psi' \subseteq \Psi$ of sets that are all Γ-equivalent but all pairwise inequivalent under $\Gamma \cap P$. Let

F_0, F_1, F_2 be distinct elements of Ψ' with $F_1 = F_0\gamma_1$, $F_2 = F_0\gamma_2$ for $\gamma_1, \gamma_2 \in \Gamma$. Since there are only finitely many rational parabolics mod Γ, we may assume γ_1 and γ_2 carry P by conjugation to the same parabolic. But then $\gamma = \gamma_2\gamma_1^{-1}$ preserves P. Hence $\gamma \in \Gamma \cap P$, and γ carries F_1 to F_2, contradicting the choice of Ψ'. $\square$

(5.5). For $F \in \mathcal{R}^*$, let $\mathcal{F}^*$ be the set of all elements of $\mathcal{R}^*$ which are contained in F^*.

Proposition. *$\mathcal{F}^*$ is a Whitney stratification of F^*.*

(As in (4.8), we make sense of this statement by covering F with a fundamental set D as in (5.2) and mapping D^* into the manifold $\mathbf{P}S(V)$.)

Proof. Condition (i) of the definition of a Whitney stratification (4.8) holds because $\mathcal{R}^*$ is a refinement of all the F^*'s. To check Whitney's Condition B, take $F_1, F_2 \in \mathcal{F}^*$ with $F_1 \subset F_2^*$. Within X itself, and within each boundary component separately, Whitney's Condition B is trivial because of the linear structure. Now say F_1, F_2 lie in different components of the space. As in (4.9), we may assume $F_2 \subset X$ and F_1 is outside X.

Assume F_1 lies in a boundary component $\hat{e}$. Let $F' = F_2^* \cap \hat{e}$. If F_1 lies in the interior of F', then Whitney's Condition B holds by Proposition 1 of (4.8). Otherwise, view F' as a polytope with respect to the linear coordinates on $\hat{e}$; then F_1 is an open subset in a proper face of (the closure of) F'. The result follows from a straightforward generalization of (4.7)–(4.8), using simplicial tents which contain a neighborhood of F_1 inside nearby proper faces of F in X. $\square$

(5.6). We now state the first version of our main theorem, which makes ${}_{\mathbf{Q}}X^*$ a regular cell complex. For facts about finite regular cell complexes and shellings, see [B-LV-S-W-Z, §4.7]. Recall that the *order complex* (or *geometric realization*) of a partially ordered set $(\mathcal{P}, \leqq)$ is the simplicial complex whose vertices are the elements of $\mathcal{P}$ and whose k-simplices are the chains $p_0 \lneqq p_1 \lneqq \cdots \lneqq p_k$ with $p_i \in \mathcal{P}$. If $\mathcal{A}$ is a finite simplicial complex of pure dimension m, a linear ordering $\sigma_1, \dots, \sigma_t$ of the maximal simplices of $\mathcal{A}$ is a *shelling* if either $m = 0$, or $m \geqslant 1$ and $\partial\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \partial\sigma_i\right)$ is of pure dimension $m-1$ for all $j = 2, \dots, t$. We say $\mathcal{A}$ is *shellable* if it has a shelling. If $\mathcal{A}$ is shellable, and every $(m-1)$-simplex is a face of exactly two m-simplices, then $\mathcal{A}$ is homeomorphic to the m-sphere [D-K].

For any $F \in \mathcal{R}^*$, $\mathcal{F}^*$ is a partially ordered set under the relation $\sigma' \subset \bar{\sigma}$ for $\sigma', \sigma \in \mathcal{F}^*$. Let $\partial\mathcal{F}^* = \mathcal{F}^* - \{F\}$ be the boundary complex of F^*, also viewed as a partially ordered set.

Theorem. *For any $F \in \mathcal{R}^*$ (5.4), assume that*

(a) the order complex of $\partial\mathcal{F}^*$ is shellable, and

(b) each codimension-one simplex of the order complex of $\partial\mathcal{F}^*$ meets exactly two top-dimensional simplices.

Then

(1) *For each $F_1 \in \mathcal{F}^*$, the closure of F_1 in F^* is homeomorphic to a closed ball. In particular, F^* itself is a closed ball.*
(2) *The balls in (1) give F^* the structure of a finite regular cell complex.*
(3) *The F^* together give ${}_{\mathbf{Q}}X^*$ the structure of a regular cell complex.*

Proof. To prove (1) and (2), we use the same double induction argument as in [M-M1, (8.3)]. Here is how to replace the named results from [M-M1] with results from this paper:

topic	*in* [M-M1]	*in this paper*
open balls	Thm. 1.12	(1.12)
$\mathcal{F}^*$ is Whitney	Sect. 6	Prop. 5.5
interior of link is open ball	Prop. 7.2	(4.8), Prop. 2

Part (3) is immediate; one checks the result finitely many balls at a time, using a fundamental set. □

(5.7). Here is the second version of our theorem about cell complexes, which applies to the maximal Satake compactification ${}_{\mathbf{Q}}X^*/\Gamma$.

Corollary. *Assume for all $F \in \mathcal{R}^*$ that the shellability hypotheses (a) and (b) of Theorem 5.6 are satisfied. Then for any Γ, there is an arithmetic subgroup $\Gamma' \subseteq \Gamma$ such that the maximal Satake compactification ${}_{\mathbf{Q}}X^*/\Gamma'$ is a finite regular cell complex, with its closed cells given by the images mod Γ' of the F^* for $F \in \mathcal{R}^*$.*

Proof. Since D is a fundamental set, at most a finite set of $\gamma \in \Gamma$ have the property that for some $F \in D$, $F^* \cdot \gamma \cap F^* \neq \varnothing$. We can choose a $\Gamma' \subseteq \Gamma$ which omits this finite set—for instance, let Γ' be the "principal congruence subgroup" of Γ that pointwise fixes L/NL, for a sufficiently large $N \in \mathbf{Z}$. A similar argument with $(\Gamma \cap Q_\Theta)/(\Gamma \cap U_\Theta)$, which is $\Gamma \cap M_\Theta$ up to commensurability, gives the same result for $F \in D^*$ lying on the boundary. □

Remark. The arithmetic group Γ' is *neat* if, for all $\gamma \in \Gamma'$, the eigenvalues of γ generate a torsion-free multiplicative subgroup of the algebraic closure of $\mathbf{Q}$. This implies Γ' acts without torsion on X and all its boundary components. This is a necessary condition for the Γ' in the Corollary.

(5.8). The previous theorems hold for any Γ-admissible polyhedral decomposition $\mathcal{R}$ of X. For the rest of Section 5, we specialize. Let $\mathfrak{D} = \mathbf{Q}$ and $G = \mathrm{SL}_n(\mathbf{R})$, and let $\mathcal{R}$ be the *Voronoï decomposition* [V1].[6] For general $\mathfrak{D}$, this is the perfect co-core decomposition of [A-M-R-T, p. 129].

[6] See also [M1, (2.6)]. The article [M2] is a survey of modern generalization and applications of this work of Voronoï. It is different from the decomposition by nearest-neighbor domains that Voronoï introduced in [V2] [V3].

One takes L to be the lattice of matrices in $\mathcal{V}$ whose diagonal entries are in $\mathbf{Z}$ and whose off-diagonal entries are in $\frac{1}{2}\mathbf{Z}$ [A-M-R-T, p. 144]. Then $\Gamma_0 = \mathrm{GL}_n(\mathbf{Z})$.[7] The $\mathcal{R}$ is Γ_0-admissible. By [B-C], every $F \in \mathcal{R}$ has the form $F = \mathrm{hull}^+(S)$ where $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_s\} \subset \mathbf{Z}^n - \{0\}$ in the notation of (1.14).

It is straightforward to compute $\mathcal{R}$ and classify its elements up to Γ_0-equivalence. The case of $n = 2$ goes back to the earliest days of quadratic forms. The classification for $n = 3$ was done in [Sou2]; $n = 4$ was done by [L-S] and (essentially) by [Što]; and $n = 5$ has been treated recently in [Ba]. When $\mathfrak{D}$ is imaginary quadratic, see [B] [S-V], [C-W] and others.

(5.9). Let P be a rational parabolic subgroup, and let W_P be the flag for P as in (1.8). We say *S respects the flag W_P* if for each k, $S \cap W_j$ spans W_j as a $\mathbf{Q}$-vector space.

Lemma. *$F^* \cap \hat{e}(P) \neq \varnothing$ if and only if S respects the flag W_P.*

Proof. We may assume $P = Q_\Theta$ for a $\mathbf{Q}$-rational Θ. Say S respects W_Θ and that the subset $\{\mathbf{x}_1, \ldots, \mathbf{x}_t\}$ of S spans W_j. Now the value of the positive semi-definite form $\varphi_{\mathbf{x}}$ on the column vector $\mathbf{y} \in \mathbf{R}^n$ is $\mathbf{y}'\varphi_{\mathbf{x}}\mathbf{y} = \mathbf{y}'(\mathbf{x}'\mathbf{x})\mathbf{y} = (\mathbf{x}\mathbf{y})'\mathbf{x}\mathbf{y}$. Here $\mathbf{x}\mathbf{y}$ is the ordinary (real) dot product of $\mathbf{x}$ (a row vector) and $\mathbf{y}$. In effect, $\mathbf{y}'\varphi_{\mathbf{x}}\mathbf{y} = (\mathbf{x}\mathbf{y})^2$, which is $\geqslant 0$. Any $\mathbf{y}$ will have $\mathbf{x}_i\mathbf{y} \neq 0$ for some $i = 1, \ldots, t$, so $\sum_{i=1}^{t} \rho_i \varphi_{\mathbf{x}_i}$ with $\rho_i > 0$ is positive definite. Since this holds for all j, every Θ-chevron for the F of (4.1.2) contains positive definite matrices. In particular, every Θ-block contains positive definite matrices. By the first remark of (4.1), then, $F^* \cap \hat{e}(P) \neq \varnothing$. The converse is similar. $\square$

(5.10). We now give some examples of the Voronoï decomposition $\mathcal{R}$ and of how to construct $\mathcal{R}^*$ using Proposition 4.1. The symmetric space X is denoted X_n. The cells in $\mathcal{R}^*$ will be called *Voronoï cells*.

Example 1. In the symmetric space $X = X_3$ for SL_3, which is of dimension 5, there is only one top-dimensional Voronoï cell F up to Γ_0-equivalence. It has the form $\mathrm{hull}^+(S)$, where $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_6\}$ and the $\mathbf{x}_i$ are the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We interpret these points in $\mathbf{P}^2(\mathbf{Q})$, where they give the complete quadrilateral $\mathbf{A}$ of the introduction (0.1). The $\varphi_{\mathbf{x}_i}$ are linearly independent, so F is an open 5-simplex. Using known properties of the decomposition (as in [M-M0, Ch. 3] [M1]), one sees that if $F' = \mathrm{hull}^+(S')$ is another top-dimensional cell that meets F in a facet, the vertices are related exactly as in the Carnot configuration. One also sees

[7] Technically, to allow G to contain $\mathrm{GL}_n(\mathbf{Z})$, we expand it up to isogeny to become $\mathrm{SL}_n^{\pm}(\mathbf{R})$, the subgroup of $\mathrm{GL}_n(\mathbf{R})$ with determinants ± 1.

that gluing the F's along facets completely determines $\mathcal{R}$. This justifies the claim in (0.1) that $\mathcal{X} = X$.

Now we must explain why $\mathcal{X}^*_{\max}$, as defined in (0.3), is the same as ${}_{\mathbf{Q}}X^*$. For rational boundary components of type $\Theta = \{\alpha_2\}$, the standard flag W_Θ is $\mathbf{Q}^3 \supset \{(0,*,*)\} \supset \{0\}$, basically a plane, which becomes a line in $\mathbf{P}^2$. Rational boundary components of this type Θ are called "line" boundary components. To understand the parts of F^* which lie in line boundary components, Lemma 5.9 says we must look at lines in $\mathbf{P}^2$ which are respected by S. Up to Γ_0-equivalence, there are two kinds of lines that S respects: those through three collinear points of S (sides of the quadrilateral), and those through only two points of S (diagonals of the quadrilateral). We look at these separately.

The flag W_Θ itself meets a side of the quadrilateral, with points $\begin{bmatrix} 0&0&0 \\ 1&0&1 \\ 0&1&1 \end{bmatrix}$. These three points are the $\mathbf{x}_{2,k}$, and the remaining points of S are the $\mathbf{x}_{1,k}$. Proposition 4.1 says that to find $F^* \cap X_\Theta$, we should take the bottom two rows of the $\mathbf{x}_{2,k}$ (the part cut out by the subspace in the flag), namely $\begin{bmatrix} 1&0&1 \\ 0&1&1 \end{bmatrix}$, and form their Voronoï cell in X_2. This cell is known to be a triangle. Thus in this and all Γ_0-equivalent cases, F^* meets a line boundary component in a triangle whenever the line lies on a side of the quadrilateral for F. The subcomplex of $\mathcal{X}^*_{\max}$ generated by the three points on the side of the quadrilateral corresponds exactly to the closure of this triangle.

The line $\{(*,*,0)\}$ in $\mathbf{P}^2$ meets a diagonal of the quadrilateral, in the points $\begin{bmatrix} 1&0 \\ 0&1 \\ 0&0 \end{bmatrix}$. If we conjugate by $g = \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$, we get the case where the flag is the standard W_Θ again, meeting the points $\begin{bmatrix} 0&0 \\ 0&1 \\ 1&0 \end{bmatrix}$. Proposition 4.1 says that the Voronoï cell for the bottom two rows $\begin{bmatrix} 0&1 \\ 1&0 \end{bmatrix}$—which is a line segment—is how F^* meets a line boundary component when the line lies on a diagonal of the quadrilateral.

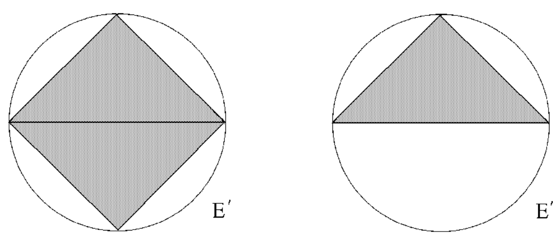
Next, we must see how F^* meets boundary components of type $\Theta = \{\alpha_1\}$. The standard flag is $\{(0,0,*)\}$, which becomes a point in $\mathbf{P}^2$. Rational boundary components of this type are "point" boundary components. We must look at flags consisting of one point which S respects—that is, at points on the quadrilateral. The six points of the quadrilateral are all equivalent, so it suffices to assume the flag is W_Θ itself. There is only one $\mathbf{x}_{2,k}$, namely $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; the other five points of S are $\mathbf{x}_{1,k}$'s. Proposition 4.1 says we should take the top two rows of the $\mathbf{x}_{1,k}$'s $\begin{bmatrix} 1&0&1&0&1 \\ 0&1&0&1&1 \end{bmatrix}$, and form their Voronoï cell in X_2. Omitting the redundant columns, we see the cell is a triangle in X_2 again. However, the top two rows of the $\mathbf{x}_{1,k}$ do not come from a subspace in the flag, but rather from the *quotient* $\mathbf{Q}^3/(0,0,*)$. Hence the triangle is specified by quotient data, in $\mathbf{P}^2$ modulo the point in the flag. This is how we get the rakes: the lines in the rake represent points in the quotient projective space $\mathbf{P}^2/(\text{point in the flag})$.

Example 2. In the symmetric space X_4 for $\mathrm{SL}_4(\mathbf{R})$ there are two different top-dimensional Voronoï cells up to Γ_0-equivalence, the *Desargues* and *Reye* cells F_D and F_R. The space X_4 has real dimension nine; the Desargues cell is the interior of a simplex in X_4 with ten vertices (all at infinity), and the Reye cell is a more complicated rational polyhedral set with twelve vertices. Write $F_D = \mathrm{hull}^+(S_D)$, $F_R = \mathrm{hull}^+(S_R)$ as in (1.14). If we view S_D and S_R as lying in $\mathbf{Q}^4$ modulo the scalars, we get configurations of points in projective three-space over $\mathbf{Q}$. These are Desargues' and Reye's configurations, which were studied in their own right by geometers of previous centuries. For pictures and more details, see [M-M0, pp. 266-267].

We may choose F_R so that its Reye configuration has the twelve points in $\mathbf{P}^3(\mathbf{Q})$ given by the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Let $\Theta = \{\alpha_1, \alpha_3\} \subset {}_{\mathbf{R}}\Delta$. By (4.1), F_R^* meets the boundary component $\hat{e}_\Theta$ in a product $E' \times E''$, where E' and E'' are cells in copies of the symmetric space X_2. The set E'' is a top-dimensional Voronoï cell for X_2, but E' is the union of *two* top-dimensional Voronoï cells for X_2. Here are pictures of E' and E'', in the Klein model of X_2:



Example 3 (Islamic tiles). The twelve points S_R of Reye's configuration can be partitioned uniquely into three sets of four, say S_1, S_2, S_3, in such a way that for each k, no two points of S_k are collinear with another point of S_R. Fix distinct $j, j' \in \{1, 2, 3\}$. There are sixteen ways to choose $\mathbf{x} \in S_j$ and $\mathbf{x}' \in S_{j'}$. Remarkably, the sixteen sets $\mathrm{hull}^+(S_R - \{\mathbf{x}, \mathbf{x}'\})$ divide F_R into a union of sixteen non-overlapping, mutually congruent pieces. Changing j, j' gives a different dissection into sixteen pieces, all congruent to those of the first dissection. A member of any one of these three dissections is called a $\frac{1}{16}$*-piece of* F_R.

Now consider $\mathrm{SL}_5(\mathbf{R})$ and its symmetric space X_5. There is a top-dimensional Voronoï cell F_0 in X_5 which meets a Satake boundary component $\hat{e}$ isomorphic to X_4, such that the rational polyhedral set $\mathrm{int}(F_0^* \cap \hat{e})$ is the union of

(1) one whole Reye cell,
(2) four whole Desargues cells, and
(3) $\frac{1}{16}$-pieces of four different Reye cells $F_{R,1}, \ldots, F_{R,4}$.

Taking translates of F_0, we find that all $\frac{1}{16}$-pieces of the $F_{R,i}$ occur in this way, for all three dissections. To form the cell complex $\mathcal{R}^*$ for SL_5, we must include the common refinement of the $\frac{1}{16}$-pieces over all three dissections. Each top-dimensional member of this refinement in the boundary component has one vertex at the center of F_R; that is, the refinement process introduces new internal vertices. This shows that, for $\mathrm{SL}_5(\mathbf{R})$ and indeed $\mathrm{SL}_n(\mathbf{R})$ for any $n \geqslant 5$, the refinement step in (5.3) is non-trivial.

(5.11). We now state the versions of our theorems for the Voronoï decomposition. These apply when $G = \mathrm{SL}_n(\mathbf{R})$ for $n = 2, 3$, or 4.

Let $\hat{e}(P)$ be a rational boundary component of type Θ with associated rational parabolic P. This $\hat{e}(P)$ is a product of symmetric spaces, one for each Θ-block. Say the Θ-blocks have sizes $\nu_1 \times \nu_1$, $\nu_2 \times \nu_2$, $\dots$, $\nu_r \times \nu_r$. The rational flag W_P is $\mathbf{Q}^n = W_1 \supset \cdots \supset W_r \supset W_{r+1} = \{0\}$, where $\dim(W_j/W_{j+1}) = \nu_j$ for $j = 1, \dots, r$.

Since W_P is rational, $\mathbf{Z}^n$ induces a full lattice in each relative quotient W_j/W_{j+1}; that is, it induces an additive subgroup Z_j, abstractly isomorphic to $\mathbf{Z}^{\nu_j}$, which lies in the relative quotient and spans it as $\mathbf{Q}$-vector space. With respect to Z_j, we define a Voronoï cell structure $\mathcal{R}_{\nu_j}$ on the symmetric space X_{ν_j}. The product $\mathcal{R}_{\nu_1} \times \cdots \times \mathcal{R}_{\nu_r}$ gives a polyhedral decomposition $\mathcal{R}(P)$ of $\hat{e}(P)$ which is $(\Gamma_0 \cap P)$-admissible and locally finite. Γ_0-equivalent boundary components have Γ_0-equivalent decompositions.

Theorem. *Let $G = \mathrm{SL}_n(\mathbf{R})$ for $n = 2$ or 3, and let $\mathcal{R}$ be the Voronoï decomposition. Then $\mathcal{R}^*$ is the union, over X and the other rational boundary components, of the induced Voronoï decompositions just defined. (No refinement is needed.) For each $F \in \mathcal{R}^*$ with associated $\mathcal{F}^*$, and for each $F_1 \in \mathcal{F}^*$, the closure of F_1 in F^* is homeomorphic to a closed ball. These balls give ${}_{\mathbf{Q}}X^*$ the structure of a regular cell complex. For suitable arithmetic $\Gamma' \subset \Gamma_0$, the (maximal) Satake compactification ${}_{\mathbf{Q}}X^*/\Gamma'$ is a finite regular cell complex, with its closed cells given by the images mod Γ' of the F^* for $F \in \mathcal{R}^*$.*

Proof. One checks by hand, using Proposition 4.1, that for every $F \in \mathcal{R}$ and every rational boundary component $\hat{e}(P)$ that F^* meets, $F^* \cap \hat{e}(P)$ is part of the decomposition $\mathcal{R}(P)$. This involves checking a representative mod Γ_0 of every pair consisting of $F = \mathrm{hull}^+(S)$ and a rational flag respecting S, as in Lemma 5.9. The same check shows that no refinement is necessary. The rest follows from Theorems 5.6 and 5.7, because the shellability hypotheses (a) and (b) have been checked. For $n = 2$, these are trivial. When $n = 3$, we have used a computer to shell the order complexes of the $\partial\mathcal{F}^*$, following the methodology in [M-M1, Remark 8.3]. (The largest $\partial\mathcal{F}^*$ was an S^4 with 1584 top-dimensional simplices. This took about fifteen minutes on a SparcStation SLC at Oklahoma State University.) $\square$

Theorem. *Let $G = \mathrm{SL}_4(\mathbf{R})$, and let $\mathcal{R}$ be the Voronoï decomposition. Then $\mathcal{R}^*$ is the union, over X and all the rational boundary components, of the induced Voronoï decompositions just defined.*

Again, the proof of this theorem is by direct calculation. Some refinement is needed, but only in the sense of Example 2 of (5.10): some of the $F^* \cap \hat{e}(P)$ are (the closures in $\hat{e}(P)$ of) the union of several different cells of $\mathcal{R}(P)$. We expect the conclusions of Theorems 5.6 and 5.7 to hold for $\mathrm{SL}_4(\mathbf{R})$, but the shellability would need to be checked.

SECTION 6—NON-MAXIMAL SATAKE COMPACTIFICATIONS

We now give an outline description of all the Satake compactifications, as opposed to just the maximal Satake, following [Z]. We establish the analogues of the theorems of (5.11) for all the compactifications. For $\mathrm{SL}_4(\mathbf{R})$ we still need the shellability hypothesis, but only for the *maximal* Satake. In (6.8), we suggest directions for how one could prove these results in a wider range of cases.

(6.1). Consider a representation $\tau : G \to SL(V)$ as in (1.7), with highest weight $\mu_0 = {}_{\mathbf{R}}\rho_{\mathbf{C}}(\lambda_0)$. Let $X^* = X^*_\tau$ be as in (2.1). This is the union of the G-translates of the boundary components X_Θ, as in the $\tau_{\max}$ case, except that the boundary component X_Θ (and its G-translates) appears if and only if Θ is τ-connected.

To get a compactification ${}_{\mathbf{Q}}X^*_\tau/\Gamma$ of X/Γ, we always assume τ satisfies certain rationality conditions (Assumptions 1 and 2 of [Z]; compare assumptions (D) and (Q) of [S2]). Then by [Z, (3.3)], the rational boundary components are the $G_{\mathbf{Q}}$-translates of the $X_{\tau\text{-conn}(\tilde{\Upsilon})}$, where Υ runs through the τ-connected subsets of ${}_{\mathbf{Q}}\Delta$, and τ-conn means τ-connected component. As before, let ${}_{\mathbf{Q}}X^*_\tau$ be the union of X and all the rational boundary components, regarded as a set. This is given a topology exactly as in (2.5). Then ${}_{\mathbf{Q}}X^*_\tau/\Gamma$ is the *Satake compactification* of X/Γ with respect to τ.

(6.2). There are natural quotient maps between the Satake compactifications. Let τ, τ' be representations with highest weights μ_0, μ_0'. Recall the definition of supp^* from (1.7).

Proposition. *Assume* $\mathrm{supp}^*(\mu_0') \subseteq \mathrm{supp}^*(\mu_0)$. *Then*

(a) The identity map on X induces a quotient map $X^*_\tau \twoheadrightarrow X^*_{\tau'}$ between the Satake compactifications of X.

(b) The identity map on X induces a quotient map ${}_{\mathbf{Q}}X^*_\tau \twoheadrightarrow {}_{\mathbf{Q}}X^*_{\tau'}$.

(c) For any arithmetic subgroup Γ, the identity map on X/Γ induces a quotient map ${}_{\mathbf{Q}}X^*_\tau/\Gamma \twoheadrightarrow {}_{\mathbf{Q}}X^*_{\tau'}/\Gamma$ between the Satake compactifications of X/Γ.

These are proved in sections (2.11), (3.6)–(3.8), and (3.9) of [Z], respectively.

(6.3). The last proposition implies that the set of Satake compactifications of X/Γ forms a poset, in which the maximal Satake is the unique maximal element. We want to describe this poset.

For the rest of Section 6, we impose the following restriction.

(C) $\mathfrak{D}$ is either a number field, or it is a quaternion algebra whose degree over its center (which is a number field) is 4.

Condition (C) is equivalent to saying that all the roots in ${}_{\mathbf{R}}\Delta$ are critical. Thus ${}_{\mathbf{Q}}\Delta = {}_{\mathbf{R}}\Delta$ and $\Theta = \tilde{\Upsilon} = \Upsilon$. Assumptions 1 and 2 follow from (C) [Z, p. 332]. If we dropped condition (C), there could a priori be several Satake compactifications for each $\Upsilon \subseteq {}_{\mathbf{Q}}\Delta$, given by varying which non-critical roots lay in $\operatorname{supp}^*(\mu_0)$.

Proposition. *Assume (C) holds. Then the poset of Satake compactifications of X/Γ is isomorphic to the poset of non-empty subsets of ${}_{\mathbf{Q}}\Delta$ ordered by inclusion.*

Proof. By part (b) of the Theorem in (2.2), the Satake compactifications X^*_τ, without rationality conditions, are classified by the poset of non-empty subsets $\Theta \subseteq {}_{\mathbf{R}}\Delta$. But (C) implies ${}_{\mathbf{Q}}\Delta = {}_{\mathbf{R}}\Delta$.

This proves there are no more compactifications than there are Υ's. To finish the proof, we must show that for every non-empty $\Upsilon \subseteq {}_{\mathbf{Q}}\Delta$, there is a τ with $\operatorname{supp}^* \mu_0 = \tilde{\Upsilon}$ that actually satisfies the rationality conditions. For this, we may take a (possibly non-rational) τ with $\operatorname{supp}^*(\mu_0) = \tilde{\Upsilon}$, and apply the Galois-averaging trick of (2.2). $\square$

Remark. The compactifications are in general homeomorphic to each other in pairs, because G has an outer automorphism that flips the Dynkin diagrams end-over-end. We have ignored this in (6.3). The two cell decompositions on such a pair will in general be different.

(6.4). The poset in (6.3) has $n-1$ minimal elements, the *minimal Satake compactifications.*

Let τ_{std} be the standard representation of G, where the matrices of (1.1) themselves give the representation. The highest weight μ_0 of τ_{std} satisfies $\operatorname{supp}^*(\mu_0) = \{\alpha_{n-1}\}$; that is, $\operatorname{supp}^*(\mu_0)$ is the rightmost dot of ${}_{\mathbf{R}}\Delta$. The space ${}_{\mathbf{Q}}X^*_{\tau_{\mathrm{std}}}/\Gamma$ is a minimal Satake compactification, the *standard minimal Satake compactification.* Here the embedding τ_0 is exactly the embedding of X in $\mathcal{V}$ in (1.2). The Peirce rational boundary components are exactly the Satake rational boundary components. For each $F \in \mathcal{R}$, F^* is simply the closure of F in $\mathcal{V}$. These ideas have also appeared in [H-Z].

(6.5). We now describe more precisely the quotient maps in Proposition 6.2. With notation as in (6.2), let $\Upsilon \subseteq {}_{\mathbf{Q}}\Delta$ be τ-connected, and let Υ' be its τ'-connected component. Write $\Theta = \tilde{\Upsilon}$ and $\Theta' = \tilde{\Upsilon}'$. By condition (C), Θ' is the τ'-connected component of Θ. Clearly $\Theta - \Theta'$ and Θ' lie in different connected components of Θ. It follows that M_Θ is the direct product (up to isogeny) of $M_{\Theta'}$ and $M_{\Theta-\Theta'}$, and that $X_\Theta = X_{\Theta'} \times X_{\Theta-\Theta'}$. The quotient maps in Proposition 6.2 are induced from the maps that collapse down the second factor of these products.

If $\tau = \tau_{\max}$, the main result in (4.1) says that an F^* meets X_Θ (if at all) in a product of rational polyhedral sets, one for each irreducible factor of X_Θ. To find the image of F^* in the smaller Satake compactification for τ', we mod out the $X_{\Theta-\Theta'}$ factors of F^*, keeping the $X_{\Theta'}$ factors. The same holds for all the rational boundary components by translating by $G_{\mathbf{Q}}$. In other words, we have the analogue of Proposition 4.1 for the non-maximal Satake compactifications.

Proposition. *Under assumption (C), the closure of F in ${}_{\mathbf{Q}}X^*_{\tau'}$ meets each rational boundary component in a rational polyhedral set, given as in Proposition 4.1 by keeping the appropriate diagonal blocks and collapsing the others down to points.*

(6.6). For the rest of Section 6, we specialize to the case $\mathfrak{D} = \mathbf{Q}$, so that $G = \mathrm{SL}_n(\mathbf{R})$ (up to isogeny). Let $\mathcal{R}$ be the Voronoï decomposition, with $X = X_n$ as in (5.10).

For $\mathrm{SL}_2(\mathbf{R})$ there is only one Satake compactification, which we have already treated ($\tau_{\max} = \tau_{\mathrm{std}}$). For $\mathrm{SL}_3(\mathbf{R})$ there are three Satake compactifications, two minimal and one maximal. We describe the cell decompositions $\mathcal{R}^*$ on the minimal ones.

Theorem. *Let $G = \mathrm{SL}_3(\mathbf{R})$. Let τ be a representation giving a minimal Satake compactification of X. Then under the map ${}_{\mathbf{Q}}X^*_{\tau_{\max}} \twoheadrightarrow {}_{\mathbf{Q}}X^*_{\tau}$ of (6.2), the cells on ${}_{\mathbf{Q}}X^*_{\tau_{\max}}$ push forward to cells in ${}_{\mathbf{Q}}X^*_{\tau}$. For suitable arithmetic $\Gamma' \subset \Gamma_0$, the cells on ${}_{\mathbf{Q}}X^*_{\tau}$ descend mod Γ' to make the minimal Satake compactification ${}_{\mathbf{Q}}X^*_{\tau}/\Gamma'$ into a finite regular cell complex.*

Proof. Assume $\mathrm{supp}^*(\mu_0) = \{\alpha_1\}$ (the case $\mathrm{supp}^*(\mu_0) = \{\alpha_2\}$ is similar). Choose $F \in \mathcal{R}^*$. Let F^* be the closure of F in ${}_{\mathbf{Q}}X^*_{\tau_{\max}}$. Since $\{\alpha_2\}$ is the only subset of Δ which is not τ-connected, let Y be the closure of the set of points of F^* which lie on a boundary component of type $\{\alpha_2\}$. We claim that Y has finitely many connected components, each of which is (the closure of) the intersection of F^* with a single type-$\{\alpha_2\}$ boundary component, and each of which is homeomorphic to a closed ball. First, we know from (4.1) that F^* meets a boundary component, if at all, in a set whose relative interior B is an open ball. By (5.6), F^* is a regular cell complex and B (by Thm. 5.11) is one of its open cells; this implies that the closure B^* of B in F^* is a closed ball. Second, in ${}_{\mathbf{Q}}X^*_{\tau_{\max}}$, an easy argument shows that any two distinct boundary components $\hat{e}, \hat{e}'$ of the same type Θ have disjoint closures. Hence the connected components of Y are exactly the B^*'s. Third, the number of connected components is finite, as in (5.4).

By Proposition 6.5, the map ${}_{\mathbf{Q}}X^*_{\tau_{\max}} \twoheadrightarrow {}_{\mathbf{Q}}X^*_{\tau}$ is given by collapsing every boundary component of type $\{\alpha_2\}$ to one point. (Distinct boundary components collapse to distinct points.) The image of F^* under this map is obtained by collapsing each connected component B^* of Y to a point. But whenever we have a big closed ball which is a finite regular cell complex, and a finite disjoint union of closed cells B^* contained in its boundary, then collapsing the B^*'s down to points does not change the big ball up to homeomorphism: it is still a closed ball. (Use the fact that every closed cell in a regular cell complex has a collared neighborhood, obtained, for example, by taking the second barycentric subdivision.) This proves the first statement of the theorem; the second follows as in (5.7). □

Example 1. We use the language of (0.1)–(0.3) and Example 1 of (5.10). By (6.4), the only non-τ-connected set for τ_{std} is $\{\alpha_1\}$. Proposition 6.5 therefore says that to pass from $X^*_{\tau_{\max}}$ to $X^*_{\tau_{\mathrm{std}}}$, we keep all the line boundary components but collapse away all the point boundary components. This corresponds to taking each cell of $\mathcal{X}^*_{\max}$ made from a rake $(p, \{m_\ell\})$ and replacing it with the cell for p alone. In short, we remove all rakes from our poset for $\mathcal{X}^*_{\max}$, and put back the singleton points of the quadrilaterals. This gives us exactly the poset for $\mathcal{X}^*$ as in (0.1). Hence $\mathcal{X}^*$ is ${}_{\mathbf{Q}}X^*_{\tau_{\mathrm{std}}}$.

Example 2. The standard *complete quadrangle cell* [M-M0, p. 247] in the symmetric space for $\mathrm{SL}_3(\mathbf{R})$ is the open 3-cell $F = \mathrm{hull}^+(S)$ for S given by the columns of the matrix $\begin{bmatrix} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&1 \end{bmatrix}$. The closure of F in the maximal Satake is a tetrahedron with its four vertices truncated. This is a polyhedron with four triangular faces and four hexagonal faces. The triangles are where F^* meets point boundary components. The edges that form the border between two hexagonal faces are where F^* meets line boundary components. When we pass to the standard minimal Satake, the point boundary components collapse, and we get a tetrahedron. When we pass to the other minimal Satake, the line boundary components collapse, and we get an octahedron. All this combinatorics can be seen in $\mathcal{X}^*_{\max}$, where $\mathcal{F}^*$ is the subcomplex generated by any set $\mathbf{B}$ of four points of the quadrilateral $\mathbf{A}$ with no three collinear, plus the rakes for $\mathbf{B}$.

(6.7). The analogue of Theorem 6.6 holds in the $\mathrm{SL}_4(\mathbf{R})$ case for all seven Satake compactifications (if we assume shellability for the maximal Satake). The maps between the compactifications are cell-preserving. The argument is more complicated than in (6.6), so we merely sketch it, in an example.

Consider the minimal Satake compactification for a τ with $\mathrm{supp}^*(\mu_0) = \{\alpha_1\}$. Let $q : {}_{\mathbf{Q}}X^*_{\tau_{\max}} \twoheadrightarrow {}_{\mathbf{Q}}X^*_{\tau}$. Let $F \in \mathcal{R}^*$. Let Y be the intersection of F^* with all rational boundary components whose types are not τ-connected. Unlike in (6.6), Y is connected. It is a union of two kinds of closed cells: (i) cells $\bar{F}'$ in boundary components $\hat{e}' \cong X_3$ of type $\{\alpha_2, \alpha_3\}$; (ii) cells $\bar{F}''$ in boundary components $\hat{e}'' \cong X_2 \times X_2$ of type $\{\alpha_1, \alpha_3\}$. Each $\hat{e}'$ collapses to a point under q. Since $\hat{e}''$ is a product, each $\bar{F}''$ has an induced product structure; q collapses the second factor of $\bar{F}''$ to a point. The $\bar{F}'$ are all disjoint from each other; the $\bar{F}''$ are all disjoint from each other.

To prove $q(F^*)$ is still a cell, we use two steps. First we do the collapsings on the second factors of the $\bar{F}''$. As in (6.6), using collared neighborhoods compatible with the product structure, we see this doesn't change the homeomorphism type of F^*; it is still a closed ball. Whenever two cells $\bar{F}'$ and $\bar{F}''$ meet, we have a smaller cell on $\partial\bar{F}'$ which collapses to one point, but—inductively—this doesn't change the homeomorphism type of $\bar{F}'$. The second step is to collapse what remains of the $\bar{F}'$. Again, this does not change F^* up to homeomorphism.

(6.8). We would like to generalize the results of (5.11), to say that the Voronoï decomposition $\mathcal{R}$ induces a regular cell decomposition for all the non-maximal Satake compactifications. We would replace $\mathcal{F}^*$ with $\mathcal{F}^*_\tau$, using only the boundary cells that arise for τ's compactification.

There is an easy way to get the generalization: simply check that $\partial\mathcal{F}^*_\tau$ satisfies the shellability hypotheses (a) and (b) of (5.6). However, it would be elegant if we did not have to shell $\partial\mathcal{F}^*_\tau$ again—if we could deduce the shellability of this complex from the shellability of $\partial\mathcal{F}^*_{\tau_{\max}}$, automatically.

To prove the theorem about $\mathrm{SL}_3(\mathbf{R})$ in (5.11), we used the fact that when a set of disjoint cells in the boundary of F^* collapsed down to distinct points, F^* was still a closed cell. It seems likely that this will be the situation whenever we pass from one Satake compactification to one immediately below it in the poset of compactifications. The example in (6.7) can be understood as applying this process twice: when we pass from $\tau_{\max}$ to the compactification with $\mathrm{supp}^*(\mu_0) = \{\alpha_1, \alpha_2\}$, the second factors of the $\bar{F}''$'s collapse to points, and when we pass from $\mathrm{supp}^*(\mu_0) = \{\alpha_1, \alpha_2\}$ to $\{\alpha_1\}$, the $\bar{F}'$'s collapse to points. Making this generalization precise for a range of groups G would require careful understanding of the combinatorics of τ-connected components.

In general, for all n, the choice of an $\Upsilon \subseteq {}_{\mathbf{Q}}\Delta$ determines a subcomplex T of the Tits building for $\mathrm{SL}_n(\mathbf{Q})$. Choosing $F \in \mathcal{R}^*$ determines a finite subcomplex T_F of the Tits building—namely, T_F is the set of all flags respecting S, where $F = \mathrm{hull}^+(S)$. The space $T \cap T_F$ is canonically embedded at infinity as a subcomplex of the order complex of $\mathcal{F}^*$. The set Y of (6.6)–(6.7) is a regular neighborhood of $T \cap T_F$ in F^*. Given any cellulated manifold (here, the sphere ∂F^*), a cellularly embedded subcomplex, and a cellularly embedded regular neighborhood of the subcomplex, collapsing the regular neighborhood down onto the subcomplex doesn't change the manifold up to homeomorphism. Thus if F^* is a closed ball in the maximal Satake, it collapses to a closed ball in all the other Satake compactifications. A rigorous proof of these assertions would require some PL topology, or decomposition theory along the lines of [D].

Section 7—Classical projective geometry and the maximal Satake compactification for $\mathrm{SL}_3(\mathbf{R})$

We now outline how to prove [M-M0, Thm. 7.2], which is the result in (0.4).

(7.1). Let k be a field of characteristic $\neq 2$. Define the set $\mathcal{C}$ of *$\mathcal{C}$-configurations in* $\mathbf{P}^2(k)$ as in [M-M0, (3.1)] [M, (1.1.2)]. Define the partial order $\preceq$ on $\mathcal{C}$ as in [M-M0, (1.3)] [M, (1.2)]. Define the *boundary $\mathcal{C}$-configurations* $\mathcal{C}'$, and the partial order $\sqsubseteq$ on $\mathcal{C} \cup \mathcal{C}'$, as in [M-M0, pp. 283–7].

(7.2). Let p be an odd prime, and let $\mathbf{F}_p$ be the field of p elements. Recall that the *principal congruence subgroup of level* p is $\Gamma(p) = \{\gamma \in \mathrm{SL}_n(\mathbf{Z}) \mid \gamma \equiv I \pmod{p}\}$, where I is the $n \times n$ identity matrix. This subgroup is neat (since $p > 2$).

The *normalization* of a simplicial complex Z is a simplicial complex $\tilde{Z}$ with a projection $\pi : \tilde{Z} \twoheadrightarrow Z$ which is uniquely characterized by the property that the points of $\pi^{-1}(z)$ are in bijection with the connected components of the complement of the codimension-two skeleton in the open stellar neighborhood of z. Informally, if the link at z has ℓ connected components, replace z with ℓ copies of itself, one for each connected component of the link, and pull them apart.

Let ${}_{\mathbf{Q}}X^*/\Gamma(p)$ be the maximal Satake compactification.

Theorem. *Assume $p \not\equiv 1 \pmod 6$. Let $\mathcal{C}$ and $\mathcal{C}'$ for $\mathbf{P}^2(\mathbf{F}_p)$ be as in the references above. Then the normalization of the order complex of $(\mathcal{C} \cup \mathcal{C}', \sqsubseteq)$ is homeomorphic to ${}_{\mathbf{Q}}X^*/\Gamma(p)$.*

Remarks. (1) If $p \equiv 1 \pmod 6$, then $(\mathcal{C}, \preceq)$ breaks up into three connected components, all homeomorphic to each other. Let $\mathcal{C}_0$ be one such component. Let $\mathcal{C}_0' = \{\mathbf{X} \in \mathcal{C}' \mid \exists \mathbf{Y} \in \mathcal{C}_0 \text{ s.t. } \mathbf{X} \sqsubseteq \mathbf{Y}\}$. Then the normalization of the order complex of $(\mathcal{C}_0 \cup \mathcal{C}_0', \sqsubseteq)$ has ${}_{\mathbf{Q}}X^*/\Gamma(p)$ as a three-sheeted covering space. Equivalently, if we replace $\Gamma(p)$ with $\tilde{\Gamma}(p)$ as in (0.4), and we take a connected component of the normalization, then the theorem holds as stated for all $p \neq 2, 3$.

(2) As in (6.6), there are two non-maximal Satake compactifications, If we remove from $\mathcal{C}'$ all elements $(\mathbf{Y}, \Phi)$ where Φ is a point—obtaining the same poset as in (0.1), but over $\mathbf{P}^2(\mathbf{F}_p)$—then the statement of Theorem 7.2 holds for the standard minimal compactification of (6.4). If we remove from $\mathcal{C}'$ all elements $(\mathbf{Y}, \Phi)$ where Φ is a line, the statement of Theorem 7.2 holds for the other minimal compactification.

Outline of proof of theorem. The corresponding result in the uncompactified case—that the order complex of $\mathcal{C}$ is homeomorphic to $X/\Gamma(p)$—is the main theorem of [M1]. The tie-in to the compactified case is essentially a direct calculation. For each $F \in \mathcal{R}$ up to $\mathrm{GL}_3(\mathbf{Z})$-equivalence, we enumerate all the cells in $\mathcal{R}^*$. One checks directly that for any fixed $F \in \mathcal{R}$, and for the $\mathbf{X} \in \mathcal{C}$ that corresponds to its class mod Γ, the set $\mathcal{B}_{\mathbf{X}} = \{\mathbf{Y} \in \mathcal{C} \cup \mathcal{C}' \mid \mathbf{Y} \sqsubseteq \mathbf{X}\}$ is combinatorially equivalent to the poset of closed faces in $\mathcal{F}^*$ ordered by inclusion. One knows, as a by-product of the shelling of $\partial\mathcal{F}^*$, that the order complex of $\mathcal{B}_{\mathbf{X}}$ is homeomorphic to F^*. Hence there is a continuous surjection α from ${}_{\mathbf{Q}}X^*/\Gamma(p)$ onto the order complex of $(\mathcal{C} \cup \mathcal{C}', \sqsubseteq)$, where α is obtained by patching together the homeomorphisms $F^* \to \mathcal{B}_{\mathbf{X}}$.

The map α is an embedding on $X/\Gamma(p)$, but it identifies certain boundary components. (See [M-M0, p. 285] for details.) However, since α is a homeomorphism on each cell separately, one can verify that α is an embedding onto the normalization of its image. $\square$

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